COUNTING LATTICE PATHS VIA A NEW CYCLE LEMMA

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Abstract. Let α, β, m, n be positive integers. Fix a line $L : y = \alpha x + \beta$, and a lattice point $Q = (m, n)$ on L . It is well known that the number of lattice paths from the origin to Q which touches L only at Q is given by

$$\frac{\beta}{m+n} \binom{m+n}{m}.$$

We extend the above formula in various ways, in particular, we consider the case when α and β are arbitrary positive reals. The key ingredient of our proof is a new variant of the cycle lemma originated from Dvoretzky–Motzkin [1] and Raney [8]. We also include a counting formula for lattice paths lying under a cyclically shifting boundary, which generalizes a result due to Irving and Ratten in [6], and a counting formula for lattice paths having given number of peaks, which contains the Narayana number as a special case.

Key words. lattice path, cycle lemma, Catalan number, Narayana number

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1. Introduction. Let α, β be positive reals, and let m, n be positive integers. Fix a line $L : y = \alpha x + \beta$, and a lattice point $Q = (m, n)$ on L , i.e., $n = \alpha m + \beta$. By a walk (or a lattice path) we will mean a path in $\mathbb{Z}^2$ with unit steps down and to the left (i.e., steps $(0, -1)$ and $(-1, 0)$, respectively). Let V be the set of walks from Q to the origin O . Clearly, we have $|V| = \binom{m+n}{m}$. Let $W \subset V$ be the set of walks which touch the line L at Q only. It is well known (for example, see Exercise 5.3.5 (b) of [2]) that if both α and β are integers, then

$$|W| = \frac{\beta}{m+n} \binom{m+n}{m}. \quad (1)$$

In particular, if $n = m + 1$ ($\alpha = \beta = 1$), then $|W| = \frac{1}{m+1} \binom{2m}{m}$ is the famous Catalan number. We will extend the above formula in various ways.

For a walk $w \in V$, we define the minimum y -distance $\delta(w)$ as follows: if w touches or crosses L after the first step, then let $\delta(w) = 0$, otherwise let $\delta(w)$ be the minimum of $\alpha m_0 + \beta - n_0$, where (m_0, n_0) runs over all lattice points on w except Q . We notice that $\delta(w) = 0$ iff $w \in V \setminus W$, so $\sum_{w \in W} \delta(w) = \sum_{w \in V} \delta(w)$. If α and β are positive integers, then $\sum_{w \in V} \delta(w)$ simply counts $|W|$ because $\delta(w) = 1$ for all $w \in W$. In this sense $\sum_{w \in V} \delta(w)$ can be viewed as a weighted sum corresponding to the number of walks.

For a real $t \geq 0$, let $W_t = \{w \in W : \delta(w) \geq t\}$. Then $|W_t|$ is a left-continuous step function of t , and it follows from the definition that

$$\int_0^1 |W_t| dt = \sum_{w \in V} \delta(w). \quad (2)$$

One can count each $|W_t|$ by a recursion as in Figure 2. On the other hand, it is somewhat surprising that the weighted sum (2) has a simple closed formula, as we

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FIG. 1. A line L and a lattice path.

FIG. 2. The number of partial paths in W_t from Q .

will soon see. Indeed one can get (2) almost effortlessly, without counting individual $|W_t|$.

EXAMPLE 1. Let $Q = (m, n)$ be a lattice point on a line $L : y = \alpha x + \beta$, where $\alpha = \sqrt{2}, \beta = 6 - 3\sqrt{2}, m = 3$, and $n = \alpha m + \beta = 6$, see Figure 1. Then we have

$$|W_t| = \begin{cases} 23 & \text{if } 0 < t \leq a, \\ 19 & \text{if } a < t \leq b, \\ 14 & \text{if } b < t \leq c, \\ 9 & \text{if } c < t \leq 1, \end{cases}$$

where $a = 3 - 2\sqrt{2}$, $b = 2 - \sqrt{2}$, $c = 5 - 3\sqrt{2}$ (see Figure 2), and

$$\int_0^1 |W_t| dt = 23a + 19(b - a) + 14(c - b) + 9(1 - c) = 56 - 28\sqrt{2}.$$

On the other hand, we have

$$\frac{\beta}{m+n} \binom{m+n}{m} = \frac{6-3\sqrt{2}}{3+6} \binom{3+6}{3} = 56 - 28\sqrt{2},$$

which verifies our main results, Theorem 1 and Corollary 1 stated below.

