ON r-CROSS INTERSECTING FAMILIES OF SETS

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ABSTRACT. Let $(r-1)n \geq rk$ and let $\mathscr{F}_1, \ldots, \mathscr{F}_r \subset \binom{[n]}{k}$. Suppose that $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds for all $F_i \in \mathscr{F}_i$, $1 \leq i \leq r$. Then we show that $\prod_{i=1}^{r} |\mathscr{F}_i| \leq \binom{n-1}{k-1}^r$.

1. INTRODUCTION

Let n, k, r be positive integers. We say that a family $\mathscr{F} \subset \binom{[n]}{k}$ is r-wise intersecting if $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds for all $F_i \in \mathscr{F}$, $1 \leq i \leq r$. Frankl [5] extended the Erdős–Ko–Rado theorem [4] as follows, see also [7, 9].

Theorem 1. *Let $(r-1)n \geq rk$ and let $\mathscr{F} \subset \binom{[n]}{k}$ be an r-cross intersecting family. Then, we have $|\mathscr{F}| \leq \binom{n-1}{k-1}$.*

We say that families $\mathscr{F}_1, \ldots, \mathscr{F}_r \subset \binom{[n]}{k}$ are r-cross intersecting if $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds for all $F_i \in \mathscr{F}_i$, $1 \leq i \leq r$. We show the following extension of Theorem 1.

Theorem 2. *Let $(r-1)n \geq rk$ and let $\mathscr{F}_1, \ldots, \mathscr{F}_r \subset \binom{[n]}{k}$ be r-cross intersecting families. Then, we have $\prod_{i=1}^{r} |\mathscr{F}_i| \leq \binom{n-1}{k-1}^r$.*

We say that families $\mathscr{G}_1, \ldots, \mathscr{G}_r \subset \binom{[n]}{\ell}$ are r-cross union if $G_1 \cup \cdots \cup G_r \neq [n]$ holds for all $G_i \in \mathscr{G}_i$, $1 \leq i \leq r$. For $\mathscr{F} \subset \binom{[n]}{k}$ we define its complement family by $\mathscr{F}^c = \{[n] \setminus F : F \in \mathscr{F}\} \subset \binom{[n]}{\ell}$ where $\ell = n-k$. Notice that $\mathscr{F}_1, \ldots, \mathscr{F}_r$ are r-cross intersecting iff $\mathscr{F}_1^c, \ldots, \mathscr{F}_r^c$ are r-cross union. To state our main result, we need one more definition. For $\mathscr{G} \subset \binom{[n]}{\ell}$ choose a unique real $x \geq \ell$ so that $|\mathscr{G}| = \binom{x}{\ell}$, and let $\|\mathscr{G}\|_\ell := x$.

Theorem 3. *Let $n \leq r\ell$ and let $\mathscr{G}_1, \ldots, \mathscr{G}_r \subset \binom{[n]}{\ell}$ be r-cross union families. Then, we have the following.*

(i) $\sum_{i=1}^{r} \|\mathscr{G}_i\|_\ell \leq r(n-1)$.
(ii) $\prod_{i=1}^{r} |\mathscr{G}_i| \leq \binom{n-1}{\ell}^r$.

By considering the complement k-uniform families, where $k = n-\ell$, we get Theorem 2 from (ii) of Theorem 3. We will see that (i) implies (ii) easily. If $n > r\ell$, then r copies of $\binom{[n]}{\ell}$ are r-cross union families which do not satisfy the conclusions of Theorem 3.

Our proof of Theorem 3 is very simple. In fact we only use well-known tools such as Katona's cycle method, the AM-GM inequality, and the Kruskal–Katona theorem on

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shadows. The novelty of the proof is to focus on the inequality (i) of Theorem 3. There are several ways to prove Theorem 1, but the authors do not know any proof for Theorem 2 without using (i). Moreover the inequality (i) seems to be applicable for some other cases. It might be interesting to obtain the corresponding vector space version of Theorem 2 via (i). (See [3] for the vector space version of Theorem 1.)

The study of cross-intersecting families has a long history, starting with Bollobás [2]. We mention some related results to Theorem 2: the case $r=2$ with different uniformity was considered by Pyber [10], Matsumoto–Tokushige [8] and Bey [1], and the non-uniform t-intersecting cases was solved by Frankl [6].

2. Proof of Theorem 3

Let $x_i = \|\mathscr{G}_i\|_\ell$, that is, $|\mathscr{G}_i| = \binom{x_i}{\ell}$, for $1 \le i \le r$.

First we show that (i) implies (ii).

Claim 1.

$$\prod_{i=1}^{r} |\mathscr{G}_i| = \binom{x_1}{\ell}\cdots\binom{x_r}{\ell} \le \binom{\frac{x_1+\cdots+x_r}{r}}{\ell}^r \le \binom{n-1}{\ell}^r.$$

Proof. The first inequality follows from the inequality of arithmetic and geometric means:

$$\text{2nd term } = \frac{1}{(\ell!)^r}\prod_{i=0}^{\ell-1}(x_1-i)\cdots(x_r-i) \le \frac{1}{(\ell!)^r}\prod_{i=0}^{\ell-1}\left(\frac{x_1+\cdots+x_r}{r}-i\right)^r = \text{ 3rd term},$$

and the second inequality follows from (i). □

So all we need is to show (i). Let $s = r\ell - n$. We prove (i) by induction on s.

First we consider the initial step $s=0$, that is, $n = r\ell$. Fix a cyclic permutation $\sigma = a_1a_2\cdots a_n \in S_n$, and let $\mathscr{A}^\sigma = \{A_1, A_2, \ldots, A_n\}$ be the set of arcs of length ℓ in σ, where $A_i = \{a_i, a_{i+1}, \ldots, a_{i+\ell-1}\}$ (the indices are read mod n). For $1 \le i \le r$, let $\mathscr{G}_i^\sigma = \mathscr{G}_i \cap \mathscr{A}^\sigma$.

