THE MAXIMUM SIZE OF 4-WISE 2-INTERSECTING AND 4-WISE 2-UNION FAMILIES

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ABSTRACT. Let $\mathscr{F}$ be an n-uniform hypergraph on $2n$ vertices. Suppose that $|F_1 \cap F_2 \cap F_3 \cap F_4| \geq 2$ and $|F_1 \cup F_2 \cup F_3 \cup F_4| \leq n-2$ holds for all $F_1, F_2, F_3, F_4 \in \mathscr{F}$. We prove that the size of $\mathscr{F}$ is at most $\binom{2n-4}{n-2}$ for n sufficiently large.

1. INTRODUCTION

A family $\mathscr{F} \subset 2^X$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathscr{F}$. A family $\mathscr{F} \subset 2^X$ is called r-wise t-union if $|F_1 \cup \cdots \cup F_r| \leq |X| - t$ holds for all $F_1, \ldots, F_r \in \mathscr{F}$. The Erdős–Ko–Rado theorem[2] states that if $n \geq 2k$ and $\mathscr{F} \subset \binom{[n]}{k}$ is 2-wise 1-intersecting then $|\mathscr{F}| \leq \binom{n-1}{k-1}$. By considering the complement, the theorem can be restated as follows: if $n \leq 2k$ and $\mathscr{F} \subset \binom{n}{k}$ is 2-wise 1-union then $|\mathscr{F}| \leq \binom{n-1}{k}$. Now what is the maximum size of a family $\mathscr{F} \subset \binom{[n]}{k}$ that is r-wise 1-intersecting and at the same time q-wise 1-union? The case $r = q = 2$ is quite easy. In fact, it follows from the Erdős–Ko–Rado theorem that

$$|\mathscr{F}| \leq \begin{cases} \binom{n-1}{k} & \text{if} \quad n < 2k \\ \binom{n-1}{k} = \binom{n-1}{k-1} & \text{if} \quad n = 2k \\ \binom{n-1}{k-1} & \text{if} \quad n > 2k. \end{cases}$$

But the case $r \geq 3$ or $q \geq 3$ is not so easy and we do not know the complete answer yet. The first result in this direction was obtained by Gronau[7] who solved the case $r \geq 6$ and $q \geq 6$ completely. Then Engel and Gronau[1] settled the case $r \geq 4$ and $q \geq 4$ as follows.

Theorem 1. *Let $r \geq 4$, $q \geq 4$ and $\mathscr{F} \subset \binom{[n]}{k}$. Suppose that $\mathscr{F}$ is r-wise 1-intersecting and q-wise 1-union, and*

$$\frac{n-1}{q} + 1 \leq k \leq \frac{r-1}{r}(n-1).$$

Then we have $|\mathscr{F}| \leq \binom{n-2}{k-1}$.

The case $r=3$ or $q=3$ is more difficult and still open. As a special case the following was proved in [6].

Theorem 2. *Let $\mathscr{F} \subset \binom{[2n]}{n}$ be a 3-wise 1-intersecting and 3-wise 1-union family. Then we have $|\mathscr{F}| \leq \binom{2n-2}{n-1}$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n-1]}{n} : 1 \in F\}$.*

In this note we consider the 4-wise 2-intersecting and 4-wise 2-union case, and our main result is the following.

Theorem 3. *Let $\mathscr{F} \subset \binom{[2n]}{n}$ be a 4-wise 2-intersecting and 4-wise 2-union family with n sufficiently large. Then we have $|\mathscr{F}| \leq \binom{2n-4}{n-2}$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n-2]}{n} : [2] \subset F\}$.*

It is most likely that the same conclusion holds for the 3-wise 2-intersecting and 3-wise 2-union case, but it seems to be much harder to prove.

We use the random walk method originated from [4] by Frankl. For $A \in \binom{[n]}{k}$ we define the corresponding walk on $\mathbb{Z}^2$, denoted by walk(A), in the following way. The walk is from $(0,0)$ to $(n-k,k)$ with n steps, and if $i \in A$ (resp. $i \notin A$) then we move one unit up (resp. one unit to the right) at the i-th step. Among $\binom{n}{k}$ walks corresponding to $\binom{[n]}{k}$, how many of them touch a given line? The next result gives an upper bound of this number, which is one of the main tools to prove Theorem 3.

