Covers in uniform intersecting families and a counterexample to a conjecture of Lovász

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Abstract

We discuss the maximum size of uniform intersecting families with covering number at least τ. Among others, we construct a large k-uniform intersecting family with covering number k, which provides a counterexample to a conjecture of Lovász. The construction for odd k can be visualized on an annulus, while for even k on a Möbius band.

1 Introduction

Let X be a finite set. $\binom{X}{k}$ denotes the family of all k-element subsets of X. We always assume that $|X|$ is sufficiently large with respect to k. A family $\mathcal{F} \subset \binom{X}{k}$ is called k-uniform. The vertex set of $\mathcal{F}$ is X and denoted by $V(\mathcal{F})$. An element of $\mathcal{F}$ is called an edge of $\mathcal{F}$. $\mathcal{F} \subset \binom{X}{k}$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every $F, G \in \mathcal{F}$. A set $C \subset X$ is called a *cover* of $\mathcal{F}$ if it intersects every edge of $\mathcal{F}$, i.e., $C \cap F \neq \emptyset$ holds for all $F \in \mathcal{F}$. A cover C is also called t-cover if $|C| = t$. The *covering number* $\tau(\mathcal{F})$ of $\mathcal{F}$ is the minimum cardinality of the covers of $\mathcal{F}$. The degree of a vertex x is defined by $\deg(x) := \#\{F \in \mathcal{F} : x \in F\}$.

For a family $\mathcal{A} \subset 2^X$ and vertices $x, y \in X$, we define

$$\begin{aligned}
\mathcal{A}(x) &:= \{A - \{x\} : x \in A \in \mathcal{A}\}, \\
\mathcal{A}(\bar{x}) &:= \{A : x \notin A \in \mathcal{A}\}, \\
\mathcal{A}(\bar{x}\bar{y}) &:= \{A : x, y \notin A \in \mathcal{A}\},\ etc,
\end{aligned}$$

and for $Y \subset X$,

$$\begin{aligned}
\mathcal{A}(Y) &:= \{A - Y : Y \subset A \in \mathcal{A}\}, \\
\mathcal{A}(\bar{Y}) &:= \{A \in \mathcal{A} : Y \cap A = \emptyset\}.
\end{aligned}$$

For a family $\mathcal{F} \subset \binom{X}{k}$ and an integer $t \geq 1$, define

$$\mathcal{C}_t(\mathcal{F}) = \{C \in \binom{X}{t} : C \cap F \neq \emptyset \text{ holds for all } F \in \mathcal{F}\}.$$

Note that $\mathcal{C}_t(\mathcal{F}) = \emptyset$ for $t < \tau(\mathcal{F})$. Define

$$p_t(k) = \max\{|\mathcal{C}_t(\mathcal{F})| : \mathcal{F} \subset \binom{X}{k} \text{ is intersecting and } \tau(\mathcal{F}) \geq t\}.$$

Note that $|\mathcal{C}_t(\mathcal{F})| \leq k^t$ was proved by Gyárfás [7] without the assumption of $\mathcal{F}$ being intersecting. In that inequality, equality is attained only if $\mathcal{F}$ consists of t pairwise disjoint sets, in particular, for $t \geq 2$ if $\mathcal{F}$ is non-intersecting.

The aim of the present paper is to attain better bounds for $p_t(k)$ and apply them to estimate the maximum size of intersecting families with fixed covering number.

Let us first derive some useful facts concerning $p_t(k)$.

(1) $p_1(k) = k$ *(take* $|\mathcal{F}| = 1$*).*

(2) $p_{t+1}(k) \leq k p_t(k)$.