THEOREM 1. Let m, n be positive integers, and let α, β be positive reals with

$n = \alpha m + \beta$. Let V be the set of walks from (m, n) to the origin. Then, we have

$$\sum_{w \in V} \delta(w) = \frac{\beta}{m+n} \binom{m+n}{m}.$$

Apparently, Theorem 1 is a generalization of (1), and it can be equivalently stated in the following integration form.

COROLLARY 1. *Under the same assumptions as in Theorem 1, we have*

$$\int_0^1 |W_t| dt = \frac{\beta}{m+n} \binom{m+n}{m}.$$

We notice that if $\alpha \in (1/\ell)\mathbb{N}$ for some $\ell \in \mathbb{N}$, then

$$\int_0^1 |W_t| dt = \frac{1}{\ell} \sum_{t \in T} |W_t|,$$

where $T = \{\delta(w) : w \in W\} = \{1/\ell, 2/\ell, \dots, (\ell-1)/\ell, 1\}$. For the general case $\alpha \in \mathbb{R}$, the following interpretation would help to understand the LHS of the formula in the corollary intuitively. If $w \in W$, then the first step is a down step. Let $\tilde{w}$ be a walk obtained from w by omitting this down step. Namely, $\tilde{w}$ is a walk of $m+n-1$ steps, from $Q - (0, 1) = (m, n-1)$ to the origin. By translating $\tilde{w}$ to the direction $(0, t)$, we get a walk $\tilde{w} + (0, t)$ from $(m, n-1+t)$ to $(0, t)$. For $0 < t \leq 1$, we see that $w \in W_t$ if and only if $\tilde{w} + (0, t)$ does not cross the line L . Thus we can think of W_t as the set of $(m+n-1)$ -step walks, from $(m, n-1+t)$ to $(0, t)$, which do not cross L .

In Section 2 we first show a new variant of the cycle lemma (Lemma 1) originated from Dvoretzky–Motzkin [1] and Raney [8] (see also [4] chapter 7.5). Then we prove Theorem 1 using the lemma. It turns out this simple looking lemma is rather strong. For example, we can show a higher dimensional version of the theorem without any extra effort. In Section 3 we apply the lemma to extend the theorem in two ways: one is counting lattice paths lying under a cyclically shifting boundary (Theorem 2), and the other is counting lattice paths having given number of peaks (Theorem 3). As a special case (Corollary 2), we get the main result of [6] due to Irving and Ratten with a much simpler proof.

Before closing the section, we remark that the formula (1) is proved and generalized in [3] by using the reflection method instead of the cycle method. Generalizations of the formula (1) are also seen in [5].

2. Proofs. We start with a variant of the cycle lemma. Let $z = (z_1, z_2, \dots, z_k)$ be a sequence of reals. The i -th partial sum will mean $z_1 + z_2 + \dots + z_i$, where $1 \leq i \leq k$. The case $i = k$ is called the total sum. We define the weight $\theta(z)$ of z as follows: if every partial sum of z is positive, then let $\theta(z)$ be the minimum partial sum, otherwise let $\theta(z) = 0$. Let $z^{(j)} = (z_{1+j}, z_{2+j}, \dots, z_{k+j})$ denote the j -th shift of z , where the indices are read modulo k .

For example, if $z = (1, 3, -1)$, then $z^{(1)} = (3, -1, 1)$ with partial sums $\{3, 2, 3\}$ etc., and we get $\theta(z^{(0)}) + \theta(z^{(1)}) + \theta(z^{(2)}) = 1 + 2 + 0 = 3$, which coincides with the total sum of z . This is not just a coincidence but a consequence of the following new cycle lemma.

LEMMA 1. *Let $z = (z_1, \dots, z_k)$ be a sequence of reals with total sum $s > 0$. Then we have*

$$\sum_{0 \leq j < k} \theta(z^{(j)}) = s.$$

FIG. 3. Partial sums of a cyclically infinite sequence. The points of Y are shown by black dots. There are four minimal points in Y .

Proof. We extend the sequence z cyclically to obtain the infinite sequence $u = (u_1, u_2, \dots)$, where $u_i = z_j$ for $j \equiv i \pmod k$. Let $s_i = u_1 + \dots + u_i$ be the i -th partial sum of u , and plot (i, s_i) for $i \geq 1$ in the plane, see Figure 3. Let $X = \{(i, s_i) : i \geq 1\}$. For each i , the line passing (i, s_i) and $(i+k, s_{i+k})$ has slope s/k . We get (at most) k lines in this way, and let L be the line in the bottom. Note that all the points (i, s_i) are above the line L .

