The maximum size of intersecting and union families of sets

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Abstract

We consider the maximal size of families of k-element subsets of an n element set $[n]$ that satisfy the properties that every r subsets of the family have non-empty intersection, and no ℓ subsets contain $[n]$ in their union. We show that for large enough n, the largest such family is the trivial one of all $\binom{n-2}{k-1}$ subsets that contain a given element and do not contain another given element. Moreover we show that unless such a family is such that all subsets contain a given element, or all subsets miss a given element, then it has size at most $.9\binom{n-2}{k-1}$.

We also obtain versions of these statements for weighted non-uniform families.

Keywords: Erdős–Ko–Rado Theorem, Brace-Daykin, intersecting union family, random walk, p-weight

1. Introduction

Since Erdős, Ko, and Rado showed in [6] that a pairwise intersecting family of k element subsets of an n element set has size at most $\binom{n-1}{k-1}$ (for $n \geq 2k$), there have been a string of variations and generalisations finding the maximum size of families satisfying various intersection conditions. We continue in this line. In these results, the maximum families, which we call extremal configurations, often have very simple constructions. It is often the case that such extremal configurations are unique, and moreover, are stable in the following sense: the size of a family satisfying the given intersection conditions is much smaller than optimal unless the family is a subfamily of the extremal configuration. (See [4, 13, 16, 19, 20, 21] for some related stability type results.) In this article, we will show such phenomenon concerning families of subsets with intersecting and union properties.

Let $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. Briefly, we say such a family $\mathcal{F}$ is $\mathrm{I}(r^t)$. A family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-union, or $\mathcal{F}$ is $\mathrm{U}(r^t)$, if $|F_1 \cup \cdots \cup F_r| \leq n - t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. Notice that $\mathcal{F}$ is $\mathrm{U}(r^t)$ if and only if the complement family

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$\mathcal{F}^c := \{[n] \setminus F : F \in \mathcal{F}\}$ is I(r^t). We say $\mathcal{F}$ is IU(r^t, ℓ^s) if $\mathcal{F}$ is I(r^t) and U(ℓ^s). It is known that

$$\text{if } \mathcal{F} \subset 2^{[n]} \text{ is IU}(2^1, 2^1), \text{ then } |\mathcal{F}| \leq 2^{n-2} \tag{1}$$

by Marica and Schonheim [18], Daykin and Lovász [3], Seymour [23], Anderson [1], Kleitman [17], etc. The following example gives several different extremal configurations.

Example 1. *Let $n = n_1 + n_2$. Choose an* I(2^1) *family $\mathcal{F}_1 \subset 2^{[n_1]}$ and a* U(2^1) *family $\mathcal{F}_2 \subset 2^{[n_1+1,\, n_1+n_2]}$. Then, $\mathcal{F}_1 \times \mathcal{F}_2 = \{F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\} \subset 2^{[n]}$ is an* IU($2^1, 2^1$) *family with size $|\mathcal{F}_1||\mathcal{F}_2|$. We notice that there are many choices of $\mathcal{F}_i$ with $|\mathcal{F}_i| = 2^{n_i-1}$.*

An IU(r^1, ℓ^1) family is *a fortiori* IU($2^1, 2^1$), so bound (1) holds also for IU(r^1, ℓ^1) families. Further, some of the families suggested in Example 1 that reach this bound are IU(r^1, ℓ^1) families. The main results of this paper are variations on the bound (1) for IU(r^1, ℓ^1) families. Our first main result extends it to a weighted version, and our second main result is a variation for k-uniform families.

1.1. The p-weight version

To extend (1), we consider a weighting of families as follows. Throughout this paper let $p, q \in (0, 1)$ with $p + q = 1$. Define the p-weight of $\mathcal{F} \subset 2^{[n]}$ by

$$w_p(\mathcal{F} : [n]) = w_p(\mathcal{F}) = \sum_{F \in \mathcal{F}} p^{|F|} q^{n-|F|}.$$

For $p = 1/2$ we have $w_{1/2}(\mathcal{F}) = |\mathcal{F}|/2^n$. So, (1) can be restated as follows: if $\mathcal{F} \subset 2^{[n]}$ is IU($2^1, 2^1$), then $w_{1/2}(\mathcal{F}) \leq 1/4$. Observe that the family $\mathcal{F} = \{F \subset [n] : 1 \in F, n \notin F\}$ is IU(r^1, ℓ^1) for any $r, \ell \geq 2$ with $w_p(\mathcal{F}) = pq$. Our first result is the following.

Theorem 1. *Let $r, \ell \geq 3$, and let $\mathcal{F} \subset 2^{[n]}$ be* IU(r^1, ℓ^1)*. If $1/\ell \leq p \leq (r-1)/r$, then $w_p(\mathcal{F}) \leq pq$.*

Intuitively, the range of p in the theorem corresponds to those values for which a family of subsets of size pn do not satisfy one or other of the intersection conditions by simple set size considerations. The following shows that this range of p cannot be extended.

Example 2. *If $p < 1/\ell$, then $\mathcal{F}_n = \{F \cup \{n\} : F \subset [n-1], |F| < (n-1)/\ell\}$ is* IU(r^1, ℓ^1) *with $w_p(\mathcal{F}_n) \to p$ as $n \to \infty$. If $p > (r-1)/r$, then $\mathcal{F}_n = \{F \subset [n-1] : |F| > \frac{r-1}{r}(n-1)\}$ is* IU(r^1, ℓ^1) *with $w_p(\mathcal{F}_n) \to q$ as $n \to \infty$.*

We believe that the extremal configurations for Theorem 1 are unique, that is, $w_p(\mathcal{F}) = pq$ iff $\mathcal{F} \cong \{F \subset [n] : 1 \in F, n \notin F\}$. Where $\bigcap \mathcal{F} = \bigcap_{F \in \mathcal{F}} F$ and $\bigcup \mathcal{F} = \bigcup_{F \in \mathcal{F}} F$, the salient properties of this family $\mathcal{F}$ are that (i) $|\bigcup \mathcal{F}| \leq n - 1$, so $\mathcal{F}$ is trivially U(ℓ^1), and (ii) $|\bigcap \mathcal{F}| \geq 1$, so that $\mathcal{F}$ is trivially I(r^1). We cannot prove the statement that "$w_p(\mathcal{F}) < pq$ unless $\mathcal{F}$ satisfies (i) and (ii)" however, we can prove the following stability result.

Theorem 2. *Let $r, \ell \geq 3$. If $\mathcal{F} \subset 2^{[n]}$ is* IU(r^1, ℓ^1) *and $1/\ell < p < (r-1)/r$, then $w_p(\mathcal{F}) \leq 0.9pq$ unless (i) $|\bigcup \mathcal{F}| \leq n - 1$, or (ii) $|\bigcap \mathcal{F}| \geq 1$.*

The following example shows that this stability result cannot be extended, as is, to families satisfying just one of (i) or (ii).

