Graph decompositions through prescribed vertices without isolates

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Abstract

Let G be a graph of order n, and let $n = \sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geq 2$. Let $v_1, \ldots, v_k$ be given distinct vertices of G. Suppose that the minimum degree of G is at least $3k$. In this paper, we prove that there exists a decomposition of the vertex set $V(G) = \bigcup_{i=1}^{k} A_i$ such that $|A_i| = a_i$, $v_i \in A_i$, and the subgraph induced by A_i contains no isolated vertices for all i, $1 \leq i \leq k$.

1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph and $x \in V(G)$ (where $V(G)$ is the vertex set of G), the neighborhood $N(x)$ of x is the set of vertices adjacent to x, and the degree $d(x)$ of x is $|N(x)|$. The minimum degree of a graph G is

$$\delta(G) := \min\{d(x) : x \in V(G)\}.$$

For a subset S of $V(G)$, $N(S) := \bigcup_{x \in S} N(x)$, $<S>$ denotes the subgraph of G induced by S, and $G - S := <V(G) - S>$. The set $\{1, \ldots, n\}$ is denoted by $[n]$.

Let G be a graph of order n, $n = \sum_{i=1}^{k} a_i$ be a partition of n, and let $\mathcal{P}$ be a property on a graph. We say that G has a decomposition property $\mathrm{DP}(n, k, \sum a_i, \mathcal{P})$ if there exists a decomposition of the vertex set $V(G) = \bigcup_{i=1}^{k} A_i$ such that $|A_i| = a_i$ and $<A_i>$ satisfies $\mathcal{P}$ for all i, $1 \leq i \leq k$. We say that G has a strong decomposition property $\mathrm{SDP}(n, k, \sum a_i, \mathcal{P})$ if, for arbitrary k vertices $v_1, \ldots, v_k$ of G, there exists a decomposition $V(G) = \bigcup_{i=1}^{k} A_i$ satisfying $\mathrm{DP}(n, k, \sum a_i, \mathcal{P})$ and $v_i \in A_i$ for all i.

Let us define two properties $\mathcal{C}$ and $\mathcal{I}$ on graphs. A graph G satisfies $\mathcal{C}$ if G is connected. A graph G satisfies $\mathcal{I}$ if G contains no isolated vertices. Let $n = \sum_{i=1}^{k} a_i$ be a partition of $n = |V(G)|$. The connectivity of G is denoted by $\kappa(G)$. Maurer proposed the following conjecture.

$$\kappa(G) \geq k \Longrightarrow \mathrm{DP}(n, k, \sum a_i, \mathcal{C}).$$

Frank posed a stronger conjecture.

$$\kappa(G) \geq k \Longleftrightarrow \mathrm{SDP}(n, k, \sum a_i, \mathcal{C}).$$

Györi and Lovász proved the above conjecture independently. Another conjecture of Frank is the following. Suppose $a_i \neq 1$ for $1 \leq i \leq k$. Then,

$$\delta(G) \geq k \Longrightarrow \mathrm{DP}(n, k, \sum a_i, \mathcal{I}).$$

The above conjecture has been proved by Enomoto.

Theorem 1 *Let G be a connected graph of order n, and let $n = \sum_{i=1}^{k} a_i$ be a partition of n with $1 \neq a_i \geq 0$. Suppose that $\delta(G) \geq k$. Then G satisfies* $\mathrm{DP}(n, k, \sum a_i, \mathcal{I})$.

The main purpose of the present paper is to prove the strong decomposition version of the above result, i.e.,

$$\delta(G) \geq 3k \Longrightarrow \mathrm{SDP}(n, k, \sum a_i, \mathcal{I}).$$

More precisely, we prove the following.

Theorem 2 *Let G be a graph of order n, and let $n = \sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geq 2$. Suppose that $\delta(G) \geq 3k$. Then G satisfies* $\mathrm{SDP}(n, k, \sum a_i, \mathcal{I})$.

As we will see in the last section, the condition imposed on the minimum degree in Theorem 2 is almost best possible.

Our proof of the above result is rather complicated because there are many cases. So, before we give the proof, we consider the following weaker but more easily proved result.

Theorem 3 *Theorem 2 is true if $\delta(G) \geq 4k - 1$.*

In section 2, we prove Theorem 3. The proof contains basic strategy for the proof of our main result. In section 3, we prove Theorem 2. In the proof, we use a lemma which is purely on integer partitions. To state the lemma, we need one more definition. Let $n = \sum_{i=1}^{k} a_i = \sum_{j=1}^{m} c_j$ be partitions of n. We say that $\sum a_i$ fits $\sum c_j$ if there exist decompositions $[n] = \bigcup_{i=1}^{k} A_i = \bigcup_{j=1}^{m} C_j$ such that $|A_i| = a_i$, $|C_j| = c_j$, and $|A_i \cap C_j| \neq 1$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$.

Lemma 4 *Let k, m, n be positive integers, and let $n = \sum_{i=1}^{k} a_i = \sum_{j=1}^{m} c_j$ be partitions of n with $a_i \geq 2$ for $1 \leq i \leq k$, and $c_j \geq k + 1$ for $1 \leq j \leq m$. Then $\sum a_i$ fits $\sum c_j$ if and only if the two partitions are different from those in the following table.*