Define a partial order on X by $(i, s_i) \succ (j, s_j)$ iff $i < j$ and $s_i \geq s_j$. Geometrically, a point x in this poset X is minimal iff X contains no point to the right and (weakly) below from x . Now the crucial observation is as follows.

CLAIM 1. *Let $i \geq 1$. A point $(i, s_i) \in X$ is one of the minimal points iff partial sums of $z^{(i)}$ are all positive.*

Proof. Let $x = (i, s_i) \in X$. Suppose that some partial sum of $z^{(i)}$ is non-positive. Then there exists an integer j with $i < j$ such that $u_{i+1} + u_{i+2} + \dots + u_j = s_j - s_i \leq 0$. Hence, we have $s_j \leq s_i$. Therefore, we have $x \succ (j, s_j)$, and x is not a minimal point of X .

Conversely, suppose that x is not a minimal point. Then there exists a point (j, s_j) such that $i < j$ and $s_i \geq s_j$. Hence, we have a non-positive partial sum $u_{i+1} + u_{i+2} + \dots + u_j$ of $z^{(i)}$. $\square$

Choose a point (a, s_a) on L . Note that both (a, s_a) and $(a+k, s_{a+k})$ are minimal points. We will look at the set of $k+1$ points $Y = \{(i, s_i) : a \leq i \leq a+k\}$. Let t be the number of minimal points of Y , and let $\{(a_1, s_{a_1}), (a_2, s_{a_2}), \dots, (a_t, s_{a_t})\}$ be the minimal points, where $a = a_1 < a_2 < \dots < a_t = a+k$.

By the minimality, we have $s_{a_i} < s_{a_{i+1}}$, and so $s_{a_1} < s_{a_2} < \dots < s_{a_t}$. Let $h > a_i$. If (h, s_h) is minimal, then $s_h \geq s_{a_{i+1}}$. If (h, s_h) is not minimal, then there is a minimal point (g, s_g) with $h < g$ and $s_h \geq s_g \geq s_{a_{i+1}}$. Consequently, we have $\min\{s_h : h > a_i\} = s_{a_{i+1}}$.

CLAIM 2. *For $a \leq j < a+k$, we have*

$$\theta(z^{(j)}) = \begin{cases} 0 & \text{if } j \notin \{a_1, a_2, \dots, a_t\}, \\ s_{a_{i+1}} - s_{a_i} & \text{if } j = a_i \text{ and } 1 \leq i < t. \end{cases}$$

Proof. If $j \notin \{a_1, a_2, \dots, a_t\}$, then we have $\theta(z^{(j)}) = 0$ by Claim 1.

Let $j = a_i$ for some i with $1 \leq i < t$. Since $\theta(z^{(j)}) = \min\{s_h - s_j : j < h\}$, it suffices to show that $s_{a_{i+1}} \leq s_h$ for all integers h with $a_i < h$, which we have just shown above. $\square$

By Claim 2, we have

$$\sum_{0 \leq j < k} \theta(z^{(j)}) = \sum_{a \leq j < a+k} \theta(z^{(j)}) = \sum_{1 \leq i < t} (s_{a_{i+1}} - s_{a_i}) = s_{a_t} - s_{a_1} = s_{a+k} - s_a = s.$$

This completes the proof of Lemma 1. $\square$

Proof of Theorem 1. For a walk $w \in V$, let w_i be the i -th step, which is one step down or to the left. For each $w \in V$, we assign a sequence $\text{seq}(w) = (z_1, \dots, z_{m+n}) \in \mathbb{R}^{m+n}$ by $z_i = 1$ if w_i is a down step, and $z_i = -\alpha$ if w_i is a left step. Finally, set $\theta(w) = \theta(\text{seq}(w))$.

CLAIM 3. $\theta(w) = \delta(w)$.