Example 3. *Let $\mathcal{F}_n = \{[2, n-1]\} \cup \{F \subset [n-1] : 1 \in F, |F| > \frac{r-2}{r-1}(n-1)\}$. Then $\mathcal{F}_n$ is an $\mathrm{IU}(r^1, \ell^1)$ family with $|\bigcup \mathcal{F}| \leq n-1$ and $|\bigcap \mathcal{F}_n| = 0$. A computation shows that $w_p(\mathcal{F}_n) \to pq$ if $(r-2)/(r-1) < p \leq (r-1)/r$. By considering the complement family, one can construct an $\mathrm{IU}(r^1, \ell^1)$ family $\mathcal{G}_n$ with $|\bigcup \mathcal{G}_n| = n$ and $|\bigcap \mathcal{G}_n| \geq 1$, and with $w_p(\mathcal{G}_n) \to pq$ for $1/\ell \leq p < 1/(\ell-1)$.*

These examples occur at the outer ends of the range of p from Theorem 2. We expect that this is necessary and ask the following.

Problem 1. *Let $r, \ell \geq 3$, and let $\mathcal{F} \subset 2^{[n]}$ be $\mathrm{IU}(r^1, \ell^1)$ with $|\bigcap \mathcal{F}| = 0$ or $|\bigcup \mathcal{F}| = n$. Then, does there exist $\epsilon > 0$ such that $w_p(\mathcal{F}) \leq (1-\epsilon)pq$ holds for all $1/(\ell-1) \leq p \leq (r-2)/(r-1)$?*

The answer is affirmative if p is close to $1/2$ and $r, \ell \geq 4$, see [26, 28]. Further, it is affirmative in the case $r = \ell = 3$ by a result of Brace and Daykin [2] which implies that the maximum $(1/2)$-weight of an $\mathrm{IU}(r^1, \ell^1)$ family with $|\bigcap \mathcal{F}| = 0$ or $|\bigcup \mathcal{F}| = n$ is $\max\{(r+2)/2^{r+2}, (\ell+2)/2^{\ell+2}\}$ for $r, \ell \geq 3$.

To give an easy proof for Theorem 1, we use a weighted version of an inequality of Kleitman. Recall that a family $\mathcal{F} \subset 2^{[n]}$ is called a complex if $F \in \mathcal{F}$ and $G \subset F$ imply $G \in \mathcal{F}$, and that a family $\mathcal{G} \subset 2^{[n]}$ is called a co-complex if the complement family $\mathcal{G}^c := \{[n] \setminus G : G \in \mathcal{G}\}$ is a complex. Kleitman proved that if $\mathcal{F}$ is a complex and $\mathcal{G}$ is a co-complex, then $|\mathcal{F} \cap \mathcal{G}| \leq |\mathcal{F}||\mathcal{G}|/2^n$. This result can be extended as follows.

Theorem 3. *Let $\mathcal{F} \subset 2^{[n]}$ be a complex and $\mathcal{G} \subset 2^{[n]}$ be a co-complex. Then,*

$$w_p(\mathcal{F} \cap \mathcal{G}) \leq w_p(\mathcal{F}) w_p(\mathcal{G}).$$

In Section 2 we prove Theorem 3 and then Theorem 1, and then observe several other consequences of Theorem 3. We also compare the asymptotic behavior of the maximum p-weight of $\mathrm{IU}(r^1, \ell^1)$ families in the cases that $p = 1/2$ and otherwise. Theorem 2 depends on our main result about k-uniform families, so is proved at the end of Section 4.

1.2. The k-uniform variation

Our second main result deals with k-uniform families– we consider the maximum size of k-uniform $\mathrm{IU}(r^1, \ell^1)$ families $\mathcal{F} \subset \binom{[n]}{k}$. The 2-wise case, $r = \ell = 2$, follows from the Erdős-Ko-Rado theorem [6] (cf. Lemma 9 for a generalized version). That is, if $\mathcal{F} \subset \binom{[n]}{k}$ is $\mathrm{IU}(2^1, 2^1)$, then

$$|\mathcal{F}| \leq \begin{cases} \binom{n-1}{k} & \text{if } n \leq 2k, \\ \binom{n-1}{k-1} & \text{if } n \geq 2k. \end{cases} \tag{2}$$

The extremal configurations are unique (up to isomorphism) unless $n = 2k$. They are $\binom{[n-1]}{k}$ for $n < 2k$, and its complement family for $n > 2k$. Theorem 5 in Section 2 can be seen as a p-weight version of (2).

Engel and Gronau showed the following in [5]. Let $r \geq 4$, $\ell \geq 4$ and $\mathcal{F} \subset \binom{[n]}{k}$. If $\mathcal{F}$ is $\mathrm{IU}(r^1, \ell^1)$ and

$$\tfrac{n-1}{\ell} + 1 \leq k \leq \tfrac{r-1}{r}(n-1), \tag{3}$$

then $|\mathcal{F}| \le \binom{n-2}{k-1}$ with equality holding iff $\mathcal{F} \cong \mathcal{F}_0(n,k,1)$, where $\mathcal{F}_0(n,k,t) = \{F \in \binom{[n-t]}{k} : [t] \subset F\}$. By constructions similar to those in Example 2, one can show that the condition (3) is necessary. The 3-wise case is more difficult, and the following is proved in [12]. If $\mathcal{F} \subset \binom{[2n]}{n}$ is $\mathrm{IU}(3^1, 3^1)$, then $|\mathcal{F}| \le \binom{2n-2}{n-1}$ with equality holding iff $\mathcal{F} \cong \mathcal{F}_0(2n,n,1)$.

Our main result is an extension of these results for large n.

Theorem 4. *Let $r, \ell \ge 3$. There exists some n_0 such that for all $n > n_0$ the following holds. If $\mathcal{F} \subset \binom{[n]}{k}$ is $\mathrm{IU}(r^1, \ell^1)$ with k satisfying (3) then $|\mathcal{F}| \le \binom{n-2}{k-1}$. Equality holds iff $\mathcal{F} \cong \mathcal{F}_0(n,k,1)$. Moreover, we have $|\mathcal{F}| < 0.9\binom{n-2}{k-1}$ unless (i) $|\bigcup \mathcal{F}| \le n-1$, or (ii) $|\bigcap \mathcal{F}| \ge 1$.*

Again, the following example shows that the stability part of the theorem cannot be extended to families satisfying (i) or (ii). See Example 2 in [28] for more general constructions. For integers $a < b$, the notation $[a,b]$ denotes the set $\{a, a+1, \ldots, b\}$.

Example 4. *Construct an $\mathrm{IU}(r^1, \ell^1)$ family $\mathcal{F}_n \subset \binom{[n]}{k}$ satisfying $|\bigcup \mathcal{F}_n| \le n-1$ and $|\bigcap \mathcal{F}_n| = 0$ by*

$$\mathcal{F}_n = \{[2,k+1]\} \cup \{F \subset [n-1] : 1 \in F,\ |F \cap [2,k+1]| > \tfrac{r-2}{r-1}k\}.$$

Standard bounds on deviations of the hypergeometric distribution (see [15]), give for fixed $p = k/n$ that $|\mathcal{F}_n|/\binom{n-2}{k-1} \to 1$ as $n \to \infty$ if $(r-2)/(r-1) < p \le (r-1)/r$. Similarly one can also construct an $\mathrm{IU}(r^1, \ell^1)$ family $\mathcal{G}_n$ with $|\bigcup \mathcal{G}_n| = n$ and $|\bigcap \mathcal{G}_n| \ge 1$, and with $|\mathcal{G}_n|/\binom{n-2}{k-1} \to 1$ for $1/\ell \le p < 1/(\ell-1)$.