Proof Take $\mathcal{F} \subset \binom{X}{k}$, $\mathcal{F}$ intersecting, $\tau(\mathcal{F}) = t + 1$ and $|\mathcal{C}_{t+1}(\mathcal{F})| = p_{t+1}(k)$. Define $\mathcal{C} = \mathcal{C}_{t+1}(\mathcal{F})$. Let $F \in \mathcal{F}$ be an arbitrary member of $\mathcal{F}$. By definition, $F \cap C \neq \emptyset$ holds for every $C \in \mathcal{C}$. Thus $|\mathcal{C}| \leq \sum_{x \in F} |\mathcal{C}(x)|$ holds. Therefore, in order to establish (2) it is sufficient to prove $|\mathcal{C}(x)| \leq p_t(k)$ for all $x \in F$. Consider $\mathcal{F}(\bar{x})$. It is intersecting with

$$t \leq \tau(\mathcal{F}(\bar{x})) \leq \tau(\mathcal{F}) = t + 1.$$

Moreover, $\mathcal{C}(x) \subset \mathcal{C}_t(\mathcal{F}(\bar{x}))$ is immediate from the definitions. Thus $|\mathcal{C}(x)| = 0$ holds if $\tau(\mathcal{F}(\bar{x})) = t + 1$ and $|\mathcal{C}(x)| \leq p_t(k)$, otherwise. ∎

(3) *For* $\mathcal{F} \subset \binom{X}{k}$*, intersecting,* $\tau(\mathcal{F}) = t$ *and an arbitrary set* $A \in \binom{X}{a}$ *with* $a < t$*, one has*

$$|\mathcal{C}_t(\mathcal{F})(A)| \leq p_{t-a}(k).$$

Proof Consider $\mathcal{F}(\bar{A}) \subset \mathcal{F}$. Then $\tau(\mathcal{F}(\bar{A})) \geq \tau(\mathcal{F}) - |A| = t - a$. Moreover, $\mathcal{C}_t(\mathcal{F})(A) \subset \mathcal{C}_{t-a}(\mathcal{F}(\bar{A}))$ holds. By definition of $p_{t-a}(k)$ the desired inequality follows. ∎

The following was proved implicitly in Frankl [3]. For a simple proof, see [4].

(4) $p_2(k) = k^2 - k + 1$.

Using a construction described in the next section, it is not difficult to check that

$$p_3(k) \geq (k-1)^3 + 3(k-1) = k^3 - 3k^2 + 6k - 4$$

holds for all $k \geq 3$. The following is the key result proved in [4]. (The proof is not simple.)

(5) *For* $k \geq 9$*,* $p_3(k) = k^3 - 3k^2 + 6k - 4$*.*

Later we prove $p_3(3) = 14$. The case $4 \leq k \leq 8$ remains open. The authors do not know an example with $p_3(k) > k^3 - 3k^2 + 6k - 4$.

The following is proved in [5].

(6) *For $k \geq k_0$, $p_4(k) = k^4 - 6k^3 + O(k^2)$.*

We will give a conjecture for $p_t(k)$ $(t \geq 5)$ in section 3.

Let us define

$$r(k) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } k\text{-uniform and intersecting with } \tau(\mathcal{F}) = k\}.$$

For example, $r(2) = 3$ and the only extremal configuration is a triangle. Note that, $\mathcal{C}_k(\mathcal{F}) \supset \mathcal{F}$ for every intersecting k-uniform hypergraph, and equality must hold if $|\mathcal{F}| = r(k)$ holds (together with $\tau(\mathcal{F}) = k$). Recall also, that $r(k) \leq k^k$ was proved by Erdős and Lovász [2].

(7) $p_k(k) \geq r(k)$.

The inequality in (7) is likely to be strict for all $k \geq 3$. E.g. for $k = 3$ consider the family

$$\mathcal{F} = \{\{1,2,3\},\{3,4,5\},\{5,6,1\},\{2,4,5\},\{4,6,1\},\{6,2,3\}\}.$$

Then $\mathcal{F} \subset \binom{[6]}{3}$ and $\tau(\mathcal{F}) = 3$ imply $|\mathcal{C}_3(\mathcal{F})| = \binom{6}{3} - |\mathcal{F}| = 14$ ($G \notin \mathcal{C}_3(\mathcal{F})$ iff G is the complement of some $F \in \mathcal{F}$). On the other hand, $r(3) = 10$ is known. (See Appendix.)