Proof. Suppose that the first $i+j$ steps of w consist of j down steps and i left steps. Then after $i+j$ steps, we are at $(m, n) - j(0, 1) - i(1, 0) = (m-i, n-j)$. The y -distance from here to the line $L : y = \alpha x + \beta$ is $\alpha(m-i) + \beta - (n-j) = 1 \cdot j - \alpha \cdot i$, where we used $\alpha m + \beta - n = 0$. This y -distance coincides with the $(i+j)$ -th partial sum of $\text{seq}(w)$. So, $\theta(w)$ is the minimum y -distance, and the desired result follows. $\square$

By Claim 3, we have

$$\sum_{w \in V} \delta(w) = \sum_{w \in V} \theta(w).$$

For $w = (w_1, \dots, w_{m+n}) \in V$ and $0 \leq j < m+n$, let $w^{(j)} = (w_{1+j}, \dots, w_{m+n+j})$, where indices are read modulo $m+n$. Then, $\text{seq}(w^{(j)}) = (\text{seq}(w))^{(j)}$. Notice that $w^{(j)} \in V$ and $(w^{(-j)})^{(j)} = w$. Thus each walk $w \in V$ appears $m+n$ times in a multiset $\{w^{(j)} : w \in V, 0 \leq j < m+n\}$ of cardinality $|V|(m+n)$. This gives

$$\sum_{w \in V} \theta(w) = \sum_{w \in V} \frac{1}{m+n} \sum_{0 \leq j < m+n} \theta(w^{(j)}).$$

For $w \in V$, the total sum of $\text{seq}(w)$ is $n - \alpha m = \beta$. Thus, by Lemma 1, we have

$$\sum_{w \in V} \frac{1}{m+n} \sum_{0 \leq j < m+n} \theta(w^{(j)}) = \sum_{w \in V} \frac{\beta}{m+n} = \frac{\beta}{m+n} |V|,$$

which finishes the proof of Theorem 1. $\square$

The proof can be extended verbatim to higher dimensions. Namely, fix a hyperplane $L : x_d = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{d-1} x_{d-1} + \beta$ in $\mathbb{R}^d$, and a lattice point $Q = (m_1, \dots, m_d) \in \mathbb{N}^d$ on L , and consider lattice paths in $\mathbb{Z}^d$ with unit steps of d types

$$e_1 = (-1, 0, \dots, 0), e_2 = (0, -1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, -1).$$

Let V be the set of walks from Q to the origin. Then, we have $|V| = \frac{(m_1 + \dots + m_d)!}{m_1! m_2! \dots m_d!}$. Let $W \subset V$ be the set of walks which touch the hyperplane L at Q only. For a lattice point $P = (m'_1, \dots, m'_d)$, let $d(P)$ be $\alpha_1 m'_1 + \dots + \alpha_{d-1} m'_{d-1} + \beta - m'_d$, which is called the x_d -distance from P to L . For a walk $w \in V$, we define the minimum x_d -distance $\delta(w)$ as follows: if w touches or crosses L after the first step, then let $\delta(w) = 0$,

FIG. 4. *Cyclically shifting boundaries with $a = (2, -\sqrt{2}, 3, 0)$, $\beta = \sqrt{2}$.*

otherwise let $\delta(w)$ be the minimum of $d(P)$, where P runs over lattice points on w except Q . For a walk $w = (w_1, w_2, \dots, w_{m_1+\dots+m_d}) \in V$, where w_i is the i -th step of w , let us assign $\text{seq}(w) = (z_1, \dots, z_{m_1+\dots+m_d})$ to w , where $z_i = 1$ if $w_i = e_d$, and $z_i = -\alpha_j$ if $w_i = e_j$ with $j \neq d$. If P is a lattice point which is s steps away from Q along the walk w , then we have $d(P) = z_1 + \dots + z_s$ because by a unit step e_j of w , the x_d -distance increases by 1 for $j = d$, and decreases by α_j for $j \neq d$. Hence, we have $\delta(w) = \theta(\text{seq}(w))$. Since $\text{seq}(w)$ has a total sum β for all $w \in V$, in the same manner as in the proof of Theorem 1, we have

$$\sum_{w \in W} \delta(w) = \int_0^1 |W_t| dt = \frac{\beta}{m_1 + \dots + m_d} |V|.$$

3. Applications. We extend Theorem 1 in two ways.

3.1. Lattice paths lying under a cyclically shifting boundary. We will count the number of lattice paths lying under a cyclically shifting piecewise linear boundary of varying slope. Let $Q = (m, n) \in \mathbb{N}^2$ be a lattice point, and let $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$ be reals with $n = \alpha_1 + \alpha_2 + \dots + \alpha_m + \beta$. Let $Q_0 = (0, \beta)$, $Q_1 = (1, \beta + \alpha_m)$, $Q_2 = (2, \beta + \alpha_m + \alpha_{m-1})$, $\dots$, $Q_i = (i, \beta + \sum_{0 \leq j < i} \alpha_{m-j}) = (i, n - \alpha_1 - \dots - \alpha_{m-i})$, $\dots$, $Q_m = Q$. For $a = (\alpha_1, \dots, \alpha_m)$, a boundary ∂a consists of line segments connecting $Q_0, Q_1, \dots, Q_{m-1}, Q_m$ in this order.