For the 4-wise case, the following is also known [26]. Let $1 \le t \le 4$. If $\mathcal{F} \subset \binom{[2n]}{n}$ is $\mathrm{IU}(4^t, 4^t)$, then $|\mathcal{F}| \le \binom{2n-2t}{n-t}$ for $n > n_0$ with equality holding iff $\mathcal{F} \cong \mathcal{F}_0(2n,n,t)$.

In Section 3 we prepare some tools to prove Theorem 4. We prove Theorem 4 in Section 4, and then use it to prove Theorem 2.

2. Extending the Kleitman inequality

Proof of Theorem 3. The proof is by induction on n. Define $\mathcal{F}(n), \mathcal{F}(\bar{n}) \subset 2^{[n-1]}$ by

$$\mathcal{F}(n) = \{F \setminus \{n\} : n \in F \in \mathcal{F}\},\ \mathcal{F}(\bar{n}) = \{F : n \notin F \in \mathcal{F}\}.$$

Define $\mathcal{G}(n)$ and $\mathcal{G}(\bar{n})$ similarly. Let $d_0 = w_p(\mathcal{F}(n) : [n-1])$, $d_1 = w_p(\mathcal{F}(\bar{n}) : [n-1])$, $u_0 = w_p(\mathcal{G}(n) : [n-1])$, $u_1 = w_p(\mathcal{G}(\bar{n}) : [n-1])$. Then,

$$w_p(\mathcal{F} \cap \mathcal{G} : [n]) = p w_p(\mathcal{F}(n) \cap \mathcal{G}(n) : [n-1]) + q w_p(\mathcal{F}(\bar{n}) \cap \mathcal{G}(\bar{n}) : [n-1]).$$

By the induction hypothesis, the RHS is $\le p d_0 u_0 + q d_1 u_1$.

Since $\mathcal{F}(n) \subset \mathcal{F}(\bar{n})$, we have $d_0 \le d_1$. Similarly, we have $u_0 \ge u_1$, and so $(d_0 - d_1)(u_0 - u_1) \le 0$. Thus,

$$\begin{aligned} w_p(\mathcal{F}) w_p(\mathcal{G}) &= (pd_0 + qd_1)(pu_0 + qu_1) \\ &\ge (pd_0 + qd_1)(pu_0 + qu_1) + pq(d_0 - d_1)(u_0 - u_1) \\ &= pd_0u_0 + qd_1u_1 \ge w_p(\mathcal{F} \cap \mathcal{G}), \end{aligned}$$

as desired. □

It is useful to introduce notation for the maximum size of a $\mathrm{IU}(r^t,\ell^s)$ family. Let

$$g(n,r^t,\ell^s)=\max\{|\mathcal{F}|:\mathcal{F}\subset 2^{[n]}\text{ is IU}(r^t,\ell^s)\}.$$

Thus, for example, (1) and Example 1 give $g(n,2^1,2^1)=2^{n-2}$. Frankl proved $g(n,2^1,2^t)=|\mathcal{K}(n-1,t)|$ in [7], where

$$\mathcal{K}(n,t)=\begin{cases}\{K\subset[n]:|K|\geq(n+t)/2\} & \text{if } n+t \text{ is even,}\\ \{K\subset[n]:|K\cap[2,n]|\geq((n-1)+t)/2\} & \text{if } n+t \text{ is odd.}\end{cases}$$

For the case $n=4s$, he conjectured $g(n,2^2,2^2)=|\mathcal{F}_2|$, where

$$\mathcal{F}_2=\{F\subset[4s]:|F\cap[2s]|\geq s+1,\text{ and }|F\cap[2s+1,4s]|\leq s-1\}.$$

The family $\mathcal{F}_2$ is 2-intersecting on $[1,2s]$ and 2-union on $[2s+1,4s]$. Notice that $|\mathcal{F}_2|/2^n\to 1/4$. For the general case, Frankl conjectures that $g(n,2^t,2^s)=\max_i|\mathcal{K}(i,t)||\mathcal{K}(n-i,s)|$. If this is true, then $g(n,2^t,2^s)/2^n\to 1/4$ as $n\to\infty$ for fixed t,s.

For the p-weight version, we analogously let

$$f_p(n,r^t,\ell^s)=\max\{w_p(\mathcal{F}):\mathcal{F}\subset 2^{[n]}\text{ is IU}(r^t,\ell^s)\}.$$

Notice that $f_p(n,r^t,\ell^s)\in[0,1]$ and $w_p(\mathcal{F})=w_q(\mathcal{F}^c)$. For $p=1/2$ we have $f_{1/2}(n,r^t,\ell^s)=g(n,r^t,\ell^s)/2^n$. Thus, for example, $g(n,2^1,2^1)=2^{n-2}$ gives $f_{1/2}(n,2^1,2^1)=1/4$.

Proof of Theorem 1. Suppose that $\mathcal{F}\subset 2^{[n]}$ is a maximal $\mathrm{IU}(r^t,\ell^s)$ family. Then $\mathcal{F}_*=\{G\subset[n]:G\subset F,F\in\mathcal{F}\}$ is an $\mathrm{U}(\ell^s)$ complex, and $\mathcal{F}^*=\{G\subset[n]:G\supset F,F\in\mathcal{F}\}$ is an $\mathrm{I}(r^t)$ co-complex. Since $\mathcal{F}$ is maximal, $\mathcal{F}=\mathcal{F}_*\cap\mathcal{F}^*$, so it follows from Theorem 3 that

$$f_p(n,r^t,\ell^s)\leq f_p(n,r^t,*)f_p(n,*,\ell^s)=f_p(n,r^t,*)f_q(n,\ell^s,*),\tag{4}$$

where '$*$' indicates the empty restriction. Combining (4) with $f_{1/2}(n,2^1,*)=1/2$, we get $f_{1/2}(n,2^1,2^1)\leq 1/4$. In fact, this was the idea for proving (1) in [17]. Similarly, Theorem 1 immediately follows from (4) and a p-weight version of the Erdős–Ko–Rado theorem from [10] which states that $f_p(n,r^1,*)=p$ for $r\geq 3$ and $p\leq(r-1)/r$. □

One can also consider the behaviour of $f_p(n,r^t,\ell^s)$ as n goes to infinity. If $\mathcal{F}\subset 2^{[n]}$ is $\mathrm{IU}(r^t,\ell^s)$, then so is $\mathcal{F}\cup\{F\cup\{n+1\}:F\in\mathcal{F}\}$. This gives $f_p(n,r^t,\ell^s)\leq f_p(n+1,r^t,\ell^s)$. So, we can define $f_p(r^t,\ell^s)=\lim_{n\to\infty}f_p(n,r^t,\ell^s)$. Frankl's conjecture being true would then imply that $f_{1/2}(2^t,2^s)=1/4$.