(8) *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathcal{F}) = k$. Then for all $x \in F \in \mathcal{F}$, there exists $F' \in \mathcal{F}$ such that $F \cap F' = \{x\}$.*

Proof Let $x \in F \in \mathcal{F}$. Suppose that for all $F \neq F' \in \mathcal{F}$, $|F \cap F'| \geq 2$. Then $F - \{x\}$ is a cover of $\mathcal{F}$, which means $\tau(\mathcal{F}) \leq k - 1$. ∎

(9) *Suppose that $\mathcal{F} \subset \binom{X}{3}$ is an intersecting family with $\tau(\mathcal{F}) = 3$. Then there exists $x \in X$ such that $\deg(x) \geq 3$, and $|\mathcal{F}| \geq 6$.*

Proof We can choose $F, F' \in \mathcal{F}$ such that $F = \{1,2,3\}$, $F' = \{1,4,5\}$. There exists $G \in \mathcal{F}$ such that $G \cap \{2,4\} = \emptyset$. If $1 \in G$, then $\deg(1) \geq 3$. Otherwise we may assume $G = \{3,5,6\}$. We can choose $G' \in \mathcal{F}$ such that $G' \cap \{3,4\} = \emptyset$. Since $F' \cap G' \neq \emptyset$, we have $G' \cap \{1,5\} \neq \emptyset$. This implies $\deg(1) \geq 3$ or $\deg(5) \geq 3$.

Next we prove $|\mathcal{F}| \geq 6$. Assume on the contrary that $|\mathcal{F}| \leq 5$. We choose $x \in X$ such that $\deg(x) \geq 3$. Thus the number of edges which do not contain x is at most 2. Let F and F' be such edges. Choose $y \in F \cap F'$. Then $\{x, y\}$ is a cover of $\mathcal{F}$, which contradicts $\tau(\mathcal{F}) = 3$. ∎

(10) $p_3(3) = 14$.

Proof Case 1. There exist $F, F' \in \mathcal{F}$ such that $|F \cap F'| = 2$.
Let $F = \{1,2,3\}$, $F' = \{1,2,4\}$, and $\mathcal{C} = \mathcal{C}_3(\mathcal{F})$. By (3) and (4), $|\mathcal{C}(1)| \leq 7$ and $|\mathcal{C}(2)| \leq 7$. Thus, since $F, F' \in \mathcal{C}(1) \cap \mathcal{C}(2)$,

$$|\mathcal{C}(1) \cup \mathcal{C}(2)| \leq 7 + 7 - 2 = 12.$$

Suppose $|\mathcal{C}| \geq 15$. Then $|\mathcal{C}(\bar{1}\bar{2})| \geq 3$. Every member of $\mathcal{C}(\bar{1}\bar{2})$ must meet F at $\{3\}$ and F' at $\{4\}$, and hence

$$\{3,4,5\},\ \{3,4,6\},\ \{3,4,7\} \in \mathcal{C}.$$

Since $\mathcal{F}(\bar{3}\bar{4}) \neq \emptyset$, we must have $\{5,6,7\} \in \mathcal{F}(\bar{3}\bar{4})$. But $F \cap \{5,6,7\} = \emptyset$, a contradiction.

Case 2. For all distinct edges $F, F' \in \mathcal{F}$, $|F \cap F'| = 1$.
Let $\mathcal{C} = \mathcal{C}_3(\mathcal{F})$. We may assume that $\deg(1) \geq 3$ (by (9)) and

$$\{1,2,3\},\ \{1,4,5\},\ \{1,6,7\} \in \mathcal{F}.$$

Note that if $F \in \mathcal{F}(\bar{1})$ then

$$F \in \binom{\{2,3\}}{1} \cup \binom{\{4,5\}}{1} \cup \binom{\{6,7\}}{1}.$$

Consequently, there exist no other edges containing 1, i.e., $\deg(1) = 3$. Hence by (9), we have $\mathcal{F}(\bar{1}) \geq 3$. Thus, we have

$$|\mathcal{C}(\bar{1})| \leq 2^3 - |\mathcal{F}(\bar{1})| \leq 5.$$

Therefore, $|\mathcal{C}| = |\mathcal{C}(1)| + |\mathcal{C}(\bar{1})| \leq 7 + 5 = 12$. ∎