Table of Exceptions

No.	m	k	$a = (a_1, \dots, a_k)$	$c = (c_1, \dots, c_m)$
1	2	$k_2 \geq 1$	$(2^{k_1}4^{k_2})$	(odd, odd)
2	2		$(23^{k-1}), (3^{k-1}5)$	$c_1 \equiv c_2 \equiv 1 \pmod 3$
3	2		(3^k)	$c_1 \equiv 1,\ c_2 \equiv 2 \pmod 3$
4	3	$k_2 \geq k_1 + 3$	$(2^{k_1}4^{k_2})$	(odd, odd, even)
5	3	$k_2 \geq k_1$	$(2^{k_1}34^{k_2}), (2^{k_1-1}4^{k_2}5), (2^{k_1}4^{k_2-1}7)$	(odd, odd, odd)
6	3	$k \equiv 0 \pmod 3$	$(3^{k-1}7), (3^{k-2}55)$	$(k+1, k+1, k+2)$
7	3	$k \equiv 0 \pmod 3$	$(23^{k-3}55), (3^{k-2}45), (23^{k-2}7), (3^{k-1}6)$	$(k+1, k+1, k+1)$
8	3	$k \equiv 2 \pmod 3$	$(3^{k-3}555), (3^{k-2}57), (3^{k-1}9)$	$(k+2, k+2, k+2)$
9	3	$k \equiv 0 \pmod 3$	$(3^{k-3}555), (3^{k-2}57), (3^{k-1}9)$	$(k+1, k+1, k+4)$
10	4	$k =$ even	$(4^{k-1}, 10), (4^{k-2}77)$	$(k+1, k+1, k+1, k+3)$
11	4	$k =$ even	$(2, 4^{k-2}, 10), (4^{k-1}8), (4^{k-2}57), (24^{k-3}77)$	$(k+1, k+1, k+1, k+1)$
12	4	6	(odd, odd, odd, odd, odd, odd)	(7777)
13	5	4	$(4, 4, 7, 10), (4, 4, 4, 13), (4777)$	(55555)
14	m	2	$a_1 \equiv a_2 \equiv 1 \pmod 3$	$(3^{m-1}5)$
15	m	2	$a_1 \equiv 1,\ a_2 \equiv 2 \pmod 3$	(3^m)
16	m	2	(odd, odd)	(4^m)
17	m	3	(odd, odd, even)	(4^m)
18	m	3	(odd, odd, odd)	$(4^{m-1}5), (4^{m-1}7)$

(In the table, $a = (2^{k_1}4^{k_2})$ means $a_1 = \cdots = a_{k_1} = 2$, $a_{k_1+1} = \cdots = a_k = 4$, $k_1 + k_2 = k$; and $a = (2^{k-2}77)$ means $a_1 = \cdots = a_{k-2} = 2$, $a_{k-1} = a_k = 7$, etc.)
To exclude overlaps in the table, we may add the following assumptions:

$$k \geq 3 \text{ in No.1, 4, 5, 10, 11.} \quad k \geq 4 \text{ in No.2, 3, 6, 7, 8, 9.}$$

Using the above lemma, we obtain a disconnected version of Theorem 1.

Theorem 5 *Let G be a graph of order n, and let $n = \sum_{i=1}^{k} a_i = \sum_{j=1}^{m} c_j$ be partitions of n with $a_i \geq 2$. Suppose that $\delta(G) \geq k$ and the orders of connected components of G are $c_1, \dots, c_m$. Then G satisfies $\mathrm{DP}(n, k, \sum a_i, \mathcal{I})$ if and only if the two partitions are different from those in the table of exceptions (see Lemma 4).*

By checking the table of exceptions, Theorem 5 implies the following.

Corollary 6 *Let G be a graph of order n, and let $n = \sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geq 2$. Suppose that $\delta(G) \geq k$. Further, suppose that $n \geq 26$ and $k = 4$, or $n \geq 4k + 7$ and $k \geq 5$. Then G satisfies $\mathrm{DP}(n, k, \sum a_i, \mathcal{I})$.*

2 The case of large minimum degree

In this section, we prove Theorem 3. First we prove a key technical lemma.

Lemma 7 *Let k, m, n be positive integers, and let $n = \sum_{i=1}^{k} a_i = \sum_{j=1}^{m} c_j$ be partitions of n with $a_i \geq 2$ for all $1 \leq i \leq k$, and $c_j \geq 2k$ for all $1 \leq j \leq m$. Then $\sum a_i$ fits $\sum c_j$ unless "$k = 2$, $c_1 = \cdots = c_m = 4$, and a_1 odd."*

Proof The lemma is true for $k = 1$, and also true for "$k = 2$, $c_1 = \cdots = c_m = 4$ and a_1 even." Let $[n] = \bigcup_{j=1}^m C_j$ be a decomposition with $|C_j| = c_j$ for $1 \le j \le m$. Applying induction on k, we shall find a decomposition $[n] = \bigcup_{i=1}^k A_i$ with $|A_i| = a_i$ such that $|A_i \cap C_j| \ne 1$ for all i and j. Note that the exception occurs only if $n = 2mk$.

Now we may assume that $k \ge 3$ or ($k = 2$ and) $c_1 > 4$. Suppose $a_1 \le \cdots \le a_k$ and set $\epsilon_j := c_j - 2(k-1)$ for $1 \le j \le m$. Note that $\epsilon_j \ge 2$. Also it follows that

$$\sum_{j=1}^m \epsilon_j = n - 2m(k-1) = (1 - \frac{1}{k})(n - 2km) + \frac{n}{k} \ge \frac{n}{k} \ge a_1.$$

For $1 \le j \le m$, we choose $D_j \subset C_j$ such that $|D_j| = \epsilon_j$.

Case 1 There exists j_0 such that $\epsilon_{j_0} \ge 3$.
In $\bigcup_{j=1}^m D_j$, we will choose A_1 with $|A_1| = a_1$ such that $|A_1 \cap C_j| \ne 1$ for all $1 \le j \le m$. We may assume that $\epsilon_1 \ge 3$ and $D_1 \supset \{x, y, z\}$. Let $a_1 = \sum_{j=1}^s \epsilon_j + \delta$ where $1 \le \delta \le \epsilon_{s+1}$, $0 \le s < m$. Choose $D' \subset D_{s+1}$ with $|D'| = \delta$ and define $A := (\bigcup_{j=1}^s D_j) \cup D'$. If $\delta \ge 2$ then set $A_1 := A$. If $\delta = 1$ then choose $w \in D_{s+1} - D'$ and set $A_1 := (A - \{x\}) \cup \{w\}$. Now we can apply induction to partitions $n - a_1 = \sum_{i=2}^k a_i = \sum_{j=1}^m |C_j - A_1|$. (Since $n = \sum \epsilon_j + 2m(k-1) > 2mk$, and $a_1 \le \ldots \le a_k$, one has $n - a_1 > 2m(k-1)$. Thus, the exception does not occur in the induction step.)