Theorem 4. *Let $p \in \mathbb{Q}$, $r,t \in \mathbb{N}$ be fixed constants with $r \geq 2$ and $p < \frac{r-1}{r+1}$, and let n and k be positive integers with $p = \frac{k}{n}$. Let $\alpha \in (p,1)$ be the unique root of the equation $(1-p)x^r - x + p = 0$ and let $f(n)$ be the number of walks from $(0,0)$ to $(n-k,k)$ which touch the line $L : y = (r-1)x + t$. Then we have*

$$f(n) \leq \alpha^t \binom{n}{k}$$

for n sufficiently large.

If $p = \frac{k}{n} > \frac{r-1}{r}$ then all walks touch the line, i.e., $f(n) = \binom{n}{k}$. The author conjectures that the conclusion of Theorem 4 still holds for $p < \frac{r-1}{r}$.

2. TOOLS

In this section we summarize some tools for the proof of Theorem 3. For integers $1 \leq i < j \leq n$ and a family $\mathscr{F} \subset 2^{[n]}$, define the (i,j)-shift S_{ij} as follows.

$$S_{ij}(\mathscr{F}) := \{S_{ij}(F) : F \in \mathscr{F}\},$$

where

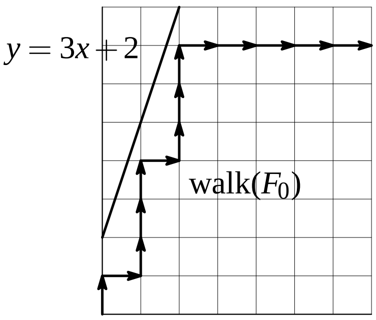
$$S_{ij}(F) := \begin{cases} (F-\{j\})\cup\{i\} & \text{if } i \notin F,\ j \in F,\ (F-\{j\})\cup\{i\} \notin \mathscr{F}, \\ F & \text{otherwise.} \end{cases}$$

A family $\mathscr{F} \subset 2^{[n]}$ is called shifted if $S_{ij}(\mathscr{F}) = \mathscr{F}$ for all $1 \le i < j \le n$. For a given family $\mathscr{F}$, one can always obtain a shifted family $\mathscr{F}'$ from $\mathscr{F}$ by applying shifting to $\mathscr{F}$ repeatedly. Then we have $|\mathscr{F}'| = |\mathscr{F}|$ because shifting keeps the size of the family. It is easy to check that if $\mathscr{F}$ is r-wise t-intersecting (resp. q-wise s-union) then $S_{ij}(\mathscr{F})$ is also r-wise t-intersecting (resp. q-wise s-union). Therefore if $\mathscr{F}$ is an r-wise t-intersecting and q-wise s-union family then we can find a shifted family $\mathscr{F}'$ which is r-wise t-intersecting and q-wise s-union and $|\mathscr{F}'| = |\mathscr{F}|$.

Next we explain how to connect Theorem 4 to bound the size of r-wise t-intersecting families. Let us begin with a toy example. Suppose that $\mathscr{F} \subset \binom{[14]}{7}$ is a shifted 4-wise 2-intersecting family. We are going to show that $F_0 := \{1,3,4,5,7,8,9\} \notin \mathscr{F}$. Suppose on the contrary that $F_0 \in \mathscr{F}$. Then by shifting F_0, we obtain $F_1, F_2, F_3 \in \mathscr{F}$:

$$\begin{aligned} F_0 &:= \{1,*,3,4,5,*,7,8,9\}, \\ F_1 &:= \{1,2,*,4,5,6,*,8,9\}, \\ F_2 &:= \{1,2,3,*,5,6,7,*,9\}, \\ F_3 &:= \{1,2,3,4,*,6,7,8,*\}, \end{aligned}$$

where "*" means visible blank space. But this is impossible because $F_0 \cap F_1 \cap F_2 \cap F_3 = \{1\}$, which contradicts the 4-wise 2-intersecting property. This proves that $F_0 \notin \mathscr{F}$. The following picture shows walk(F_0).



Note that walk(F_0) is the "maximal" walk which does not touch the line $L : y = 3x + 2$. In other words, if walk(G), $G \in \binom{[14]}{7}$, does not touch L then we can obtain F_0 from G by shifting (a sequence of shiftings). Since $\mathscr{F}$ is shifted we have $G \notin \mathscr{F}$. Equivalently, if $F \in \mathscr{F}$ then walk(F) must touch L. For the general case, i.e., a shifted r-wise t-intersecting family $\mathscr{F} \subset \binom{[n]}{k}$, we consider the line $y = (r-1)x + t$ and $F_0, F_1, \dots, F_r$, where F_i consists of the first k elements of $[n] - \{t+i, t+r+i, t+2r+i, \dots\}$. Then we have the following.