(11) *$p_{t+1}(k+1) \geq (k+1)p_t(k)$ holds for $t < k$ and $p_{k+1}(k+1) \geq (k+1)p_k(k) + 1$ for $t = k$.*

Proof Take an intersecting family $\mathcal{F} \subset \binom{X}{k}$ with $\tau(\mathcal{F}) = t$ and $\mathcal{C} = \mathcal{C}_t(\mathcal{F})$ of size $p_t(k)$. Let Y be a $(k+1)$-element set which is disjoint to X. Define

$$\mathcal{H} = \{F \cup \{y\} : F \in \mathcal{F},\ y \in Y\} \cup \{Y\}.$$

Then $\mathcal{H}$ is intersecting, $(k+1)$-uniform with $\tau(\mathcal{H}) = t+1$. Also $\{C \cup \{y\} : C \in \mathcal{C},\ y \in Y\} \subset \mathcal{C}_{t+1}(\mathcal{H})$ holds, proving the first inequality. To prove the second, note that Y is a cover of $\mathcal{H}$, too. ∎

Let us remark that the same proof yields

(12) *$r(k+1) \geq (k+1)r(k) + 1$.*

Using the above inequality together with $r(2) = 3$, we obtain

(13) *$r(k) \geq \lfloor k!(e-1) \rfloor$.*

Actually, (13) was proved by Erdős and Lovász [2].

(14) *Let $k > k_0(\tau)$, $|X| > n_0(k)$. Suppose that $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family with covering number τ. Then,*

$$|\mathcal{F}| \leq p_{\tau-1}(k)\binom{|X| - \tau}{k - \tau} + O\left(|X|^{k-\tau-1}\right)$$

holds.

The above claim is proved in [4] for $\tau = 4$. One can prove the general case in the same way.

2 A counterexample to a conjecture of Lovász

Erdős and Lovász[2] proved that the maximum size of k-uniform intersecting families with covering number k is at least $\lfloor k!(e-1)\rfloor$ and at most k^k. Lovász[10] conjectured that $\lfloor k!(e-1)\rfloor$ is the exact bound. This conjecture is true for $k=2,3$. However, for the case $k\geq 4$, this conjecture turns out to be false. In this section, we will construct k-uniform intersecting family with covering number k whose size is greater than $(\frac{k+1}{2})^{k-1}$.

The constructions are rather complicated, therefore we first give an outline of them. There is a particular element x_0 which will have the unique highest degree in general. We construct an intersecting family $\mathcal{G}\subset\binom{X-\{x_0\}}{k}$ with $\tau(\mathcal{G})=\tau-1$. ($\tau=k$ in the Erdős–Lovász case, and $\tau\leq k$ in general.) Next we define

$$\mathcal{B}:=\{\{x_0\}\cup C : C\in\bigcup_{t=\tau-1}^{k-1}\mathcal{C}_t(\mathcal{G})\}.$$

Finally, the family $\mathcal{F}_0=\mathcal{F}_0(k,\tau)$ is defined as

$$\mathcal{F}_0:=\mathcal{G}\cup\{F\in\binom{X}{k}, \exists B\in\mathcal{B}, B\subset F\}.$$

Now we give the two examples, according to the parity of τ.