Theorem 5. *$f_p(2^1,2^1)=\min\{p,q\}$ for $p\neq 1/2$.*

Proof. First, let $p<1/2$. Choose $0<\epsilon<p$ so that $p+\epsilon<1/2$. Let $I=((p-\epsilon)n,(p+\epsilon)n)\cap\mathbb{N}$. As the binomial distribution $B(n,p)$ is concentrated around pn, we have

$$\lim_{n\to\infty}\sum_{k\in I}\binom{n-1}{k-1}p^kq^{n-k}=p,\ \text{ and }\ \lim_{n\to\infty}\sum_{k\notin I}\binom{n}{k}p^kq^{n-k}=0.$$

Let $\mathcal{F}\subset 2^{[n]}$ be $\mathrm{IU}(2^1,2^1)$, and let $\mathcal{F}_k$ be the subfamily of all k element sets. Then by (2) we have $|\mathcal{F}_k|\leq\binom{n-1}{k-1}$ for $k\in I$. As $n\to\infty$, we have

$$w_p(\mathcal{F})\leq\sum_{k\in I}|\mathcal{F}_k|p^kq^{n-k}+\sum_{k\notin I}\binom{n}{k}p^kq^{n-k}\leq\sum_{k\in I}\binom{n-1}{k-1}p^kq^{n-k}+o(1)=p+o(1).$$

On the other hand, $\mathcal{F} = \{F \subset [n] : 1 \in F, |F| < n/2\}$ is $\mathrm{IU}(2^1, 2^1)$ with $w_p(\mathcal{F}) > p - o(1)$. Thus we have $f_p(2^1, 2^1) = p$ for $p < 1/2$. The proof for the case $p > 1/2$ is similar. $\square$

We omit the details, but a similar proof would show that

$$f_p(r^1, \ell^1) = \begin{cases} p & \text{if } p < 1/\ell, \\ q & \text{if } p > (r-1)/r. \end{cases}$$

for $r, \ell \geq 2$. Thus $f_p(r^1, \ell^1)$ is not a continuous function of p at $p \in \{1/\ell, (r-1)/r\}$.

Problem 2. *Find conditions on n, r, ℓ, t, s, p so that*

$$f_p(n, r^t, \ell^s) = p^t q^s. \tag{5}$$

For $t = s = 1$, (5) is true if $1/\ell \leq p \leq (r-1)/r$ by Theorem 1 and false otherwise, by Example 2. Using Theorem 3 and results concerning the maximum p-weight of $I(r^t)$ families, (5) can be shown to hold for several other special cases. Some such cases are: [9] $r = \ell = 3$, $t = s = 2$, and $|p - 1/2| < 0.0018$; [25] $r = \ell = 3$, $t, s \geq 26$, and $1 - \frac{2}{\sqrt{4s+9}-1} \leq p \leq \frac{2}{\sqrt{4t+9}-1}$; [26] $r = \ell = 4$, $s, t \in [1, 7]$ and $|p - 1/2| < \epsilon$ for some $\epsilon > 0$; [26, 27] $r, \ell \geq 5$, $1 \leq t \leq 2^r - r - 1$, $1 \leq s \leq 2^\ell - \ell - 1$, and $|p - 1/2| < \epsilon$ for some $\epsilon = \epsilon(r, \ell, t, s)$; [30] for all $p \in (0, 1)$, $r > r(p)$, $\ell > \ell(p)$, $1 \leq t \leq (p^{1-r} - p)/q - r$, and $1 \leq s \leq (q^{1-\ell} - q)/p - \ell$. It would also be interesting to get corresponding results for k-uniform families.

3. Tools

In this section we present some tools that will be used in the proof of Theorem 4.

The (i, j)-shift $S_{i,j}(\mathcal{F})$ of a family $\mathcal{F} \subset 2^{[n]}$ is the family $\{S_{i,j}(F) : F \in \mathcal{F}\}$ where

$$S_{i,j}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } F \cap \{i, j\} = \{j\} \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

One can easily verify that $|S_{i,j}(\mathcal{F})| = |\mathcal{F}|$ and that if $\mathcal{F}$ is $\mathrm{IU}(r^t, \ell^s)$ then $S_{i,j}(\mathcal{F})$ is. A family $\mathcal{F}$ is called *shifted* if $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$. For any given family, one can always obtain a shifted family by repeatedly shifting the family (finitely many times). So to estimate the size of a maximum $\mathrm{IU}(r^t, \ell^s)$ family, we can always assume that it is shifted. See [8] for more details.

Now we introduce Frankl's random walk method. Associate $\binom{[n]}{k}$ with the set of n-step walks from $(0, 0)$ to $(n - k, k)$ on $\mathbb{Z}^2$ as follows. Let $F \in \binom{[n]}{k}$ correspond to the n-step walk in which the i-th step is one unit up if $i \in F$ and one unit right otherwise.

Roughly speaking, shifting a set F moves the corresponding walk up and to the left, as is demonstrated in Figure 1. The bold line is the walk corresponding to the set $F = \{2, 3, 6, 8, 9, 11, 12\} \in \binom{[12]}{7}$ and the dotted line is the walk corresponding to a subset in $S_{4,11}(\mathcal{F})$.

The main idea behind the random walk method is that a walk in a shifted $\mathrm{I}(r^t)$ family hits some line determined by r and t. For example, the line for $\mathrm{I}(3^1)$ is $y = 2x + 1$. To see this, consider $F = \{2, 3, 5, 6\}$ whose walk does not hit the line. If F is in a shifted

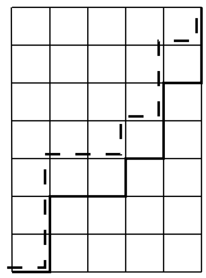

Figure 1: Shifting moves a walk up and to the left

$\mathrm{I}(3^1)$ family $\mathcal{F}$, then $\{1,3,4,6\}$ and $\{1,2,4,5\}$ would also be in $\mathcal{F}$, but these 3 subsets do not satisfy the intersection property $\mathrm{I}(3^1)$. Thus F cannot be in $\mathcal{F}$. More generally, we have the following, see [8, 29], which enables us to bound the size of an $\mathrm{I}(r^t)$ family by counting the number of walks touching the corresponding line.

Lemma 6. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted $\mathrm{I}(r^t)$ family. Then for every $F \in \mathcal{F}$ the walk associated to F touches the line $y = (r-1)x + t$.*

The next lemma estimates the number of walks touching a line $y = 2x + t$. To state the result, let $p \in (0,1)$, $q = 1 - p$, and let $\alpha_{r,p} \in (p,1)$ be the root of the equation $X = p + qX^r$. Observing that this is equivalently the root in the interval $(0,1)$ of the equation $1/(1-p) = 1 + X + X^2 + \cdots + X^{r-1}$, one can show that $\alpha_{r,p}$ is continuous as a function of p on this interval. (In fact one can expand $\alpha_{r,p}$ as a power series of p, that is, $\alpha_{r,p} = \sum_{k\ge 0} \frac{1}{rk+1}\binom{rk+1}{k} p^{(r-1)k+1} q^k$, but we will not use this.)