Fix a lattice point $P = (m - m', n - n') \in \mathbb{N}^2$ below ∂a , that is, the y -coordinate of P is at most that of $Q_{m-m'}$. Let V' be the set of walks from Q to P , so $|V'| = \binom{m'+n'}{m'}$. For a walk $w = (w_1, \dots, w_{m'+n'}) \in V'$, let us define the minimum y -distance of w with respect to ∂a , denoted by $\delta(w, \partial a)$, in the same manner as in Section 1: if w touches or crosses ∂a after the first step, then $\delta(w, \partial a) = 0$, otherwise $\delta(w, \partial a)$ is the minimum of the difference between Q_{m_0} and (m_0, n_0) , where (m_0, n_0) runs over lattice points on w with $(m_0, n_0) \neq Q$. Also, for each $w \in V'$, we assign the corresponding sequence $\text{seq}(w, \partial a) = (z_1, \dots, z_{m'+n'})$ as follows: if w_i is a down step, then let $z_i = 1$; if w_i is the j -th left step, then let $z_i = -\alpha_j$. Finally, we define the weight of w with respect to ∂a by $\theta(w, \partial a) = \theta(\text{seq}(w, \partial a))$. (See the definition of θ before Lemma 1.) Recall that $a^{(t)} = (\alpha_{1+t}, \alpha_{2+t}, \dots, \alpha_{m+t})$ where the indices are read modulo m . Similarly to Claim 3, we have

$$\theta(w, \partial a^{(t)}) = \delta(w, \partial a^{(t)}) \quad (3)$$

for all $w \in V'$ and $0 \leq t < m$.

EXAMPLE 2. Let $m = m' = 4$, $n = n' = 5$, $a = (2, -\sqrt{2}, 3, 0)$ and $\beta = \sqrt{2}$. In this case, $\delta(w, \partial a^{(t)})$ is one of 0, $d_1 = \sqrt{2} - 1$, $d_2 = 1$, $d_3 = \sqrt{2}$ for $0 \leq t < 4$ and for

FIG. 5. $n_t(s) = \#\{w \in V' : \delta(w, \partial a^{(t)}) \geq s\}$.

$w \in V'$. (See Figure 4 and Figure 5.) Then we have

$$\begin{aligned}
 \sum_{w \in V'} \delta(w, \partial a^{(0)}) &= (12 - 6)d_1 + 6d_2 = 6\sqrt{2}, \\
 \sum_{w \in V'} \delta(w, \partial a^{(1)}) &= (62 - 28)d_1 + (28 - 6)d_2 + 6d_3 = 40\sqrt{2} - 12, \\
 \sum_{w \in V'} \delta(w, \partial a^{(2)}) &= 0, \\
 \sum_{w \in V'} \delta(w, \partial a^{(3)}) &= (32 - 22)d_1 + 22d_2 = 10\sqrt{2} + 12.
 \end{aligned}$$

Hence, we have

$$\sum_{0 \leq t < 4} \sum_{w \in V'} \delta(w, \partial a^{(t)}) = 56\sqrt{2} = \frac{4 \cdot 5 - (5 - \sqrt{2})4}{4 + 5} \binom{4 + 5}{4},$$

which verifies Theorem 2.

THEOREM 2. Let m, n be positive integers and let $\alpha_1, \dots, \alpha_m, \beta$ be (possibly negative) reals with $n = \alpha_1 + \alpha_2 + \dots + \alpha_m + \beta$. Fix a lattice point $P = (m - m', n - n') \in \mathbb{N}^2$ below ∂a , where $a = (\alpha_1, \dots, \alpha_m)$, and let V' be the set of walks from (m, n) to P . If P is below $\partial a^{(t)}$ for all $0 \leq t < m$, then we have

$$\sum_{0 \leq t < m} \sum_{w \in V'} \delta(w, \partial a^{(t)}) = \frac{mn' - (n - \beta)m'}{m' + n'} \binom{m' + n'}{m'}.$$

It is worth noting that the RHS of the above formula is independent of the decomposition of $a = (\alpha_1, \dots, \alpha_m)$, and it depends only on the sum $\alpha_1 + \dots + \alpha_m$. We can deduce Theorem 1 from Theorem 2 by setting $\alpha = \alpha_1 = \dots = \alpha_m$, $m' = m$, and $n' = n$.