Case 2 For all $1 \le j \le m$, $\epsilon_j = 2$. I.e., $c_1 = \cdots = c_m = 2k$.
By our assumption, $k \ge 3$. If $a_1 = 2m$ then $a_1 = \cdots = a_k = 2m$ holds. So the desired decomposition is trivial in this case. Now we may assume $a_1 < 2m$. We choose $x_1^j, x_2^j, y_1^j, y_2^j \in C_j$ for $1 \le j \le m$. If $a_1 = 2s$ ($s < m$) then we set $A_1 := \bigcup_{j=1}^s \{x_1^j, x_2^j\}$ and apply induction. (Since $n - a_1 > 2m(k-1)$, the exception does not occur in the induction step.) So we may assume $a_1 = 2s + 1$, $s < m$. Define $A_1 := (\bigcup_{j=1}^s \{x_1^j, x_2^j\}) \cup \{y_1^1\}$. Note that $a_2 \le |(\bigcup_{j=2}^s \{y_1^j, y_2^j\}) \cup (\bigcup_{j=s+1}^m \{x_1^j, x_2^j, y_1^j, y_2^j\})|$. Thus, using the same argument as in the previous case, we can choose an appropriate A_2. Note that $n - a_1 - a_2 > 2m(k-2)$. Thus we can apply induction to partitions $n - a_1 - a_2 = \sum_{i=3}^k a_i = \sum_{j=1}^m |C_j - A_1 - A_2|$.
∎

Example 8 *Let $m = 2$, $a_1 = 2$, $a_2 = \cdots = a_k = 4$, and $c_1 = c_2 = 2k - 1$. Then $\sum a_i$ does not fit $\sum c_j$.*

Using the lemma we prove a disconnected version of Theorem 1.

Theorem 9 *Let G be a graph of order n, and $n = \sum_{i=1}^k a_i$ be a partition of n with $a_i \ge 2$. Suppose that $\delta(G) \ge k$, and every connected component of G has at least $2k$ vertices. Then G satisfies $\mathrm{DP}(n, k, \sum a_i, \mathcal{I})$ unless "$k = 2$, a_1 is odd, and every connected component of G is order 4."*

Proof Let $C_1, \ldots, C_m$ be connected components of G with $c_j := |C_j| \ge 2k$ for $1 \le j \le m$. Using the lemma, we can find a decomposition $V(G) = \bigcup_{i=1}^k B_i$ such that $|B_i| = a_i$ and $a_{ij} := |B_i \cap C_j| \ne 1$. Let us consider C_j. Applying Theorem 1 to C_j and $c_j = \sum_{i=1}^k a_{ij}$, we see that $<C_j>$ satisfies $\mathrm{DP}(c_j, k, \sum a_{ij}, \mathcal{I})$, and hence there is a partition $C_j = \bigcup_{i=1}^k A_{ij}$ such that $|A_{ij}| = a_{ij}$ and $\delta(<A_{ij}>) \ge 1$ for all i. Now define $A_i := \bigcup_{j=1}^m A_{ij}$. Then we get a desired decomposition $V(G) = \bigcup_{i=1}^k A_i$. ∎

The same argument is valid for the proof of "Lemma 4 implies Theorem 5."

Proof of Theorem 3 Let $v_1, \dots, v_k$ be given distinct vertices. We may assume $a_1 \le \cdots \le a_k$. Suppose that $a_1 \le 3$. In this case we choose a path P in $G - \{v_2, \dots, v_k\}$ satisfying that $|V(P)| = a_1$ and v_1 is an endpoint of P. Since $\delta(G - V(P)) \ge (4k-1) - 3 > 4(k-1) - 1$, we can apply induction.

Next suppose that $a_1 \ge 4$. Choose k independent edges $v_1w_1, \dots, v_kw_k$ in G. Delete these $2k$ vertices. The remaining graph G' satisfies $\delta(G') \ge 2k - 1$. Thus every connected component of G' has at least $2k$ vertices. First we consider the exceptional case. Let $W := \{v_1, w_1, v_2, w_2\}$. Note that every connected component of $G - W$ is order 4, and $\delta(G) \ge 4k - 1 = 7$ in this case. Thus, every $x \in W$ and $y \in V(G) - W$ are adjacent. Therefore, we can easily get a desired decomposition.

Now we may assume that G' is not the exceptional case. Applying Theorem 9 to G' and the partition $n - 2k = \sum_{i=1}^{k}(a_i - 2)$, we get an appropriate decomposition $V(G') = \bigcup_{i=1}^{k} A'_i$. Define $A_i := A'_i \cup \{v_i, w_i\}$. Then we obtain a desired decomposition $V(G) = \bigcup_{i=1}^{k} A_i$. ∎

3 Proof of the main result

In this section, we prove Theorem 2.

A partition $n = \sum_{i=1}^{k} a_i$ is called $\{2,3\}$-partition if $a_i \in \{2,3\}$ for all $1 \le i \le k$. There is a unique $\{2,3\}$-partition for $n = 2, 3, 4, 5, 7$, and there are two $\{2,3\}$-partitions for $n = 6$. For a given partition $n = \sum_{i=1}^{k} b_i$, a refinement of this partition $n = \sum_{i=1}^{k} \sum_{j=1}^{k_i} a_{i,j}$ ($b_i = \sum_{j=1}^{k_i} a_{i,j}$) is called a $\{2,3\}$-refinement if $a_{i,j} \in \{2,3\}$ for all i and j. The next lemma immediately follows from definitions.

Lemma 10 *Let $n = \sum_{i=1}^{k} a_i = \sum_{j=1}^{m} c_j$ be partitions of n. Then $\sum a_i$ fits $\sum c_j$ if and only if there exists a common $\{2,3\}$-refinement of these partitions.*

The following lemma gives a necessary and sufficient conditions for fitness.