Lemma 7 ([11])**.** *Let $p \in \mathbb{Q}$ and $t \in \mathbb{N}$ be fixed constants. Then, for every $\epsilon > 0$ there exists n_0 such that for all $n, k \in \mathbb{N}$ with $n \ge n_0$, $p = k/n$, and $p < 2/3$, the following holds: the number of walks from $(0,0)$ to $(n-k, k)$ which touch the line $L : y = 2x + t$ is at most $(\alpha_{3,p}^t + \epsilon)\binom{n}{k}$. Moreover, if $p < 0.51$, then one can take $\epsilon = 0$.*

Lemma 6 defines a property of the walks corresponding to the sets in a shifted $\mathrm{I}(r^t)$ family. As the complement family of a $\mathrm{U}(\ell^s)$ family is $\mathrm{I}(\ell^s)$ one gets a dual statement about the walks corresponding to sets in such a family. Making the coordinate transformation $x' = k - y$ and $y' = (n-k) - x$, we get that every walk in a shifted $\mathrm{U}(\ell^s)$ family touches the line $y' = (\ell - 1)x' + s$, which is equivalently $y = \frac{1}{\ell-1}(x - (n-k-s)) + k$. So walks in an $\mathrm{IU}(r^t, \ell^s)$ family touch both lines, see Figure 2. The number of such walks for the case $r = \ell = 3$ can be bounded by the following result.

Lemma 8. *Let $0 < p \le 1/2$ and $s, t \in \mathbb{N}$ be fixed constants. Then, for every $\epsilon > 0$, there exists n_0 such that for all $n, k \in \mathbb{N}$ with $n \ge n_0$, $p = k/n$. the following holds: the number of walks from $(0,0)$ to $(n-k, k)$ which touch both of the lines $L_1 : y = 2x + t$ and $L_2 : y = \frac{1}{2}(x - (n-k-s)) + k$ is at most $(\alpha_{3,p}^t \alpha_{3,q}^s + \epsilon)\binom{n}{k}$.*

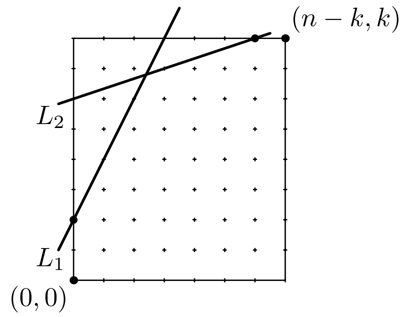

Figure 2: The two lines for IU$(3^2, 4^1)$

Proof. Let $\epsilon > 0$ be given. First assume that n is even. The idea of the proof is as follows. All walks from $(0,0)$ to $(n-k,k)$ hit the line $L_0 : x+y = n/2$ after $n/2$ steps. Most of them hit L_0 near the center point $c = (\frac{n-k}{2}, \frac{k}{2})$. We will choose constants and define "near the center" $C_n \subset L_0$ so that

$$N_1 := \#\{\text{walks from } (0,0) \text{ to } (n-k,k) \text{ that do not hit } C_n\} < \tfrac{\epsilon}{2}\tbinom{n}{k} \tag{6}$$

and

$$N_2 := \#\{\text{walks that hit } L_1,\, L_2, \text{ and } C_n\} < (\alpha^t_{3,p}\alpha^s_{3,q} + \tfrac{\epsilon}{2})\tbinom{n}{k}. \tag{7}$$

Thus the number of walks that hit L_1 and L_2 is at most $N_1 + N_2 < (\alpha^t_{3,p}\alpha^s_{3,q} + \epsilon)\binom{n}{k}$, as needed.

Let $\mu_j\binom{n}{k}$ be the number of walks from $(0,0)$ to $(n-k,k)$ which cross L_0 at the point $(n/2-j, j)$, namely, let

$$\mu_j = \frac{\binom{n/2}{j}\binom{n/2}{k-j}}{\binom{n}{k}}.$$

For $c > 0$, let $J_n = \{j \in \mathbb{N} : |j - k/2| \le c\sqrt{n}\}$. Then a variant of the central limit theorem gives

$$\lim_{n\to\infty} \sum_{j\in J_n} \mu_j = \frac{1}{\sqrt{2\pi}} \int_{\frac{-2c}{\sqrt{pq}}}^{\frac{2c}{\sqrt{pq}}} \exp(-x^2/2)dx.$$

Choose $c > 0$ so that the above quantity is more than $1 - \epsilon/4$, and choose n_1 so that $\sum_{j\notin J_n} \mu_j < \epsilon/2$ holds for all $n > n_1$. Let $C_n = \{(n/2-j, j) : j \in J_n\} \subset L_0$. Then we have (6) for $n > n_1$.

Now we look at the walks that do hit C_n. Since $\alpha_{3,z}$ is a continuous function of z around p, we can choose $\delta > 0$ so that

$$\alpha^t_{3,z} < (1+\epsilon/5)\alpha^t_{3,p} \text{ and } \alpha^s_{3,1-z} < (1+\epsilon/5)\alpha^s_{3,q} \tag{8}$$

hold for all z with $|z-p| < \delta$. Choose n_2 so that $2c/\sqrt{n_2} < \delta$. For $j \in J_n$ and $n > n_2$, we have $|j-k/2| \le c\sqrt{n}$. Let $z = 2j/n$. Then we have $|z-p| = |\frac{2j}{n} - \frac{k}{n}| = \frac{2}{n}|j - \frac{k}{2}| < \frac{2c}{\sqrt{n}} < \delta$. That is, if $j \in J_n$ and $z = 2j/n$, then $|z-p| < \delta$.

Now we choose n_3 from Lemma 7 so that for all n, j, z with $n > n_3$, $z = j/(n/2)$ and $|z - p| < \delta$, the following holds: the number of walks from $(0,0)$ to $(n/2 - j, j)$ that hit L_1 is $\le \alpha^t_{3,z}$, and the number of walks from $(n/2 - j, j)$ to $(n-k, k)$ that hit L_2 is $\le \alpha^s_{3,1-z}$. Finally choose n sufficiently large, i.e., $n > n_0 := \max\{n_1, n_2, n_3\}$. Then, letting $z = 2j/n$, we have $N_2 < \sum_{j \in J_n} \alpha^t_{3,z} \binom{n/2}{j} \alpha^s_{3,1-z} \binom{n/2}{k-j}$. By (8) we get

$$N_2 < \sum_{j \in J_n} (1 + \epsilon/5)^2 \alpha^t_{3,p} \binom{n/2}{j} \alpha^s_{3,q} \binom{n/2}{k-j} < (1 + \epsilon/2) \alpha^t_{3,p} \alpha^s_{3,q} \binom{n}{k},$$

where we used $\sum_{j=0}^{n} \binom{n/2}{j}\binom{n/2}{k-j} = \binom{n}{k}$. This gives (7), and this completes the proof for the case that n is even.