Proof. Let $0 \leq t < m$ and $w \in V'$. Recall that $w^{(j)} \in V'$ is a cyclic shift (modulo $m' + n'$) of w starting from $(j + 1)$ -th step. Since w consists of n' down steps and m' left steps, the total sum of $\text{seq}(w, \partial a^{(t)})$ is $n' - (\alpha_{1+t} + \dots + \alpha_{m'+t})$, where the indices are read modulo m . Thus, by (3) and Lemma 1, we have

$$\sum_{0 \leq j < m' + n'} \delta(w^{(j)}, \partial a^{(t)}) = \sum_{0 \leq j < m' + n'} \theta(w^{(j)}, \partial a^{(t)}) = n' - (\alpha_{1+t} + \dots + \alpha_{m'+t}).$$

Using

$$\sum_{0 \leq t < m} (\alpha_{1+t} + \dots + \alpha_{m'+t}) = \sum_{1 \leq i \leq m'} \sum_{0 \leq t < m} \alpha_{i+t} = \sum_{1 \leq i \leq m'} (n - \beta) = m'(n - \beta),$$

we have

$$\sum_{0 \leq t < m} \sum_{0 \leq j < m' + n'} \delta(w^{(j)}, \partial a^{(t)}) = \sum_{0 \leq t < m} (n' - (\alpha_{1+t} + \dots + \alpha_{m'+t})) = mn' - (n - \beta)m'.$$

Since each walk $w \in V'$ appears $m' + n'$ times in a multiset $\{w^{(j)} : w \in V', 0 \leq j < m' + n'\}$, we have

$$(m' + n') \sum_{w \in V'} \delta(w, \partial a^{(t)}) = \sum_{w \in V'} \sum_{0 \leq j < m' + n'} \delta(w^{(j)}, \partial a^{(t)}).$$

Therefore, we have

$$\begin{aligned} (m' + n') \sum_{0 \leq t < m} \sum_{w \in V'} \delta(w, \partial a^{(t)}) &= \sum_{w \in V'} \sum_{0 \leq t < m} \sum_{0 \leq j < m' + n'} \delta(w^{(j)}, \partial a^{(t)}) \\ &= \sum_{w \in V'} (mn' - (n - \beta)m') = (mn' - (n - \beta)m')|V'|, \end{aligned}$$

which completes the proof of Theorem 2. $\square$

Next we consider the case when $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$ are (possibly negative) integers. For an integer t , $0 \leq t < m$, let U_t be the set of walks in V' which touch the shifted boundary $\partial a^{(t)}$ at Q only. By definition, we have $\delta(w, \partial a^{(t)}) = 1$ if $w \in U_t$, and $\delta(w, \partial a^{(t)}) = 0$ otherwise. This gives $|U_t| = \sum_{w \in V'} \delta(w, \partial a^{(t)})$. Then Theorem 2 implies the following.

COROLLARY 2. *Under the same assumptions as in Theorem 2, if $\alpha_1, \dots, \alpha_m, \beta$ are (possibly negative) integers, then we have*

$$\sum_{0 \leq t < m} |U_t| = \frac{mn' - (n - \beta)m'}{m' + n'} \binom{m' + n'}{m'}.$$

We get the main result of Irving–Rattan, Theorem 1 of [6], from Corollary 2 by setting our parameters (m, n, m', n', β) to $(m, n + 1, \ell, k + 1, 1)$. (They proved the case that $\alpha_1, \dots, \alpha_m, \beta$ are all non-negative.) For comparison, we remark that the roles of x -axis and y -axis in Corollary 1 are the opposite of those in their result, and our condition “ P is below $\partial a^{(t)}$ for all t ” is equivalent to their condition “ $t' = (k + 1, \ell)$ lies weakly to the right of $\partial a^{(j)}$ for all j .”