Example 1 (The case $\tau=2s+2$.) *Let $h=k-s$. First we define an infinite k-uniform family $\mathcal{G}^*=\mathcal{G}^*(h)$ as follows. Let*

$$\begin{aligned} V(\mathcal{G}^*) &:= \{(2i,2j): i\in\mathbf{Z}, 0\leq j<h\}\\ &\quad\cup\{(2i+1,2j+1): i\in\mathbf{Z}, 0\leq j<h\}.\end{aligned}$$

We define a broom structure $\mathcal{G}_i$ as follows. A broom $\mathcal{G}_i$ has a broomstick

$$S_i:=\{(i,j):(i,j)\in V(\mathcal{G}^*)\},\quad (|S_i|=h)$$

and tails

$$\begin{aligned}\mathcal{T}_i &:= \{\{(i,j_0),(i+1,j_1),(i+2,j_2),\dots,(i+s,j_s)\}:\\ &\qquad j_{t+1}-j_t\in\{1,-1\} \text{ for } 0\leq\forall t<s\}\end{aligned}$$

where

$$j_0:=\begin{cases} h & \text{if } h+i \text{ is even}\\ h-1 & \text{if } h+i \text{ is odd.}\end{cases}$$

Set $\mathcal{G}_i:=\{S_i\cup T : T\in\mathcal{T}_i\}$. Note that $\mathcal{G}_i$ is a k-uniform family with size $|\mathcal{T}_i|=2^s$. Now define $\mathcal{G}^*:=\bigcup_{i\in\mathbf{Z}}\mathcal{G}_i$.

Next we define an equivalence relation $R(s)$ on $V(\mathcal{G}^*)$ induced by

$$(i,j)\equiv(i+2s+1,2h-1-j)\quad \text{for all } i\in\mathbf{Z} \text{ and } 0\leq j\leq 2h-1.$$

Note that this equivalence transforms the infinite tape into a Möbius band. Finally, we define $\mathcal{G}$ as a quotient family of $\mathcal{G}^*$ by $R(s)$, that is,

$$\mathcal{G}:=\mathcal{G}^*/R(s).$$

Note that $|V(\mathcal{G})|=(2s+1)h$. □

Example 2 (The case $\tau = 2s+1$.) *Let $h = k - s$, and*

$$\begin{aligned} V(\mathcal{G}) \;:=\; & \{(2i,2j) : i \in \mathbf{Z}_{2s}, 0 \le i < s, 0 \le j \le h\} \\ & \cup\{(2i+1,2j+1) : i \in \mathbf{Z}_{2s}, 0 \le i < s, 0 \le j \le h\} \\ & -\{(2i,0) : i \in \mathbf{Z}_{2s}, s \le 2i < 2s,\} \\ & -\{(2i+1,2h+1) : i \in \mathbf{Z}_{2s}, s \le 2i+1 < 2s,\} \end{aligned}$$

Note that $|V(\mathcal{G})| = s(2h+1)$. We define a broom structure $\mathcal{G}_i$ as follows. A broom $\mathcal{G}_i$ has a broomstick

$$S_i := \{(i,j) : (i,j) \in V(\mathcal{G})\},$$
$$(|S_0| = \cdots = |S_{s-1}| = h+1,\ |S_s| = \cdots = |S_{2s-1}| = h)$$

and tails

$$\begin{aligned} \mathcal{T}_i \;:=\; & \{\{(i,j_0),(i+1,j_1),(i+2,j_2),\dots,(i+u,j_u)\} : \\ & j_{t+1} - j_t \in \{1,-1\} \ \textit{for}\ 0 \le \forall t < u\} \end{aligned}$$

where

$$u := \begin{cases} s-1 & \textit{if } i \in \{0,1,\dots,s-1\} \ (\textit{mod. } 2s) \\ s & \textit{if } i \in \{s,s+1,\dots,2s-1\} \ (\textit{mod. } 2s), \end{cases}$$

and

$$j_0 := \begin{cases} h & \textit{if } h+i \textit{ is even} \\ h+1 & \textit{if } h+i \textit{ is odd.} \end{cases}$$

Set $\mathcal{G}_i := \{S_i \cup T : T \in \mathcal{T}_i\}$, and define $\mathcal{G} := \bigcup_{0 \le i < 2s} \mathcal{G}_i$. □

Remark 1 *In both examples, any edge of type*

$$\{x_0, x_1, \dots, x_{\tau-1}\} \quad (x_j \in S_j \textit{ for all } 0 \le j \le \tau - 1)$$

is a cover of $\mathcal{G}$. This implies that

$$|\mathcal{C}_{\tau-1}(\mathcal{G})| \ge \prod_{i=0}^{\tau-2} |S_i|. \qquad \square$$

Now we check that the above constructions satisfy the required conditions. It is easy to see that the family $\mathcal{G}$ is intersecting. But $\tau(\mathcal{G}) = \tau - 1$ is not trivial. We only prove the case $\tau = 2s+2$, because the proof for the case $\tau = 2s+1$ is very similar.