Fact 5 ([4])**.** *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. Then for all $F \in \mathscr{F}$, walk(F) must touch the line $L_1 : y = (r-1)x+t$.*

Fact 5 and Theorem 4 gives $|\mathscr{F}| \leq \alpha^t \binom{n}{k}$ if $\frac{k}{n} < \frac{r-1}{r+1}$ and n is sufficiently large.

If $\mathscr{F} \subset \binom{[n]}{k}$ is a shifted q-wise s-union family then the complement family $\mathscr{F}^c = \{[n]-F : F \in \mathscr{F}\} \subset \binom{[n]}{n-k}$ is a shifted (in the reverse direction) q-wise s-intersecting family. Changing the coordinate system by $x' = k-y$ and $y' = (n-k)-x$, one obtains from Fact 5 that walks corresponding to $\mathscr{F}^c$ touch the line $y' = (q-1)x'+s$. Namely we have the following.

Fact 6. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted q-wise s-union family. Then for all $F \in \mathscr{F}$, walk(F) must touch the line $L_2 : y = \frac{1}{q-1}(x-n+k+s)+k$.*

If $\mathscr{F} \subset \binom{[2n]}{n}$ is a shifted r-wise t-intersecting and q-wise s-union family then the corresponding walks of the family touch the both lines of L_1 and L_2. In this situation, we can use the following result which is deduced from Theorem 4 by setting $p = \frac{1}{2}$.

Corollary 7. *Let $q,r,s,t \in \mathbb{N}$ be fixed constants with $q \geq 4$ and $r \geq 4$. Let $\alpha_j \in (\frac{1}{2},1)$ be the unique root of the equation $\frac{1}{2}x^j - x + \frac{1}{2} = 0$. Let $h(n)$ be the number of walks from $(0,0)$ to (n,n) which touch both of the lines $L_1 : y = (r-1)x+t$ and $L_2 : y = \frac{1}{q-1}(x-n+s)+n$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\frac{h(n)}{\binom{2n}{n}} \leq (1+\varepsilon)\alpha_r^t \alpha_q^s$$

holds for all $n > n_0$.

One can not remove ε from the above inequality. (Numerical experiments suggest that $h(n)/\binom{2n}{n} \geq \alpha_r^t \alpha_q^s$ always holds.) In our application, we also need a slight modification of Theorem 4 and Corollary 7 stated below.

Corollary 8. *Let $p \in \mathbb{Q}$, $r,t,u,v \in \mathbb{N}$ be fixed constants with $r \geq 2$ and $p < \frac{r-1}{r+1}$, and let n and k be positive integers with $p = \frac{k}{n}$. Let $\alpha \in (p,1)$ be the unique root of the equation $(1-p)x^r - x + p = 0$ and let $g(n)$ be the number of walks from $(0,0)$ to $(n-k-u, k-v)$ which touch the line $y = (r-1)x+t$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\frac{g(n)}{\binom{n-u-v}{k-v}} \leq (1+\varepsilon)\alpha^t$$

holds for all $n > n_0$. Moreover if $u = 0$ then we can choose $\varepsilon = 0$.

Corollary 9. *Let $q,r,s,t,u,v \in \mathbb{N}$ be fixed constants with $q \geq 4$, $r \geq 4$ and $t+(r-1)u-v > 0$. Let $\alpha_j \in (\frac{1}{2},1)$ be the unique root of the equation $\frac{1}{2}x^j - x + \frac{1}{2} = 0$. Let $m(n)$ be the number of walks from (u,v) to (n,n) which touch both of the lines $L_1 : y = (r-1)x+t$ and $L_2 : y = \frac{1}{q-1}(x-n+s)+n$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\frac{m(n)}{\binom{2n-u-v}{n-v}} \leq (1+\varepsilon)\alpha_r^{t+(r-1)u-v}\alpha_q^s$$

holds for all $n > n_0$.

Finally we list the following Erdős–Ko–Rado type results for multiply intersecting families which we will use to prove Theorem 3.