Claim 2. *Let σ be an arbitrary cyclic permutation. Then $\sum_{i=1}^r |\mathscr{G}_i^\sigma| \le r(n-\ell)$.*

Proof. Let $\sigma = a_1a_2\cdots a_n \in S_n$ be given. For $1 \le i \le r$ and $j \in \mathbb{Z}_n$, let

$$\varepsilon_j^i = \begin{cases} 1 & \text{if } A_{j+(i-1)\ell} \in \mathscr{G}_i^\sigma, \\ 0 & \text{if } A_{j+(i-1)\ell} \not\in \mathscr{G}_i^\sigma. \end{cases}$$

Then $|\mathscr{G}_i^\sigma| = \sum_{j=1}^n \varepsilon_j^i$. Notice that $[n] = A_j \cup A_{j+\ell} \cup A_{j+2\ell} \cup \cdots \cup A_{j+(r-1)\ell}$ is a partition. Since $\mathscr{G}_1, \ldots, \mathscr{G}_r$ are r-cross union, we have $\#\{i : A_{j+(i-1)\ell} \in \mathscr{G}_i\} \le r-1$. This gives $\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3 + \cdots + \varepsilon_j^r \le r-1$ for all $j \in \mathbb{Z}_n$. Thus we have

$$\sum_{i=1}^r |\mathscr{G}_i^\sigma| = \sum_{j=1}^n (\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3 + \cdots + \varepsilon_j^r) \le (r-1)n = r(n-\ell),$$

where we used $n = r\ell$ in the last equality. □

Claim 3. *If $n = r\ell$, then we have $\sum_{i=1}^r |\mathscr{G}_i| \le r\binom{n-1}{\ell}$.*

Proof. Each $G \in \mathscr{G}_i$ is counted $\ell!(n-\ell)!$ times in $\sum_{\sigma\in\mathscr{C}_n} |\mathscr{G}_i^\sigma|$, where $\mathscr{C}_n$ is the set of all cyclic permutations. This gives

$$\sum_{\sigma\in\mathscr{C}_n}\sum_{i=1}^{r}|\mathscr{G}_i^\sigma| = \ell!(n-\ell)!\sum_{i=1}^{r}|\mathscr{G}_i|.$$

On the other hand, since $|\mathscr{C}_n| = (n-1)!$, it follows from Claim 2 that

$$\sum_{\sigma\in\mathscr{C}_n}\sum_{i=1}^{r}|\mathscr{G}_i^\sigma| \le (n-1)!\,r(n-\ell).$$

Thus we have

$$\sum_{i=1}^{r}|\mathscr{G}_i| \le \frac{(n-1)!\,r(n-\ell)}{\ell!(n-\ell)!} = r\binom{n-1}{\ell}$$

as desired. □

We notice that $f(x) = \binom{x}{\ell}$ is convex for $x \ge \ell$. In fact, one can show $f''(x) > 0$ for $x \ge \ell$ by a direct computation. So, we have

$$\binom{\frac{x_1+\cdots+x_r}{r}}{\ell} \le \frac{1}{r}\sum_{i=1}^{r}\binom{x_i}{\ell} = \frac{1}{r}\sum_{i=1}^{r}|\mathscr{G}_i| \le \binom{n-1}{\ell},$$

where we used Claim 3 for the last inequality. Thus we get $\frac{x_1+\cdots+x_r}{r} \le n-1$, that is, (i) of the theorem for the initial step $s = 0$.

Next we deal with the induction step. Let $s > 0$. Suppose that (i) is true for the case $r\ell - n = s$, and we will consider the case $r\ell - n = s+1$.

So, let $\mathscr{G}_1,\dots,\mathscr{G}_r \subset \binom{[n]}{\ell}$ be r-cross union families with $r\ell - n = s+1$. Recall that $x_i = \|\mathscr{G}_i\|_\ell$ for $1 \le i \le r$, and we shall show that

$$\sum_{i=1}^{r} x_i \le r(n-1). \tag{1}$$

Define $\mathscr{H}_i = \mathscr{G}_i \cup \mathscr{D}_i \subset \binom{[n+1]}{\ell}$ by

$$\mathscr{D}_i = \{D \cup \{n+1\} : D \in \Delta_{\ell-1}(\mathscr{G}_i)\},$$

where $\Delta_j(\mathscr{G}_i) = \{J \in \binom{[n]}{j} : J \subset \exists G \in \mathscr{G}_i\}$ is the j-th shadow of $\mathscr{G}_i$. Then, by the Kruskal–Katona theorem, we have $|\mathscr{D}_i| = |\Delta_{\ell-1}(\mathscr{G}_i)| \ge \binom{x_i}{\ell-1}$, and

$$|\mathscr{H}_i| = |\mathscr{G}_i| + |\mathscr{D}_i| \ge \binom{x_i}{\ell} + \binom{x_i}{\ell-1} = \binom{x_i+1}{\ell},$$

namely,

$$z_i := \|\mathscr{H}_i\|_\ell \ge x_i + 1. \tag{2}$$

Now we notice that $\mathscr{H}_1,\dots,\mathscr{H}_r \subset \binom{[n+1]}{\ell}$ are r-cross union families. Moreover, since $r\ell - (n+1) = s$, we can apply induction hypothesis to $\mathscr{H}_1,\dots,\mathscr{H}_r$, and we get

$$\sum_{i=1}^{r} z_i \le r((n+1)-1) = rn. \tag{3}$$

Finally, (1) follows from (2) and (3). This completes the proof of the theorem. □

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