Lemma 11 *Let $n = \sum_{i=1}^{k} a_i = \sum_{j=1}^{m} c_j$ be partitions of n. Suppose that $a_1, \dots, a_s$ are odd and $a_{s+1}, \dots, a_k$ are even, and suppose that $c_1, \dots, c_p$ are odd and $c_{p+1}, \dots, c_m$ are even. Further suppose that $a_i, c_j \ge 2$ for all i, j. Set*

$$b_i := \begin{cases} a_i - 3 & 1 \le i \le s \\ a_i & s < i \le k, \end{cases}$$

$$d_j := \begin{cases} c_j - 3 & 1 \le j \le p \\ c_j & p < j \le m. \end{cases}$$

Then the following hold.

(i) *If $s = p$ then $\sum a_i$ fits $\sum c_j$.*

(ii) *If $s < p$ then $\sum a_i$ fits $\sum c_j$ iff $\sum_{i=1}^{k} \lfloor b_i/6 \rfloor \ge (p-s)/2$.*

(iii) *If $s > p$ then $\sum a_i$ fits $\sum c_j$ iff $\sum_{j=1}^{m} \lfloor d_j/6 \rfloor \ge (s-p)/2$.*

Proof Note that both b_i and d_j are even. Since

$$n = 3s + \sum_{i=1}^{k} b_i = 3p + \sum_{j=1}^{m} d_j,$$

we have $s \equiv p$ (mod. 2).

(i) In this case, $n = 2 \times \frac{n-3s}{2} + 3 \times s$ is a common $\{2,3\}$-refinement.

(ii) Suppose that $\sum_{i=1}^k \lfloor b_i/6 \rfloor \geq (p-s)/2$. Then $n-3s = \sum_{i=1}^k b_i = 2 \times \frac{n-3p}{2} + 3 \times (p-s)$ is a $\{2,3\}$-refinement of $\sum_{i=1}^k b_i$. Thus, $n = 2 \times \frac{n-3p}{2} + 3 \times p$ is a common $\{2,3\}$-refinement of $n = \sum_{i=1}^k a_i = \sum_{j=1}^m c_j$.

Now suppose that there exists a common $\{2,3\}$-refinement for $\sum a_i = \sum c_j$. Let m_i be the number of 3s used in the refinement of a_i, i.e., $a_i = 2 \times \frac{n-3m_i}{2} + 3 \times m_i$. Then m_i is odd for $1 \leq i \leq s$, and m_i is even for $s < i \leq k$. Thus, we have $(m_i - 1)/2 \leq b_i/6$ for $i \leq s$, and $m_i/2 \leq b_i/6$ for $i > s$. On the other hand, $\sum_{i=1}^k m_i \geq p$ holds. Therefore,

$$\sum_{i=1}^k \lfloor b_i/6 \rfloor \geq \sum_{i=1}^s (m_i - 1)/2 + \sum_{i=s+1}^k m_i/2 \geq (p-s)/2.$$

(iii) Same as the previous case. ∎

Now we prove Lemma 4.

Proof Suppose that $a_1, \ldots, a_s$ are odd and $a_{s+1}, \ldots, a_k$ are even, where $0 \leq s \leq k$, and suppose that $c_1, \ldots, c_p$ are odd and $c_{p+1}, \ldots, c_m$ are even, where $0 \leq p \leq m$. Define b_i and d_j as in Lemma 11. Note that both b_i and d_j are even, and $s \equiv p$ (mod 2). By Lemma 11, we can find an appropriate decomposition iff

$$\begin{aligned} & s = p \quad \text{or} \\ & s < p \quad \text{and} \quad \sum_{i=1}^k \lfloor b_i/6 \rfloor \geq (p-s)/2 \quad \text{or} && (1) \\ & s > p \quad \text{and} \quad \sum_{j=1}^m \lfloor d_j/6 \rfloor \geq (s-p)/2. && (2) \end{aligned}$$

From now on, we classify all exceptional parameters. Let $b_i = 2\alpha_i$ for $1 \leq i \leq k$, and $d_j = 2\gamma_j$ for $1 \leq j \leq m$. If $c_j \geq k+1$ is odd then $\gamma_j = d_j/2 = (c_j - 3)/2 \geq (k-2)/2$, otherwise $\gamma_j \geq (k+1)/2$. Since

$$n = 3s + 2\sum_{i=1}^k \alpha_i = 3p + 2\sum_{j=1}^m \gamma_j, \tag{3}$$

we have

$$p - s = \frac{2}{3}\left(\sum_{i=1}^k \alpha_i - \sum_{j=1}^m \gamma_j\right). \tag{4}$$

Case 1 $s < p$.
By (1) and (4), exceptions occur if and only if

$$\sum_{i=1}^k \lfloor \alpha_i/3 \rfloor \leq \left(\sum_{i=1}^k \alpha_i - \sum_{j=1}^m \gamma_j\right)/3 - 1,$$

or equivalently,

$$3 + \sum \gamma_j \leq \sum \left(\alpha_i - 3\left\lfloor \frac{\alpha_i}{3} \right\rfloor\right).$$

Let $k_l := \#\{i : \alpha_i \equiv l \text{ (mod 3)}\}$. Note that $k_0 + k_1 + k_2 = k$. Consequently, exceptions occur if and only if

$$3 + \sum \gamma_j \leq k_1 + 2k_2. \tag{5}$$

If $s = 0$, then

$$2k_1 + 4k_2 + 6k_0 \le 2\sum \alpha_i = n = 3p + 2\sum \gamma_j \le 3(p-2) + 2k_1 + 4k_2,$$

which implies $2k_0 \le p - 2$. Further, if $p = 2$ then

$$n = 2k_1 + 4k_2. \quad (\text{if } s = 0,\ p = 2) \tag{6}$$

Case 1.1 $k = 2$.
By (5) and $k_1 + k_2 = 2$, we have $(k_1, k_2) = (0, 2)$ or $(1, 1)$.
Case 1.1.1 $k_2 = 2$.
Since $\alpha_1 \equiv \alpha_2 \equiv 2 \pmod 3$, we have $b_1 \equiv b_2 \equiv 4 \pmod 6$ and $a_1 \equiv a_2 \equiv 1 \pmod 3$. Thus,

$$n \equiv a_1 + a_2 \equiv 2 \pmod 3.$$

This together with (5), i.e., $\sum \gamma_j \le 1$ implies $c = (3^{m-1}5)$.