Theorem 10. [3] *If $\mathscr{F} \subset \binom{[n]}{k}$ is r-wise 1-intersecting and $(r-1)n \geq rk$ then $|\mathscr{F}| \leq \binom{n-1}{k-1}$. If $r \geq 3$ then equality holds iff $\mathscr{F} \cong \{F \in \binom{[n]}{k} : 1 \in F\}$.*

The equivalent complement version is the following: If $\mathscr{F} \subset \binom{[n]}{k}$ is r-wise 1-union and $rk \geq n$ then $|\mathscr{F}| \leq \binom{n-1}{k}$.

Theorem 11. [5] *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family with $k/n \leq 0.501$, n sufficiently large. Then we have $|\mathscr{F}| \leq \binom{n-2}{k-2}$, and equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n]}{n} : [2] \subset F\}$.*

3. Proof of Theorem 4

Let $I := \{0,1,\ldots,\lfloor \frac{k-t}{r-1} \rfloor\}$ and for $i \in I$ let a_i be the number of walks of length $ri+t$ from $(0,0)$ to $(i,(r-1)i+t)$ which touch the line L only at $(i,(r-1)i+t)$. Then we have

$$f(n) = \sum_{i \in I} a_i \binom{n-ri-t}{k-(r-1)i-t}. \tag{1}$$

We also use the following fact (cf. (7) and Fact 3 in [8]):

$$\sum_{i \in I} a_i p^{(r-1)i+t}(1-p)^i \leq \sum_{i=0}^{\infty} a_i p^{(r-1)i+t}(1-p)^i = \alpha^t. \tag{2}$$

Comparing (1) and (2) it suffices to show that

$$\binom{n-ri-t}{k-(r-1)i-t} \Big/ \binom{n}{k} \leq p^{(r-1)i+t}(1-p)^i \tag{3}$$

holds for all $i \in I$.

Claim 12. *Let $S(t) := \binom{n-ri-t}{k-(r-1)i-t}/p^t$. Then $S(t)$ is a decreasing function of t.*

Proof. Since $S(t+1)=\binom{n-ri-t-1}{k-(r-1)i-t-1}/p^{t+1}=S(t)\frac{k-(r-1)i-t}{p(n-ri-t)}$, it suffices to show

$$1>\frac{k-(r-1)i-t}{p(n-ri-t)}=\frac{n(k-(r-1)i-t)}{k(n-ri-t)},$$

or equivalently,

$$\frac{(r-1)i+t}{ri+t}>\frac{k}{n}.$$

This is certainly true because

$$\frac{(r-1)i+t}{ri+t}>\frac{r-1}{r}>\frac{r-1}{r+1}>\frac{k}{n}. \quad \square$$

Due to the claim, it suffices to show (3) for $t=1$, that is,

$$\binom{n-ri-1}{k-(r-1)i-1}/\binom{n}{k}\le p^{(r-1)i+1}(1-p)^i \quad \text{for} \quad i\in I.$$

The LHS of the above inequality is rewritten as $p\prod_{j=0}^{i-1}T(j)$ where

$$T(j):=\frac{n-k-j}{n-rj-1}\prod_{\ell=2}^{r}\frac{k-(r-1)j-\ell+1}{n-rj-\ell}.$$

Thus we have to show

$$\prod_{j=0}^{i-1}T(j)\le (p^{r-1}(1-p))^i. \tag{4}$$

Claim 13. *We have $T(j)>T(j+1)$ for $0\le j\le i-2$.*

Proof. Comparing

$$T(j)=\frac{n-k-j}{n-rj-1}\prod_{\ell=2}^{r}\frac{k-(r-1)j-\ell+1}{n-rj-\ell}=\frac{n-k-j}{n-rj-r}\prod_{\ell=2}^{r}\frac{k-(r-1)j-\ell+1}{n-rj-\ell+1}$$

and

$$T(j+1)=\frac{n-k-(j+1)}{n-r(j+1)-1}\prod_{\ell=2}^{r}\frac{k-(r-1)(j+1)-\ell+1}{n-r(j+1)-\ell},$$

it suffices to show the following inequalities:

$$\frac{n-k-j}{n-rj-r}>\frac{n-k-(j+1)}{n-r(j+1)-1} \tag{5}$$

and, for $2\le\ell\le r$,

$$\frac{k-(r-1)j-\ell+1}{n-rj-\ell+1}>\frac{k-(r-1)(j+1)-\ell+1}{n-r(j+1)-\ell}. \tag{6}$$