Let us consider properties of covers of $\mathcal{T}_0$. Define $I_t := \bigcup_{T \in \mathcal{T}_0} (S_t \cap T)$, $J_t := \bigcup_{l=0}^{t} I_l$, and fix a cover $C \in \mathcal{C}(\mathcal{T}_0)$. A vertex $y_i \in S_i$ is called suspicious (under C) if there exists

$$T = \{y_0, y_1, \dots, y_s\} \in \mathcal{T}_0 \quad (y_j \in S_j \text{ for all } 0 \le j \le s)$$

such that

$$\{y_0, y_1, \dots, y_i\} \cap C = \emptyset.$$

Let $L = L(C)$ be the set of all suspicious vertices.

Let us start with a trivial but useful fact.

Claim 1 *If $C \cap I_{i+1} = \emptyset$ then $|L \cap I_{i+1}| \ge |L \cap I_i| + 1$ and equality holds only if $L \cap I_i$ consists of consecutive vertices on I_i.* □

The following fact is easily proved by induction on i.

Claim 2 *Let $a = |C \cap I_i|$. Suppose that $|C \cap J_l| \le l$ for all $0 \le l < i$. Then $|L \cap I_i| \ge i - a + 1$ and equality holds only if $L \cap I_i$ consists of consecutive vertices on I_i.* $\square$

The following is a direct consequence of the above fact.

Proposition 1 *Suppose that $|C \cap J_l| \le l$ for all $0 \le l < i$ and $L \cap I_i = \emptyset$. Then $|C \cap J_i| \ge i + 1$ and equality holds only if $C \cap I_i$ consists of consecutive vertices on I_i.* $\square$

Proposition 2 $\tau(\mathcal{G}) = 2s + 1$.

Proof Let C be any cover for $\mathcal{G}$. For each $0 \le i \le 2s$, we define the interval $W_i = [i, i + r]$ (mod $2s + 1$) so that r is the minimum non-negative integer satisfying

$$|C \cap (S_i \cup S_{i+1} \cup \ldots \cup S_{i+r})| \ge r + 1.$$

In fact, such an integer r exists by Proposition 1. The following claim can be shown easily.

Claim 3 *If W_i and W_j have non-empty intersection, then $W_i \subset W_j$ or $W_j \subset W_i$ holds.* $\square$

Using this, we can choose disjoint intervals from $W_0, W_1, \ldots, W_{2s}$ whose union is exactly $[0, 2s]$. And so, $|C| \ge 2s + 1$. This completes the proof of $\tau(\mathcal{G}) = 2s + 1$. $\blacksquare$

Now we know that

$$\mathcal{F}_0 := \mathcal{G} \cup \{F \in \binom{X}{k}, \exists B \in \mathcal{B}, B \subset F\}$$

is intersecting, and $\tau - 1 \le \tau(\mathcal{F}_0) \le \tau$. We can check that $\tau(\mathcal{F}_0) = \tau$ using the following easy fact.