$$\boxed{\text{No.14: } k = 2,\ a_1 \equiv a_2 \equiv 1 \pmod 3,\ c = (3^{m-1}5).}$$

Case 1.1.2 $k_1 = k_2 = 1$.
By (5), we have $\sum \gamma_j = 0$, and $c = (3^m)$.

$$\boxed{\text{No.15: } k = 2,\ \{a_1, a_2\} \equiv \{1, 2\} \pmod 3,\ c = (3^m).}$$

From now on, we may assume $k \ge 3$.
Case 1.2 $m = 2$.
Since $2 \le 2 + s \le p \le m = 2$, we have $s = 0$, $p = 2$. By (6), $n = 2k_1 + 4k_2$. (Since $2k_1 + 4k_2 = n \ge 2(k+1) = 2(k_1 + k_2 + 1)$, we have $k_2 \ge 1$.)

$$\boxed{\text{No.1: } m = 2,\ a = (2^{k_1}4^{k_2}),\ c = (\text{odd, odd}).}$$

Case 1.3 $m = 3$.
Since $2 \le 2 + s \le p \le m = 3$, we have "$s = 0$, $p = 2$" or "$s = 1$, $p = 3$."
Case 1.3.1 $s = 0$, $p = 2$.
By (6), we have $n = 2k_1 + 4k_2$. (Since $2k_1 + 4k_2 \ge 3(k_1 + k_2 + 1)$, we have $k_2 \ge k_1 + 3$.)

$$\boxed{\text{No.4: } m = 3,\ a = (2^{k_1}4^{k_2}),\ c = (\text{odd, odd, even}).}$$

Case 1.3.2 $s = 1$, $p = 3$.
By (5), we have

$$\sum \gamma_j \le k_1 + 2k_2 - 3. \tag{7}$$

Case 1.3.2.1 $k_0 = 0$.
Using (3) and (7), we have

$$3 + 2k_1 + 4k_2 \le 3 + 2\sum \alpha_i = n = 9 + 2\sum \gamma_j \le 3 + 2k_1 + 4k_2.$$

This implies $n = 3 + 2k_1 + 4k_2$. (Since $3 + 2k_1 + 4k_2 \ge 3(k_1 + k_2 + 1)$, we have $k_2 \ge k_1$.)

$$\boxed{\text{No.5: } m = 3,\ a = (52^{k_1-1}4^{k_2}) \text{ or } (72^{k_1}4^{k_2-1}),\ c = (\text{odd, odd, odd}).}$$

Case 1.3.2.2 $k_0 \geq 1$.
Using (3) and (7), we have

$$3 + 2k_1 + 4k_2 + 6(k_0 - 1) \leq 3 + 2\sum \alpha_i = n = 9 + 2\sum \gamma_j \leq 3 + 2k_1 + 4k_2.$$

This implies $k_0 = 1$ and $n = 3 + 2k_1 + 4k_2$. ($k_2 \geq k_1$.)

$$\boxed{\text{No.5: } m = 3,\ a = (32^{k_1}4^{k_2}),\ c = (\text{odd, odd, odd}).}$$

Case 1.4 $m = 4$.
First suppose $p \leq 3$. Then,

$$\begin{aligned} \text{LHS of (5)} &\geq 3 + 3 \times \frac{k-2}{2} + \frac{k+1}{2} = 2k + \frac{1}{2}, \\ \text{RHS of (5)} &\leq 2k, \end{aligned}$$

a contradiction. So, we may assume that $p = 4$, which implies $s = 0$ or 2. Now we have

$$\text{LHS of (5)} \geq 3 + 4 \times \frac{k-2}{2} = 2k - 1, \tag{8}$$

$$\text{RHS of (5)} \leq (k - k_2) + 2k_2 = k + k_2. \tag{9}$$

Thus, we have $k_2 = k$ or "$k_2 = k - 1$, $k_1 = 1$."
Case 1.4.1 $k_2 = k$.
By (8) and (9), we have $2k - 1 \leq 3 + \sum \gamma_j \leq 2k$. Thus, k is even and "$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = (k-2)/2$" or "$\gamma_1 = \gamma_2 = \gamma_3 = (k-2)/2$, $\gamma_4 = k/2$."
Case 1.4.1.1 $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = (k-2)/2$.
Since $c_1 = c_2 = c_3 = c_4 = k + 1$, we have $n = 4(k+1) \equiv k + 1 \pmod 3$. On the other hand, using $\alpha_i \equiv 2 \pmod 3$, we have

$$n = 3s + \sum a_i \equiv \sum a_i \equiv 2\sum \alpha_i \equiv 4k \equiv k \pmod 3.$$

This is a contradiction.
Case 1.4.1.2 $\gamma_1 = \gamma_2 = \gamma_3 = (k-2)/2$, $\gamma_4 = k/2$.
In this case, we have $c_1 = c_2 = c_3 = k + 1$, $c_4 = k + 3$, and $n = 4k + 6$. If $s = 0$ then $a = (4^{k-1}, 10)$, if $s = 2$ then $a = (774^{k-2})$.

$$\boxed{\text{No.10: } m = 4,\ k = \text{even},\ a = (4^{k-1}, 10) \text{ or } (774^{k-2}),\ c = (k+1, k+1, k+1, k+3).}$$

Case 1.4.2 $k_1 = 1$, $k_2 = k - 1$.
By (8) and (9), we have $2k - 1 \leq 3 + \sum \gamma_j \leq 2k - 1$. This implies that k is even and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = (k-2)/2$. Thus, $c_1 = c_2 = c_3 = c_4 = k + 1$ and $n = 4(k+1)$. If $s = 0$ then $a = (24^{k-2}, 10)$ or $(4^{k-1}8)$, if $s = 2$ then $a = (574^{k-2})$ or (7724^{k-3}).

$$\boxed{\begin{array}{l}\text{No.11: } m = 4,\ k = \text{even},\ a = (24^{k-2}, 10) \text{ or } a = (4^{k-1}8) \text{ or } (574^{k-2}) \text{ or } (7724^{k-3}), \\ c = (k+1, k+1, k+1, k+1).\end{array}}$$

Case 1.5 $m = 5$.
By (5), we have $3 + 5 \times \frac{k-2}{2} \leq 3 + \sum \gamma_j \leq k_1 + 2k_2 \leq 2k$, and so $k \leq 4$. If $k = 3$, then using $\gamma_j \geq 1$, we get $8 \leq 3 + \sum \gamma_j \leq 2k = 6$, a contradiction. Thus we may assume that

$k = 4$. Then we have $\gamma_1 = \cdots = \gamma_5 = 1$, $k_2 = 4$, $p = 5$. Since $s \leq p - 2 = 3$, we have $s = 1$ or 3. If $s = 1$ then $a = (744, 10)$ or $(13, 4, 4, 4)$, if $s = 3$ then $a = (7774)$.