The inequality (5) is equivalent to $j < \frac{k-1}{r-1} - 1$, which follows from our assumption $j \le i-2 \le \lfloor \frac{k-1}{r-1} \rfloor - 2$. Since $k = pn$, inequality (6) is equivalent to

$$(r-1-p(r+1))n + (r-1)j + 2(\ell-1) > 0.$$

Since $p < \frac{r-1}{r+1}$, the coefficient of n in the LHS is positive and so the above inequality clearly holds. □

By the claim we have $\prod_{j=0}^{i-1} T(j) \le T(0)^i$. Thus to prove (4) it suffices to show $T(0) \le p^{r-1}(1-p)$ or equivalently,

$$p^{r-1}(1-p)(n-1)\cdots(n-r) - (pn-1)\cdots(pn-r+1)(n-pn) \ge 0.$$

The LHS can be rewritten as

$$\frac{1}{2} r p^{r-2}(1-p)(r-1-(r+1)p)n^{r-1} + O(n^{r-2}).$$

Since $p < \frac{r-1}{r+1}$, the coefficient of n^{r-1} is positive and we are done.

4. Proof of Corollary 7

Let $\varepsilon > 0$ be given. We choose $\delta_1, \delta_2 > 0$ so that

$$\delta_1 < (\varepsilon/2)\alpha_r^t \alpha_q^s, \tag{7}$$

$$(1+\delta_2)^2 < 1 + (\varepsilon/2). \tag{8}$$

Let $K_n := \{k \in \mathbb{N} : |k - \frac{n}{2}| \le c\sqrt{n}\}$ where we choose $c > 0$ so that

$$\lim_{n\to\infty} \sum_{k \in K_n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = \frac{2}{\sqrt{\pi}} \int_{-c}^{c} \exp(-4x^2)dx > 1 - \frac{\delta_1}{2}.$$

(The first equality follows from the de Moivre–Laplace limit Theorem. In fact one has $\binom{n}{k}^2 / \binom{2n}{n} = \frac{2}{\sqrt{\pi n}} \exp(-4x^2 + o(1))$ by setting $x = (k - \frac{n}{2})/\sqrt{n}$.) Then we can choose $n_1 \in \mathbb{N}$ so that

$$\sum_{k \notin K_n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} < \delta_1 \tag{9}$$

holds for all $n > n_1$.

For $0 < p < 1$ let $\alpha_j(p) \in (p, 1)$ be the unique root of the equation $(1-p)x^j - x + p = 0$. Then $\alpha_j(p)$ is a continuous function of p at $p = 1/2$, and $\alpha_j(1/2) = \alpha_j$. Therefore we can choose $\delta_3 > 0$ so that

$$\alpha_r(p)^t < (1+\delta_2)\alpha_r^t, \quad \alpha_q(p)^s < (1+\delta_2)\alpha_q^s \tag{10}$$

holds for all p with $|p - \frac{1}{2}| < \delta_3$. Choose $n_2 \in \mathbb{N}$ so that $\frac{c}{\sqrt{n_2}} < \delta_3$, and let $n_0 := \max\{n_1, n_2\}$. Finally we choose n sufficiently large, i.e., $n > n_0$.

Now we consider a walk from $(0,0)$ to (n,n). After $n/2$ steps this walk arrives at the line $x+y=n$. Roughly speaking, a typical walk arrives at a point near the center $(\frac{n}{2},\frac{n}{2})$. More precisely we are interested in the walks which go through the center zone $\{(n-k,k) : k \in K_n\}$ and touch the lines L_1 and L_2 both. We will estimate the number of those walks by using Theorem 4. The number of walks outside the center zone is so small that we do not need a serious estimation for this type of walks.

Let $k \in K_n$ and $p = k/n$. Then we have $|p-\frac{1}{2}| < \delta_3$, which guarantees (10). Also, since $r \geq 4$ and δ_3 is small we may assume that $p < \frac{1}{2}+\delta_3 < \frac{r-1}{r+1}$. Thus by Theorem 4 and (10) the number of walks from $(0,0)$ to $(n-k,k)$ which touch the line L_1 is at most $\alpha_r(p)^t\binom{n}{k} < (1+\delta_2)\alpha_r^t\binom{n}{k}$.