In the case that n is odd, we use the line $x + y = (n+1)/2$ instead of $x + y = n/2$ to define C_n. The rest of the proof is almost identical. □

With a more careful choice of constants, one can prove Lemma 8 not only for $p \le 1/2$ but also for $p < 2/3$. Moreover, one can extend Lemma 8 to $r \ge 4$ and $p \le (r-1)/(r+1)$. The number of walks in this case is at most $(\alpha^t_{r,p} \alpha^s_{r,q} + \epsilon)\binom{n}{k}$, because Lemma 7 holds with $\epsilon = 0$ in this situation as well. However, it is known that $\epsilon > 0$ is necessary in Lemma 8, see [24]. For the proof of Theorem 4 we will only need Lemma 8 in the generality given, so we refrain from proving a more general version.

The corresponding p-weight version of Lemma 8 is easier. In fact, if $\mathcal{F} \subset 2^{[n]}$ is $\mathrm{I}(r^t)$, then $w_p(\mathcal{F}) \le \alpha^t_{r,p}$, see [28, 29]. Thus, using (4), we have

$$f_p(n, r^t, \ell^s) \le \alpha^t_{r,p} \alpha^s_{\ell,q}. \tag{9}$$

4. Extending the Engel–Gronau inequality

Let $0 < p < 1$ and $q = 1 - p$ be fixed. Let $k = pn$. We will frequently use the fact that for fixed $a, b \in \mathbb{N}$, $\lim_{n\to\infty} \binom{n-a}{k-b}/\binom{n}{k} = p^b q^{a-b}$. We also use the following version of the Erdős–Ko–Rado Theorem. For the proof, see [8, 14], and also [21] for a stronger version.

Lemma 9. *Let $r \ge 2$. If $k \le \frac{r-1}{r} n$ and $\mathcal{F} \subset \binom{[n]}{k}$ is $\mathrm{I}(r^1)$, then $|\mathcal{F}| \le \binom{n-1}{k-1}$. If $r \ge 3$, then equality holds iff $\mathcal{F} \cong \mathcal{F}_0(n, k, 1)$.*

Proof of Theorem 4. Let $r, \ell \ge 3$, and n and k be integers satisfying (3). Let $\mathcal{F} \subset \binom{[n]}{k}$ be $\mathrm{IU}(r^1, \ell^1)$. If $\mathcal{F}$ is not $\mathrm{U}((\ell-1)^2)$, then we have $F_1, \ldots, F_{\ell-1} \in \mathcal{F}$ such that $|F_1 \cup \cdots \cup F_{\ell-1}| = n - 1$, say, $F_1 \cup \cdots \cup F_{\ell-1} = [n-1]$. Then, since $\mathcal{F}$ is $\mathrm{U}(\ell^1)$, we have $\mathcal{F} \subset \binom{[n-1]}{k}$, which is case (i) from the statement of the theorem. Since $\mathcal{F}$ is $\mathrm{I}(r^1)$ and $k \le \frac{r-1}{r}(n-1)$, it follows from Lemma 9 that $|\mathcal{F}| \le \binom{n-2}{k-1}$ with equality holding iff $\mathcal{F} \cong \mathcal{F}_0(n, k, 1)$. Similarly, if $\mathcal{F}$ is not $\mathrm{I}((r-1)^2)$, then we have $\mathcal{F}^c \subset \binom{[n-1]}{n-k}$, and this is case (ii). Since $\mathcal{F}$ is $\mathrm{U}(\ell^1)$ and $\frac{n-1}{\ell} + 1 \le k$, Lemma 9 gives the desired inequality.

So, we may assume that $\mathcal{F}$ is $\mathrm{IU}((r-1)^2, (\ell-1)^2)$, on top of being $\mathrm{IU}(r^1, \ell^1)$. This implies, by definition, that $\mathcal{F}$ is $\mathrm{IU}(2^2, 2^2)$ and $\mathrm{IU}(3^1, 3^1)$. From this, we will show for large enough n, that $|\mathcal{F}| < 0.9\binom{n-2}{k-1}$, or equivalently, that $|\mathcal{F}|/\binom{n}{k} < 0.9pq + o(1)$, where $p = k/n$ and $q = 1 - p$.

When $p \le 0.3$ we use a result of Ray-Chaudhuri and Wilson [22], which states that if $\mathcal{H} \subset 2^{[n]}$ is L-intersecting, that is, $H \cap H' \in L \subset \mathbb{N}$ for all distinct $H, H' \in \mathcal{H}$, then $|\mathcal{H}| \le \binom{n}{|L|}$. Now our family $\mathcal{F}$ is $\mathrm{I}(2^2)$ and therefore $L = \{2, 3, \dots, k-1\}$-intersecting. Thus we have $|\mathcal{F}| \le \binom{n}{k-2} = (p/q^3 + o(1))\binom{n-2}{k-1} < 0.9\binom{n-2}{k-1}$ for $p \le 0.3$ as needed.

Now we consider the case that $0.3 \le p \le 1/2$. (If $p > 1/2$, then we consider $\mathcal{F}^c$ instead of $\mathcal{F}$.) Assume that $\mathcal{F}$ is shifted. Let

$$\mathcal{A}_i = \{A \in \tbinom{n}{k} : |A \cap [1+3x]| \ge 1+2x \text{ first holds at } x = i\},$$
$$\mathcal{A}_{\bar{j}} = \{A \in \tbinom{n}{k} : |A \cap [n-3x, n]| \le x \text{ first holds at } x = j\}.$$

So if $A \in \mathcal{A}_i$ then the walk corresponding to A touches the line $y = 2x+1$ at $(i, 2i+1)$ for the first time, and if $A \in \mathcal{A}_{\bar{j}}$ then the walk touches the line $y = \frac{1}{2}(x - n + k + 1) + k$ at $(n-k-2j-1, k-j)$ for the last time.

Let $\mathcal{A}_{i\bar{j}} = \mathcal{A}_i \cap \mathcal{A}_{\bar{j}}$. Then, $\mathcal{F} \subset \bigcup_{ij} \mathcal{A}_{i\bar{j}}$. Set

$$\mathcal{F}_i := \mathcal{A}_i \cap \mathcal{F}, \quad \mathcal{F}_{\bar{j}} := \mathcal{A}_{\bar{j}} \cap \mathcal{F}, \quad \mathcal{F}_{i\bar{j}} := \mathcal{A}_{i\bar{j}} \cap \mathcal{F},$$

$$f_i := |\mathcal{F}_i|/\tbinom{n-2}{k-1}, \quad f_{\bar{j}} := |\mathcal{F}_{\bar{j}}|/\tbinom{n-2}{k-1}, \quad f_{i\bar{j}} := |\mathcal{F}_{i\bar{j}}|/\tbinom{n-2}{k-1},$$
$$a_i := |\mathcal{A}_i|/\tbinom{n-2}{k-1}, \quad a_{\bar{j}} := |\mathcal{A}_{\bar{j}}|/\tbinom{n-2}{k-1}, \quad a_{i\bar{j}} := |\mathcal{A}_{i\bar{j}}|/\tbinom{n-2}{k-1},$$

and

$$\mathcal{G}_{i\bar{j}} = \{F \cap [3i+2, n-3j-1] : F \in \mathcal{F}_{i\bar{j}}\}.$$

If $F \in \mathcal{F}_{i\bar{j}}$, then $|F \cap [3i+1]| = 2i+1$ and $|F \cap [n-3j, n]| = j$. This gives $\mathcal{G}_{i\bar{j}} \subset \binom{[3i+2, n-3j-1]}{k-2i-j-1}$. In particular, for $i, j \in \{0, 1\}$, we have

$$|\mathcal{G}_{i\bar{j}}| = |\mathcal{F}_{i\bar{j}}| \le |\mathcal{A}_{i\bar{j}}| = \tbinom{n-3i-3j-2}{k-2i-j-1} = (p^{2i+j+1}q^{i+2j+1} + o(1))\tbinom{n}{k},$$

and $a_{i\bar{j}} = p^{2i+j}q^{i+2j} + o(1)$. (For i or $j \notin \{0, 1\}$, things are more complicated.) The following two claims are valid for all j, but we will use the case $j \in \{0, 1\}$ only.