No.13: $m = 5$, $k = 4$, $a = (447, 10)$, $(13, 4, 4, 4)$ or (7774), $c = (55555)$.

Case 1.6 $m \geq 6$.
By (5), we have $3 + 6 \times \frac{k-2}{2} \leq 3 + \sum \gamma_j \leq k_1 + 2k_2 \leq 2k$, and so $k = 3$. Using $\gamma_j \geq 1$, we get $9 \leq 3 + \sum \gamma_j \leq 2k = 6$, a contradiction.

Case 2 $s > p$.
By (2) and (4), exceptions occur if and only if

$$\sum_{j=1}^{m} \lfloor \gamma_j/3 \rfloor \leq (\sum_{j=1}^{m} \gamma_j - \sum_{i=1}^{k} \alpha_i)/3 - 1,$$

or equivalently,

$$3 + \sum \alpha_i \leq \sum (\gamma_j - 3\lfloor \gamma_j/3 \rfloor).$$

Let $m_l := \#\{j : \gamma_j \equiv l \pmod 3\}$. Note that $m_0 + m_1 + m_2 = m$. Consequently, exceptions occur if and only if

$$3 + \sum \alpha_i \leq m_1 + 2m_2. \tag{10}$$

This implies that

$$3 + \sum \alpha_i \leq m_1 + 2m_2 \leq 2m \leq \frac{2n}{k+1}. \tag{11}$$

By (3) and $s \leq k$, we have

$$(k+1)m \leq \sum c_j = n \leq 3k + 2\sum \alpha_i \leq 3k + 2(2m - 3) \tag{12}$$

By (11) and (12), one has

$$\frac{n - 3k}{2} \leq \sum \alpha_i \leq \frac{2n}{k+1} - 3. \tag{13}$$

Case 2.1 $k = 2$.
Since $s \geq p + 2 \geq 2$, we have $a_1 \equiv a_2 \equiv 1 \pmod 2$ and $a_1 = 2\alpha_1 + 3$, $a_2 = 2\alpha_2 + 3$. By (10), we have

$$3 + \alpha_1 + \alpha_2 = \frac{n}{2} \leq m_1 + 2m_2,$$

which implies $c = (4^m)$.

No.16: $k = 2$, $a = (\text{odd}, \text{odd})$, $c = (4^m)$.

Case 2.2 $k = 3$.
By (12), we have

$$4m \leq n \leq 4m + 3. \tag{14}$$

Since $p + 2 \leq s \leq k = 3$, we have $(p, s) = (0, 2)$ or $(1, 3)$. If $p = 0$ then $4m \leq n = 3s + 2\sum \alpha_i \leq 4m$, and so $c = (4^m)$. Now we may assume $p = 1$. In general,

$$4m \leq n \leq 3k + 2\sum \alpha_i \leq 9 + 2(m_1 + 2m_2 - 3) \leq 3 + 2m + 2m_2,$$

which implies $m_2 \geq m-1$. This together with (14) implies $c = (54^{m-1})$ or (74^{m-1}).

$$\boxed{\text{No.17: } k=3,\ a = (\text{odd, odd, even}),\ c = (4^m).}$$

$$\boxed{\text{No.18: } k=3,\ a = (\text{odd, odd, odd}),\ c = (54^{m-1}) \text{ or } (74^{m-1}).}$$

From now on, we may assume that $k \geq 4$.

Case 2.3 $m = 2$.
Using (10), we have $3 \leq m_1 + 2m_2$. Thus, $m_2 = 2$ or $m_1 = m_2 = 1$.

Case 2.3.1 $m_2 = 2$.
In this case, we have $\gamma_1 \equiv \gamma_2 \equiv 2 \pmod 3$. By (3), we have

$$2\sum \alpha_i \equiv 3s + 2\sum \alpha_i = n = 3p + 2\sum \gamma_j \equiv 2 \pmod 3.$$

Thus, $\sum \alpha_i \equiv 1 \pmod 3$. By (10), we have $3 + \sum \alpha_i \leq 4$, and so $\sum \alpha_i = 1$. Then (12) implies that $n \leq 3k+2$. Thus, $a = (3^{k-1}2)$ or $(3^{k-1}5)$, and we get $c_1 \equiv c_2 \equiv 1 \pmod 3$ from $\gamma_1 \equiv \gamma_2 \equiv 2 \pmod 3$.

$$\boxed{\text{No.2: } m=2,\ a = (3^{k-1}2) \text{ or } (3^{k-1}5),\ c_1 \equiv c_2 \equiv 1 \pmod 3.}$$

Case 2.3.2 $m_1 = m_2 = 1$.
By (10), we have $\sum \alpha_i = 0$, and by (12) we have $n \leq 3k$. Thus, $a = (3^k)$, and we get $\{c_1, c_2\} \equiv \{1, 2\} \pmod 3$ from $m_1 = m_2 = 1$.

$$\boxed{\text{No.3: } m=2,\ a = (3^k),\ \{c_1, c_2\} \equiv \{1,2\} \pmod 3.}$$

Case 2.4 $m = 3$.
By (10) and (12), we have

$$3(k+1) \leq n \leq 3k + 2\sum \alpha_i \leq 3(k-2) + 2m_1 + 4m_2, \tag{15}$$

which implies $9 \leq 2m_1 + 4m_2$. Thus, we have "$m_1 = 1$, $m_2 = 2$," or $m_2 = 3$. Note that $m_l := \#\{j : c_j \equiv 2l \pmod 3\}$.