Next we consider the walks from $(n-k,k)$ to (n,n) which touch the line L_2. Changing the coordinate system by $x' = n-y$ and $y' = n-x$, we find that the number of these walks is equal to the number of walks from $(0,0)$ to $(k,n-k)$ which touch the line $y' = (q-1)x'+s$, and this number is at most $(1+\delta_2)\alpha_q^s\binom{n}{k}$ if $k \in K_n$.

Therefore we have

$$h(n) \leq \sum_{k\in K_n}(1+\delta_2)\alpha_r^t\binom{n}{k}(1+\delta_2)\alpha_q^s\binom{n}{k} + \sum_{k\notin K_n}\binom{n}{k}^2. \tag{11}$$

Dividing the both sides by $\binom{2n}{n}$, and using $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ and (9), we have

$$h(n)/\binom{2n}{n} < (1+\delta_2)^2\alpha_r^t\alpha_q^s+\delta_1.$$

By (7) and (8) the RHS is less than $(1+\varepsilon)\alpha_r^t\alpha_q^s$.

5. PROOF OF COROLLARY 8

Let $\varepsilon > 0$ be given. Let $\alpha(w) \in (w,1)$ be the unique root of the equation $(1-w)x^r - x + w = 0$. Choose n and k with $p = \frac{k}{n}$ and set $n' := n-u-v$, $k' := k-v$ and $p' := \frac{k'}{n'}$. Then by Theorem 4 we have

$$f(n')/\binom{n'}{k'} \leq \alpha(p')^t.$$

We also have $p' \to p$ as $n \to \infty$. Since $\alpha(w)$ is a continuous function it follows that $\alpha(p') \to \alpha(p) = \alpha$ as $n \to \infty$. Thus we can choose n_0 such that $\alpha(p')^t < (1+\varepsilon)\alpha^t$ holds for all $n > n_0$. Then we have

$$\frac{g(n)}{\binom{n-u-v}{k-v}} = \frac{f(n')}{\binom{n'}{k'}} \leq \alpha(p')^t < (1+\varepsilon)\alpha^t.$$

Moreover if $u = 0$ then we have $p' = \frac{k-v}{n-v} < \frac{k}{n} = p$. Since $\alpha(w)$ is an increasing function, we have $\alpha(p') < \alpha(p)$ and $\alpha(p')^t < \alpha^t$.

6. PROOF OF COROLLARY 9

The proof is almost identical to the proof of Corollary 7. The only difference is that we consider walks $(u,v) \to (n-k,k) \to (n,n)$ in this case instead of $(0,0) \to (n-k,k) \to (n,n)$. For the part $(u,v) \to (n-k,k)$ we apply Corollary 8. To do so, we translate the walks by $(-u,-v)$, in other words, we consider walks from $(0,0)$ to $(n-k-u,k-v)$ with (translated) new line $y = (r-1)(x+u)+t-v = (r-1)x+t+(r-1)u-v$. (We need $t+(r-1)u-v > 0$ here.) The number of walks which touch this line is at most $(1+\varepsilon')\alpha_r(p)^{t+(r-1)u-v}\binom{n-u-v}{k-v}$. So we have to change the first inequality in (10) by $(1+\varepsilon')\alpha_r(p)^{t+(r-1)u-v} < (1+\delta_2)\alpha_r^{t+(r-1)u-v}$. Then inequality (11) is replaced by the following:

$$m(n) \le \sum_{k\in K_n} (1+\delta_2)^2 \alpha_r^{t+(r-1)u-v} \alpha_q^s \binom{n-u-v}{k-v}\binom{n}{k} + \sum_{k\notin K_n} \binom{n-u-v}{k-v}\binom{n}{k}.$$

We omit the remaining details which can be checked by routine calculation.

7. PROOF OF THEOREM 3

Let $\mathscr{F} \subset \binom{[2n]}{n}$ be a 4-wise 2-intersecting and 4-wise 2-union family. Suppose that $\mathscr{F}$ is not 3-wise 3-union. Then there exist $A,B,C \in \mathscr{F}$ such that $|A\cup B\cup C| = 2n-2$, say, $A\cup B\cup C = [2n-2]$. Since $\mathscr{F}$ is 4-wise 2-union, we have $\mathscr{F} \subset \binom{[2n-2]}{n}$. On the other hand, $\mathscr{F}$ is 4-wise 2-intersecting (and so 3-wise 2-intersecting). Then by Theorem 11 we have $|\mathscr{F}| \le \binom{2n-4}{n-2}$ and equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n-2]}{n} : [2] \subset F\}$. This means that the theorem is true if $\mathscr{F}$ is not 3-wise 3-union. Considering the complement, the theorem is also true if $\mathscr{F}$ is not 3-wise 3-intersecting. Therefore from now on we assume that

$\mathscr{F}$ is 3-wise 3-intersecting and 3-wise 3-union.