Claim 10. *$\mathcal{G}_{0\bar{j}}$ is* $\mathrm{I}(2^1)$.

Proof. Suppose, to the contrary, that there are $G_1, G_2 \in \mathcal{G}_{0\bar{j}} \subset \binom{[2, n-3j-1]}{k-j-1}$ such that $G_1 \cap G_2 = \emptyset$. These sets come from sets in $\mathcal{F}_{0\bar{j}} \subset \mathcal{A}_{\bar{j}}$, which means they can be shifted to sets containing $\{n - 3m - s : 0 \le m < j\}$ for $s = 1, 2$.

Thus, by the shiftedness of $\mathcal{F}$, we have $F_s \in \mathcal{F}$ for $s = 1, 2$ where $F_s = \{1\} \cup G_s \cup \{n - 3m - s : 0 \le m < j\}$. But $F_1 \cap F_2 = \{1\}$ contradicts our assumption that $\mathcal{F}$ is $I(2^2)$. ☐

Claim 11. *If $\mathcal{G}_{1\bar{j}}$ is not* $\mathrm{I}(2^1)$*, then $|\mathcal{F}| < 0.8\binom{n-2}{k-1}$.*

Proof. Choose $G_1, G_2 \in \mathcal{G}_{1\bar{j}} \subset \binom{[5, n-3j-1]}{k-j-3}$ such that $G_1 \cap G_2 = \emptyset$. Then, by the shiftedness, we have $F_s \in \mathcal{F}$ for $s = 1, 2$ where $F_s = ([4] \setminus \{s\}) \cup G_s \cup \{n - 3m - s : 0 \le m < j\}$. Since $F_1 \cap F_2 = \{3, 4\}$ and $\mathcal{F}$ is $\mathrm{I}(3^1)$, we must have $|F \cap [4]| \ge 3$ for all $F \in \mathcal{F}$. Indeed, if $F \in \mathcal{F}$ satisfies $|F \cap [4]| \le 2$, then by the shiftedness we can assume that $F \cap [4] \subset [2]$, which implies $F_1 \cap F_2 \cap F = \emptyset$, a contradiction.

The walks from $(0,0)$ to $(n-k,k)$, corresponding to sets $F \in \mathcal{F}$, hit either $(0,4)$ or $(1,3)$. Notice that the walk from $(0,0)$ to $(0,4)$ is unique but there are 4 walks from $(0,0)$ to $(1,3)$. Thus we have $|\mathcal{F}| \leq \binom{n-4}{k-4} + 4\binom{n-4}{k-3} = (p^3/q + 4p^2 + o(1))\binom{n-2}{k-1}$, which is less than $0.8\binom{n-2}{k-1}$ for $p \leq 0.4$. So we may assume that $p > 0.4$.

As $\mathcal{F}$ is $\mathrm{U}(3^1)$ the walks also hit the line $L : y = \frac{1}{2}(x - (n-k-1)) + k$. Using the change of variables $x' = k - y$ and $y' = n - k - x$, we see that $(x, y) = (0, 4)$ or $(1, 3)$ corresponds to $(x', y') = (k-4, n-k)$ or $(k-3, n-k-1)$, and L corresponds to $L' : y' = 2x' + 1$. Now we apply Lemma 7 to the complement of $\mathcal{F}$. Namely, we count the number of walks from $(x', y') = (0, 0)$ to $(n-k, k)$ passing through $(k-4, n-k)$ or $(k-3, n-k-1)$, which hit L'. Then it follows that

$$|\mathcal{F}| \leq (\alpha_{3,q} + o(1))\left(\binom{n-4}{k-4} + 4\binom{n-4}{k-3}\right) = \alpha_{3,q}(p^3/q + 4p^2 + o(1))\binom{n-2}{k-1},$$

which is less than $0.8\binom{n-2}{k-1}$ for $0.4 < p \leq 1/2$. □

So we may assume that $\mathcal{G}_{1\bar{j}}$ is $\mathrm{I}(2^1)$. Recall that $\mathcal{G}_{1\bar{j}}$ is a subfamily of $\binom{[5,\, n-3j-1]}{k-2-j-1}$ and that $\mathcal{G}_{0\bar{j}}$ is an $\mathrm{I}(2^1)$ subfamily of $\binom{[2,\, n-3j-1]}{k-j-1}$. Thus, for $i, j \in \{0, 1\}$, Lemma 9 gives

$$|\mathcal{G}_{i\bar{j}}| \leq \binom{n-3i-3j-3}{k-2i-j-2} = (p^{2i+j+2}q^{i+2j+1} + o(1))\binom{n}{k},$$

and

$$f_{i\bar{j}} \leq p\, a_{i\bar{j}} = p(p^{2i+j}q^{i+2j}) + o(1).$$

Therefore we have

$$H_1 := \sum_{i,j \leq 1} f_{i\bar{j}} = f_{00} + f_{0\bar{1}} + f_{1\bar{0}} + f_{1\bar{1}} < p(1 + pq^2 + p^2q + p^3q^3) + o(1).$$

Since $\sum_{i,j\geq 0} |\mathcal{A}_{i\bar{j}}|$ is the number of walks from $(x, y) = (0, 0)$ to $(n-k, k)$ which touch both lines $y = 2x + 1$ and $y = \frac{1}{2}(x - (n-k-1)) + k$, it follows from Lemma 8 that

$$\sum_{i,j\geq 0} |\mathcal{A}_{i\bar{j}}|/\binom{n}{k} = \sum_{i,j\geq 0} a_{i\bar{j}}\binom{n-2}{k-1}/\binom{n}{k} < \alpha_{3,p}\alpha_{3,q} + o(1).$$

Then,

$$\begin{aligned}\sum_{i\geq 2 \text{ or } j\geq 2} f_{i\bar{j}} &\leq \sum_{i\geq 2 \text{ or } j\geq 2} a_{i\bar{j}} = \sum_{i,j\geq 0} a_{i\bar{j}} - \sum_{i,j\leq 1} a_{i\bar{j}} \\ &< \alpha_{3,p}\alpha_{3,q}/(pq) - (1 + pq^2 + p^2q + p^3q^3) + o(1) =: H_2,\end{aligned}$$

and

$$|\mathcal{F}|/\binom{n-2}{k-1} = \sum_{i,j\geq 0} f_{i\bar{j}} = \sum_{i,j\leq 1} f_{i\bar{j}} + \sum_{i\geq 2 \text{ or } j\geq 2} f_{i\bar{j}} \leq H_1 + H_2.$$

The RHS is less than 0.9 for $0.3 \leq p \leq 1/2$. This completes the proof of Theorem 4. □

One can show Theorem 2 along the same lines, using (9) instead of Lemma 8. The proof would be slightly easier. Here we deduce Theorem 2 from Theorem 4.