Case 2.4.1 $m_1 = 1$, $m_2 = 2$.
By (15), we have

$$n \leq 3(k-2) + 2 + 8 = (k+1) + (k+1) + (k+2) = n \equiv 1 \pmod 3.$$

Since $n \equiv 2m_1 + 4m_2 = 10 \equiv 1 \pmod 3$, this implies $n = 3k+4$, and we therefore get $s = k$ and $\sum \alpha_i = m_1 + 2m_2 - 3 = 2$ from (15). Thus, $a = (3^{k-1}7)$ or $(3^{k-2}55)$. Since $c = (k+1, k+1, k+2)$, we also have $k \equiv 0 \pmod 3$ by the assumption that $m_1 = 2$ and $m_2 = 2$.

$$\boxed{\text{No.6: } m=3,\ k \equiv 0 \pmod 3,\ a = (3^{k-1}7) \text{ or } (3^{k-2}55),\ c = (k+1, k+1, k+2).}$$

Case 2.4.2 $m_2 = 3$.
By (9), we have $\sum \alpha_i \leq 3$, and by (12), we have $n \leq 3k+6$. Thus it follows that $c_1 = c_2 = c_3 = k+1$, or $c_1 = c_2 = c_3 = k+2$, or $\{c_1, c_2, c_3\} = \{k+1, k+1, k+4\}$.

Case 2.4.2.1 $c_1 = c_2 = c_3 = k+1$.
Since $n = 3k+3 = 3s + 2\sum \alpha_i \leq 3s + 6$, we have $k-1 \leq s \leq k$. If $s = k$, then

$n = 3k + 2\sum \alpha_i \equiv k \pmod 2$, which contradicts $n = 3k + 3 \equiv k + 1 \pmod 2$. Thus, we may assume that $s = k - 1$. Therefore $a = (3^{k-3}552)$ or $(3^{k-2}54)$ or $(3^{k-2}72)$ or $(3^{k-1}6)$.

No.7: $m = 3$, $k \equiv 0 \pmod 3$, $a = (3^{k-3}552)$ or $(3^{k-2}54)$ or $(3^{k-2}72)$ or $(3^{k-1}6)$, $c = (k + 1, k + 1, k + 1)$.

Case 2.4.2.2 $c_1 = c_2 = c_3 = k + 2$.
Since $n = 3k + 6 = 3s + 2\sum \alpha_i \le 3s + 6$, we have $s = k$. Thus, $a = (3^{k-3}555)$ or $(3^{k-2}57)$ or $(3^{k-1}9)$.

No.8: $m = 3$, $k \equiv 2 \pmod 3$, $a = (3^{k-3}555)$ or $(3^{k-2}57)$ or $(3^{k-1}9)$, $c = (k + 2, k + 2, k + 2)$.

Case 2.4.2.3 $\{c_1, c_2, c_3\} = \{k + 1, k + 1, k + 4\}$.
Since $n = 3k + 6 = 3s + 2\sum \alpha_i \le 3s + 6$, we have $s = k$. Thus, $a = (3^{k-3}555)$ or $(3^{k-2}57)$ or $(3^{k-1}9)$.

No.9: $m = 3$, $k \equiv 0 \pmod 3$, $a = (3^{k-3}555)$ or $(3^{k-2}57)$ or $(3^{k-1}9)$, $c = (k + 1, k + 1, k + 4)$.

Case 2.5 $m = 4$.
By (12), we have

$$4(k + 1) \le n \le 3k + 10, \tag{16}$$

and so $k \le 6$.
Case 2.5.1 $k = 6$.
By (16), we have $n = 7 \times 4$, and $c = (7777)$, $(p, s) = (4, 6)$.

No.12: $m = 4$, $k = 6$, $a = (\text{odd, odd, odd, odd, odd, odd})$, $c = (7777)$.

Case 2.5.2 $k = 5$.
By (10) and (12), we have $24 \le n \le 3k + 2(m_1 + 2m_2 - 3)$, which implies

$$(m_1, m_2) = (0, 4). \tag{17}$$

By (12), we have $24 \le n \le 25$, and $c = (6666)$ or (6667), but neither of these two parameters satisfies (17).
Case 2.5.3 $k = 4$.
By (10) and (12), we have $20 \le n \le 3k + 2(m_1 + 2m_2 - 3)$, which implies

$$(m_1, m_2) = (1, 3) \text{ or } (0, 4). \tag{18}$$

By (12), we have $20 \le n \le 22$, and $c = (5555)$, (5556), (5557), or (5566), but none of these parameters satisfy (18).
Case 2.6 $m \ge 5$.
By (13), we have

$$5 \le m \le \frac{n}{k + 1} \le \frac{3(k - 2)}{k - 3}, \tag{19}$$

and so $k \le 4$. If $m \ge 7$ then (19) implies $k \le 3$. Thus, only possible cases are $(m, k) = (5, 4)$ or $(6, 4)$.

Case 2.6.1 $(m, k) = (5, 4)$.
By (12), we have $25 \leq n \leq 26$. Thus $c = (55555)$ or (55556), but neither of these two parameters satisfies $p + 2 \leq s \leq k = 4$.

Case 2.6.2 $(m, k) = (6, 4)$.
By (12), we have $n = 30$. Thus $c = (555555)$, which contradicts $p + 2 \leq s \leq k = 4$. ∎

The same argument we used in the proof of Theorem 9 using Lemma 7 is also valid for the proof of Theorem 5 using Lemma 4.

Proof of Theorem 2 Let $v_1, \ldots, v_k$ be given k vertices. We may assume $a_1 \leq \cdots \leq a_k$. We apply induction on k. The induction is trivially true at $k = 1$. Suppose that $a_1 \leq 3$. In this case we choose a path P satisfying that $|V(P)| = a_1$ and v_1 is an endpoint of P (and $v_i \notin A_1$ for $i \neq 1$). Since $\delta(G - V(P)) \geq 3k - 3 = 3(k - 1)$, we can apply induction.