We also assume that $\mathscr{F}$ is shifted. Now suppose that

$$|\mathscr{F}| \ge \binom{2n-4}{n-2} \tag{12}$$

and we shall prove that there is no such $\mathscr{F}$.

Recall that for $A \in \binom{[2n]}{n}$ we define walk(A) on $\mathbb{Z}^2$ in the following way. The walk is from $(0,0)$ to (n,n) with $2n$ steps, and if $i \in A$ (resp. $i \notin A$) then we move one unit up (resp. one unit to the right) at the i-th step. Let us define

$$\mathscr{A}_i := \{A \in \binom{[2n]}{n} : |A \cap [2+4\ell]| \geq 2+3\ell \text{ first holds at } \ell = i\},$$

$$\mathscr{A}_{\bar{j}} := \{A \in \binom{[2n]}{n} : |A \cap [2n-4\ell-1, 2n]| \leq \ell \text{ first holds at } \ell = j\}.$$

(Here we say a property $P(\ell)$ first holds at $\ell = i$ if $P(\ell)$ does not hold for $0 \leq \ell < i$ but $P(i)$ holds.) If $A \in \mathscr{A}_i$ then, after starting from the origin, walk(A) touches the line $L_1 : y = 3x+2$ at $(i, 3i+2)$ for the first time. If $A \in \mathscr{A}_{\bar{j}}$ then walk(A) touches the line $L_2 : y = \frac{1}{3}(x-(n-2))+n$ at $(n-3j-2, n-j)$ and after passing this point this walk never touches the line again. By Fact 5 and Fact 6 every walk corresponding to a member of $\mathscr{F}$ touches both L_1 and L_2. Thus we have $\mathscr{F} \subset \bigcup_{i,j}(\mathscr{A}_i \cap \mathscr{A}_{\bar{j}})$. Set $\mathscr{A}_{i\bar{j}} := \mathscr{A}_i \cap \mathscr{A}_{\bar{j}}$,

$$\mathscr{F}_i := \mathscr{A}_i \cap \mathscr{F}, \quad \mathscr{F}_{\bar{j}} := \mathscr{A}_{\bar{j}} \cap \mathscr{F}, \quad \mathscr{F}_{i\bar{j}} := \mathscr{A}_{i\bar{j}} \cap \mathscr{F},$$

and

$$\mathscr{G}_{i\bar{j}} := \{F \cap [4i+3, 2n-4j-2] : F \in \mathscr{F}_{i\bar{j}}\}.$$

Since $\mathscr{F}_{0\bar{0}}$ is 3-wise 3-intersecting, $\mathscr{G}_{0\bar{0}} \subset \binom{[3,2n-2]}{n-2}$ is 3-wise 1-intersecting, and it follows from Theorem 10 that

$$|\mathscr{F}_{0\bar{0}}| = |\mathscr{G}_{0\bar{0}}| \leq \binom{2n-5}{n-3}. \tag{13}$$

Claim 14. *$\mathscr{G}_{1\bar{0}} \subset \binom{[7,2n-2]}{n-5}$ is 3-wise 1-intersecting.*

Proof. Suppose on the contrary that there exist $A, B, C \in \mathscr{G}_{1\bar{0}}$ such that $A \cap B \cap C = \emptyset$. If $F \in \mathscr{F}_{1\bar{0}}$ then $F \cap [6] = \{1,3,4,5,6\}$ or $\{2,3,4,5,6\}$. By the shiftedness we may assume that the following three subsets A', B', C' belong to $\mathscr{F}$:

$$A' := \{1,3,4,5,6\} \cup A, \; B' := \{1,2,4,5,6\} \cup B, \; C' := \{1,2,3,5,6\} \cup C.$$

If there exists $F \in \mathscr{F}$ such that $|F \cap [6]| \leq 4$ then using the shiftedness we may assume that $F \cap [6] \subset [4]$. But this is impossible because $A' \cap B' \cap C' \cap F = \{1\}$, contradicting the 4-wise 2-intersecting property. So we may assume that $|F \cap [6]| \geq 5$ holds for all $F \in \mathscr{F}$.