Proof of Theorem 2. Let $\mathcal{F} \subset 2^{[n]}$ be IU(r^1, ℓ^1). If $\mathcal{F}$ is not U$((\ell-1)^2)$, then as in the proof of Theorem 4 we get (i). Similarly, if $\mathcal{F}$ is not I$((r-1)^2)$, then we have (ii). So, we may assume that $\mathcal{F}$ is IU$((r-1)^2, (\ell-1)^2)$.

Let $\epsilon > 0$ be given. Then we can choose $\delta > 0$ small enough and n_0 large enough so that the following three conditions hold for all $n > n_0$:

- $I := ((1-\delta)pn, (1+\delta)pn) \cap \mathbb{N} \subset (\frac{n}{\ell}, (1-\frac{1}{r})n)$,
- $S_1 := \sum_{k \notin I} \binom{n}{k} p^k q^{n-k} < \epsilon$, and
- $\binom{n-2}{n-1} < (pq+\epsilon)\binom{n}{k}$ for all $k \in I$.

Let $\mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k}$. Then by Theorem 4, we have $|\mathcal{F}_k| < 0.9\binom{n-2}{k-1} < 0.9(pq+\epsilon)\binom{n}{k}$ for $k \in I$, and $S_2 := \sum_{k \in I} |\mathcal{F}_k| p^k q^{n-k} < 0.9(pq+\epsilon)$. Thus, we have

$$w_p(\mathcal{F}) = \sum_{k=0}^{n} |\mathcal{F}_k| p^k q^{n-k} \le S_1 + S_2 < 0.9pq + 2\epsilon.$$

Now let X_n be the collection of families $\mathcal{F} \subset 2^{[n]}$ which are IU(r^1, ℓ^1) with $\bigcup \mathcal{F} = [n]$ and $\bigcap \mathcal{F} = \emptyset$, and let $f_n = \max\{w_p(\mathcal{F}) : \mathcal{F} \in X_n\}$. Notice that if $\mathcal{F} \in X_n$ then $\mathcal{F} \cup \{F \cup \{n+1\} : F \in \mathcal{F}\} \in X_{n+1}$, and $f_n \le f_{n+1}$. On the other hand, we have just showed that for all $\epsilon > 0$, $f_n < 0.9pq + 2\epsilon$ for large enough n. Thus $\lim_{n\to\infty} f_n \le 0.9pq$, and so $f_n \le 0.9pq$ must hold for all n. □

References

[1] I. Anderson. Intersection theorems and a lemma of Kleitman. *Discrete Math. 16* (1976), no. 3, 181–185.

[2] A. Brace, D. E. Daykin. A finite set covering theorem. *Bull. Austral. Math. Soc.* 5:197–202, 1971.

[3] D. E. Daykin, L. Lovasz. The number of values of a Boolean function. *J. London Math. Soc.* (2) 12 (1975/76), no. 2, 225–230.

[4] I. Dinur, E. Friedgut. Intersecting families are essentially contained in juntas. *Combin. Probab. Comput.* 18 (2009), 107–122.

[5] K. Engel, H.-D.O.F. Gronau. An Erdős–Ko–Rado type theorem II. *Acta Cybernet.*, 4:405–411, 1986.

[6] P. Erdős, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.

[7] P. Frankl. The proof of a conjecture of G. O. H. Katona. *J. Combinatorial Theory Ser. A* 19 (1975), no. 2, 208–213.

[8] P. Frankl. The shifting techniques in extremal set theory, "Surveys in Combinatorics 1987" (C. Whitehead, Ed. LMS Lecture Note Series 123), 81–110, Cambridge Univ. Press (1987).

[9] P. Frankl, N. Tokushige. Weighted 3-wise 2-intersecting families. *J. Comb. Theory (A)*, Vol. 100 (2002) 94–115.

[10] P. Frankl, N. Tokushige. Weighted multiply intersecting families. *Studia Sci. Math. Hungarica*, Vol. 40 (2003) 287–291.

[11] P. Frankl, N. Tokushige. Random walks and multiply intersecting families. *J. Combin. Theory (A)* 109:121-134, 2005.

[12] P. Frankl, N. Tokushige. The maximum size of 3-wise intersecting and 3-wise union families. *Graphs and Combinatorics*, 22:225–231, 2006.

[13] E. Friedgut. On the measure of intersecting families, uniqueness and stability. *Combinatorica* 28 (2008) 503–528.

[14] H.-D.O.F. Gronau. On Sperner families in which no k sets have an empty intersection III. *Combinatorica*, 2:25–36, 1982.

[15] S. Janson, T. Łuczak, A. Rucinski. Random Graphs. *Wiley-Interscience Series in Discrete Mathematics and Optimization.* Wiley-Interscience, New York, 2000.
[16] P. Keevash, D. Mubayi. Set systems without a simplex or a cluster. *Combinatorica* 30 (2010) 573–606.
[17] D. J. Kleitman. Families of non-disjoint subsets. *J. Combinatorial Theory* 1 (1966) 153–155.
[18] J. Marica, J. Schonheim. Differences of sets and a problem of Graham. *Canad. Math. Bull.* 12 1969 635–637.
[19] D. Mubayi. An intersection theorem for four sets. *Adv. Math.* 215 (2007) 601–615.
[20] D. Mubayi, R. Ramadurai. Simplex stability. *Combin. Probab. Comput.* 18 (2009) 441–454.
[21] D. Mubayi, J. Verstraëte. Proof of a conjecture of Erdős on triangles in set-systems. *Combinatorica* 25 2005 599–614.
[22] D. K. Ray-Chaudhuri, R. M. Wilson. On t-designs. *Osaka J. Math.* 12 (1975), no. 3, 737–744.
[23] P. D. Seymour. On incomparable collections of sets. *Mathematika* 20 (1973), 208–209.
[24] N. Tokushige. The maximum size of 4-wise 2-intersecting and 4-wise 2-union families. *European J. of Comb.* 27:814–825, 2006.
[25] N. Tokushige. The maximum size of 3-wise t-intersecting families. *European J. of Comb*, Vol 28 (2007) 152–166.
[26] N. Tokushige. EKR type inequalities for 4-wise intersecting families. *J. Comb. Theory (A)*, Vol 114, (2007) 575–596.
[27] N. Tokushige. Multiply-intersecting families revisited, *J. Comb. Theory (B)* Vol 97 (2007) 929–948.
[28] N. Tokushige. Brace–Daykin type inequalities for intersecting uniform families. *European J. of Comb*, Vol 29 (2008) 273–285.
[29] N. Tokushige. The random walk method for intersecting families, in Horizons of combinatorics, Bolyai society mathematical studies vol 17 (2008) 215–224.
[30] N. Tokushige. A multiply intersecting Erdos-Ko-Rado theorem — The principal case. *Discrete Mathematics* vol 310 (2009) 453–460.