Next suppose that $a_1 \geq 4$. Choose k independent edges $v_1w_1, \ldots, v_kw_k$ in G. Delete these $2k$ vertices. The remaining graph G' has minimum degree at least k. If G' is connected, we can apply Theorem 1 to G' and the partition $n - 2k = \sum_{i=1}^{k}(a_i - 2)$.

Let $C_1, \ldots, C_m$ be connected components of G'. We may suppose that $m \geq 2$. Let $V := \{v_1, \ldots, v_k\}$, $W := \{w_1, \ldots, w_k\}$, and $c_j := |C_j|$ for $1 \leq j \leq m$. Suppose that $c_1 \geq c_2 \geq \cdots \geq c_m$. We choose W so that $(c_1, \ldots, c_m)$ is maximal with respect to the lexicographic order. This order is defined by setting $(c_1, \ldots, c_m) > (d_1, \ldots, d_l)$ $(c_1 \geq \cdots \geq c_m,\ d_1 \geq \cdots \geq d_l)$ if there exists i such that $c_j = d_j$ for all $1 \leq j < i$ and $c_i > d_i$. We define $\mathrm{lex}(G - W) := (c_1, \ldots, c_m)$.

Let $s \leq t$ (i.e., $c_s \geq c_t$). Choose $x \in C_s$ and $y \in C_t$.

Lemma 12 *If $xw_i \in E(G)$ then $yv_i \notin E(G)$.*

Proof Suppose, on the contrary, $yv_i \in E(G)$. Define $w_i' := y$ and $W' := W - \{w_i\} \cup \{w_i'\}$. Then $\mathrm{lex}(G - W) < \mathrm{lex}(G - W')$, which contradicts our assumption. ∎

Lemma 13 *$|N(x) \cap W| + |N(y) \cap V| \leq k$ holds for every $x \in C_s$ and $y \in C_t$.*

Proof Suppose, on the contrary, that $|N(x) \cap W| + |N(y) \cap V| > k$. Then by the pigeonhole principle, there exists i such that $xw_i, yv_i \in E(G)$. This contradicts Lemma 12. ∎

Lemma 14 *$c_s + c_t \geq 3k + 2$.*

Proof Since

$$3k \leq d(x) \leq |N(x) \cap V| + |N(x) \cap W| + (c_s - 1),$$

we have $2k \leq |N(x) \cap W| + (c_s - 1)$. In the same way, one has $2k \leq |N(y) \cap V| + (c_t - 1)$. Using the above two inequalities and Lemma 13, we have $4k \leq k + (c_s - 1) + (c_t - 1)$, i.e., $c_s + c_t \geq 3k + 2$. ∎

We continue the proof of Theorem 2. Let $n' := n - 2k$ and $a_i' := a_i - 2$ for $1 \leq i \leq k$. We apply Theorem 5 to G' and $n' = \sum a_i'$. This way we can get an appropriate decomposition except in the case of exceptional parameters. Note that we assume $c_s + c_t \geq 3k + 2$ for all $s \neq t$. (Thus, we must have $2n = (c_1 + c_2) + (c_2 + c_3) + \cdots + (c_m + c_1) \geq m(3k + 2)$.) Therefore it is sufficient to consider the following exceptional cases: No. 1, 2, 14, 16.

If $c_s + c_t \geq 4k + 2$, then these exceptions can not occur. So we may assume that $c_s + c_t \leq 4k + 1$. Under this assumption, we have

$$6k \leq d(x) + d(y) \leq |N(x) \cap (V \cup W)| + |N(y) \cap (V \cup W)| + (c_s - 1) + (c_t - 1),$$

that is,

$$2k+1 \leq |N(x) \cap (V \cup W)| + |N(y) \cap (V \cup W)|.$$

By the pigeonhole principle, there exists i such that

$$|N(x) \cap \{v_i, w_i\}| + |N(y) \cap \{v_i, w_i\}| \geq 3.$$

Using Lemma 12, we may assume that $xv_i, yw_i \in E(G)$.

Recall that we are considering the exceptional cases No. 1, 2, 14, 16. If $C_s - \{x\}$ is connected, then we can escape from these cases by setting $w'_i := x$, $c'_s := c_s - 1$, and $c'_t := c_t + 1$. So we may assume that $C_s - \{x\}$ is disconnected. Let D_1, D_2 be connected components of $C_s - \{x\}$. Choose $z_1 \in D_1$, $z_2 \in D_2$. Then we have

$$\begin{aligned} d(z_1) &\leq |N(z_1) \cap V| + |N(z_1) \cap W| + |\{x\} \cup D_1 - \{z_1\}| \\ &\leq |N(z_1) \cap W| + |D_1| + k, \\ d(y) &\leq |N(y) \cap V| + k + (c_t - 1). \end{aligned}$$

Using Lemma 13, the above inequalities imply

$$6k \leq d(z_1) + d(y) \leq 3k + |D_1| + c_t - 1,$$

that is $3k \leq |D_1| + c_t - 1$. In the same way, we have $3k \leq |D_2| + c_t - 1$. Consequently, we have

$$\begin{aligned} 6k &\leq |D_1| + |D_2| + 2c_t - 2 \\ &\leq c_s + 2c_t - 3 \\ &\leq 4k - 2 + c_t. \end{aligned}$$

This implies $c_t \geq 2k+2$, which contradicts our earlier assumption $c_s \geq c_t$ and $c_s + c_t \leq 4k+1$.

This completes the proof of Theorem 2. ∎

4 Open problem

Theorem 2 requires the condition $\delta(G) \geq 3k$. Is this condition sharp? The following example shows that one can not replace this condition by $\delta(G) \geq 3k-3$.

Example 15 *Let $G := K_{3k-2} \cup K_{3k-2}$, i.e., the disjoint union of two complete graphs of order $3k-2$. Choose $v_1, \ldots, v_k$ in the same connected component. Let $a_1 = \cdots = a_{k-1} = 3$, $a_k = 3k-1$. Then any decomposition $V(G) = \bigcup A_i$ with $|A_i| = a_i$, $v_i \in A_i$ contains some j such that $\delta(<A_j>) = 0$.*

Problem 16 *Does Theorem 2 hold under the assumption $\delta(G) \geq 3k-2$?*

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