For $S \subset [6]$ let $\mathscr{F}(S) := \{F \in \mathscr{F} : F \cap [6] = S\}$. We consider the case $|S| = 5, 6$ and the corresponding walks clearly touch the line L_1 in the beginning. If $|S| = 5$ then the corresponding walks from $(1,5)$ to (n,n) must touch L_2, or equivalently we have to count the number of walks from $(0,0)$ to $(n-5, n-1)$ which touch L_1. (Here we change the coordinate system by $x' = n-y$ and $y' = n-x$.) Then by Corollary 8 ($r=4, t=2, u=5, v=1$) we have

$$\sum_{S \in \binom{[6]}{5}} |\mathscr{F}(S)| < 6(1+\varepsilon)\alpha^2 \binom{2n-6}{n-1},$$

where $\alpha \approx 0.543689$ is the root of the equation $x^4 - 2x + 1 = 0$. If $S = [6]$ then the corresponding walk from $(0,6)$ to (n,n) must touch L_2, and we count the number of walks from $(0,0)$ to $(n-6,n)$ which touch L_1. Again by Corollary 8 ($r = 4, t = 2, u = 6, v = 0$) we have

$$|\mathscr{F}([6])| < (1+\varepsilon)\alpha^2 \binom{2n-6}{n}.$$

Consequently, for sufficiently large n, we have

$$\frac{|\mathscr{F}|}{\binom{2n-4}{n-2}} < (6\alpha^2 + \alpha^2)\frac{1+\varepsilon'}{4} < 0.52,$$

which contradicts (12). □

By Claim 14 and Theorem 10 we have

$$|\mathscr{F}_{1\bar{0}}| \le 2|\mathscr{G}_{1\bar{0}}| \le 2\binom{2n-9}{n-6}. \tag{14}$$

By considering the complement we also have

$$|\mathscr{F}_{0\bar{1}}| \le 2\binom{2n-9}{n-6}. \tag{15}$$

Let $\sum_*$ denote the summation over all $i,j \ge 0$ except $(i,j) = (0,0),(1,0),(0,1)$. Then we have

$$|\mathscr{F}| = \sum_{i,j\ge 0} |\mathscr{F}_{i\bar{j}}| = |\mathscr{F}_{0\bar{0}}| + |\mathscr{F}_{1\bar{0}}| + |\mathscr{F}_{0\bar{1}}| + \sum_* |\mathscr{F}_{i\bar{j}}|,$$

and

$$\sum_* |\mathscr{F}_{i\bar{j}}| \le \sum_* |\mathscr{A}_{i\bar{j}}| \le \sum_{i,j\ge 0} |\mathscr{A}_{i\bar{j}}| - \{|\mathscr{A}_{0\bar{0}}| + |\mathscr{A}_{1\bar{0}}| + |\mathscr{A}_{0\bar{1}}|\}.$$

Since $|\mathscr{A}_{0\bar{0}}| = \binom{2n-4}{n-2}$ and $|\mathscr{A}_{1\bar{0}}| = |\mathscr{A}_{0\bar{1}}| = 2\binom{2n-8}{n-5}$, Corollary 7 implies that

$$\sum_* |\mathscr{A}_{i\bar{j}}| < (1+\varepsilon)\alpha^4\binom{2n}{n} - \left\{\binom{2n-4}{n-2} + 4\binom{2n-8}{n-5}\right\}. \tag{16}$$

Finally using (13), (14), (15) and (16), we have

$$\begin{aligned} |\mathscr{F}| &\le |\mathscr{F}_{0\bar{0}}| + |\mathscr{F}_{1\bar{0}}| + |\mathscr{F}_{0\bar{1}}| + \sum_* |\mathscr{A}_{i\bar{j}}| \\ &< \binom{2n-5}{n-3} + 4\binom{2n-9}{n-6} + (1+\varepsilon)\alpha^4\binom{2n}{n} \\ &\qquad - \left\{\binom{2n-4}{n-2} + 4\binom{2n-8}{n-5}\right\} \\ &< 0.78\binom{2n-4}{n-2}, \end{aligned}$$

for n sufficiently large, which contradicts (12). This completes the proof of Theorem 3.

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