3.2. Walks having k peaks. We give a refinement of Theorem 1, cf. Theorem 8 of [6]. Let α, β be positive reals, and let m, n be positive integers. Fix a line $L : y = \alpha x + \beta$, and a lattice point $Q = (m, n)$ on L . We use the notation $V, W, \delta(w)$ for $w \in V$, as the same meaning as in Section 1. For a walk $w \in V$, a down step followed by a left step in w is called a peak of w , and a left step followed by a down step in w is called a valley of w . For example, a walk described in Figure 1 has three peaks and two valleys. Let $V(k)$ (resp. $W(k)$) be the set of walks $w \in V$ (resp. $w \in W$) having k peaks for a non-negative integer k .

In the case α is a positive integer and $\beta = 1$, the following result is given as Theorem 3.4.3 of [7] (see also Theorem 7 of [3]).

THEOREM 3. *Let m, n be positive integers, and let α, β be positive reals with $n = \alpha m + \beta$. Let V be the set of walks from (m, n) to the origin. Then, we have*

$$\sum_{w \in V(k)} \delta(w) = \frac{\beta}{k} \binom{m-1}{k-1} \binom{n-1}{k-1}.$$

If $n = m + 1$ ($\alpha = \beta = 1$), then the RHS becomes $\frac{1}{m} \binom{m}{k} \binom{m}{k-1}$, which is called the Narayana number, for example, see Exercise 6.36 of [9]. We notice that Theorem 1 is derived from Theorem 3. Indeed, by taking sum over $k \geq 1$, we have

$$\begin{aligned} \sum_{w \in V} \delta(w) &= \sum_{k \geq 1} \sum_{w \in V(k)} \delta(w) \\ &= \sum_{k \geq 1} \frac{\beta}{k} \binom{m-1}{k-1} \binom{n-1}{k-1} = \sum_{k \geq 1} \frac{\beta}{m} \binom{m}{k} \binom{n-1}{n-k} \\ &= \frac{\beta}{m} \binom{m+n-1}{n} = \frac{\beta}{m+n} \binom{m+n}{m}. \end{aligned}$$

For a non-negative integer s , the subset of integers $\{1, 2, \dots, s\}$ is denoted by $[1, s]$. For a set X , the family of all k -element subsets of X is denoted by $\binom{X}{k}$.

Proof of Theorem 3. Let $U(k) \subset V(k)$ be the set of walks in which the first step is a down step and the last step is a left step. For $\{x_1, \dots, x_{k-1}\} \in \binom{[m-1]}{k-1}$ ($x_1 < \dots < x_{k-1} < x_k := m$) and $\{y_1, \dots, y_{k-1}\} \in \binom{[n-1]}{k-1}$ ($y_0 := 0 < y_1 < \dots < y_{k-1}$), we can associate a path $w \in U(k)$ with peaks at (x_i, y_{i-1}) ($1 \leq i \leq k$). This gives

$$|U(k)| = \binom{m-1}{k-1} \binom{n-1}{k-1}.$$

For all $w \in V$, the total sum of $\text{seq}(w)$ is $n - \alpha m = \beta$. Thus, by Claim 3 and Lemma 1, we have $\sum_{0 \leq j < m+n} \delta(w^{(j)}) = \sum_{0 \leq j < m+n} \theta(w^{(j)}) = \beta$. Hence, we have

$$\sum_{w \in U(k)} \sum_{0 \leq j < m+n} \delta(w^{(j)}) = \beta |U(k)|.$$

For a walk $u \in U(k)$ and for $0 \leq j < m + n$, we notice that if $\delta(u^{(j)}) > 0$ then $u^{(j)}$ also has exactly k peaks, because $u^{(j)}$ starts with a down step. Thus we have $\delta(u^{(j)}) > 0$ iff $u^{(j)} \in W(k)$. On the other hand, for a given walk $w \in W(k)$, there exist exactly k pairs (u, j) with $u \in U(k)$, $0 \leq j < m + n$ such that $w = u^{(j)}$. In fact, if $w \in W(k) \cap U(k)$, then w has exactly $k - 1$ valleys from which we get walks

$u \in U(k)$ with $w = u^{(j)}$ for some j ; if $w \in W(k) \setminus U(k)$, then w has exactly k valleys from which we get walks satisfying the same property. Therefore, we have

$$k \sum_{w \in V(k)} \delta(w) = k \sum_{w \in W(k)} \delta(w) = \sum_{u \in U(k)} \sum_{0 \leq j < m+n} \delta(u^{(j)}) = \beta |U(k)|,$$

as desired. $\square$

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