Proposition 3 *Let $\mathcal{G} \subset \binom{X - \{x_0\}}{k}$ be an intersecting family with $\tau(\mathcal{G}) = \tau - 1$. Define*

$$\begin{aligned} \mathcal{B} &:= \{\{x_0\} \cup C : C \in \bigcup_{t=\tau-1}^{k-1} \mathcal{C}_t(\mathcal{G})\}, \\ \mathcal{F} &:= \mathcal{G} \cup \{F \in \binom{X}{k}, \exists B \in \mathcal{B}, B \subset F\}. \end{aligned}$$

Then $\tau(\mathcal{F}) = \tau$ if and only if for all $C \in \mathcal{C}_{\tau-1}(\mathcal{G})$ there exists $C' \in \mathcal{C}_{\tau-1}(\mathcal{G})$ such that $C \cap C' = \emptyset$. $\square$

Lovász conjectured that $r(k) = \lfloor k!(e - 1) \rfloor < e^2(\frac{k+1}{e})^{k+1}$. Our construction beats this conjecture as follows. Let $\mathcal{G}$ be a k-uniform intersecting family defined in Example 1 or Example 2. Then $\tau(\mathcal{G}) = k$. By Remark 1, we have the following lower bound.

Theorem 1

$$r(k) > |\mathcal{C}_{k-1}(\mathcal{G})| > \begin{cases} \left(\frac{k}{2} + 1\right)^{k-1} & \text{if } k \text{ is even,} \\ \left(\frac{k+3}{2}\right)^{\frac{k-1}{2}} \left(\frac{k+1}{2}\right)^{\frac{k-1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

Thus, our construction is exponentially larger than Erdős–Lovász construction.

3 Open problems

Problem 1 *Determine the maximum size of* 4*-uniform intersecting families with covering number four. Does* $r(4) = 42$ *hold?* □

Problem 2 *Determine* $p_3(k)$ *for* $4 \le k \le 8$*. Does* $p_3(k) = k^3 - 3k^2 + 6k - 4$ *hold in these cases?* □

Conjecture 3 *Let* $\mathcal{F} \subset \binom{X}{k}$ *be an intersecting family with covering number* τ*. If* $k > k_0(\tau)$*,* $|X| > n_0(k)$*, then we have*

$$|\mathcal{F}| \le \left(k^{\tau-1} - \binom{\tau-1}{2} k^{\tau-2} + c(k,\tau)\right) \binom{|X| - \tau}{k - \tau} + O\left(|X|^{k-\tau-1}\right),$$

where $c(k,\tau)$ *is a polynomial of* k *and* τ*, and the degree of* k *is at most* $\tau - 3$*.* □

Using (14), the above conjecture would follow from the following conjecture by setting $\tau = t + 1$.

Conjecture 4 *Let* $k \ge k_0(t)$*. Then*

$$p_t(k) = k^t - \binom{t}{2} k^{t-1} + O(k^{t-2})$$

holds. □

This conjecture holds for $t \le 4$. It seems that the coefficient of k^{t-2} in the above conjecture is

$$\frac{t}{4} \left\lfloor \frac{(t+1)(t^2 - 4t + 7)}{2} \right\rfloor .$$

For the case $\tau = k$, we conjecture the following.

Conjecture 5 *For some absolute constant* $\frac{1}{2} \le \mu < 1$*,* $r(k) < (\mu k)^k$ *holds.* □

We close this section with a bold conjecture.

Conjecture 6 *Let* $k \ge \tau \ge 4$ *and* $n > n_0(k)$*. Let* $\mathcal{F}_0$ *be the family defined in Example 1 or Example 2. Suppose that* $\mathcal{F} \subset \binom{X}{k}$ *is an intersecting family with covering number* τ*, then*

$$|\mathcal{F}| \le |\mathcal{F}_0|$$

holds. Equality holds if and only if $\mathcal{F}$ *is isomorphic to* $\mathcal{F}_0$*.* □

This conjecture is true if "$k \ge 4$ and $\tau = 2$ [9]," or "$k \ge 4$ and $\tau = 3$ [3]," or "$k \ge 10$ and $\tau = 4$ [4]." (Inequality holds even if "$k = 3$ and $\tau = 2$," or "$k = 3$ and $\tau = 3$," but the uniqueness of the extremal configuration does not hold in these cases.) Of course, this conjecture is much stronger than Conjecture 3. Note that for $k = \tau$ this conjecture would give the solution to the problem of Erdős–Lovász, and in particular, it would show that the answer to Problem 1 is 42.

4 Appendix

4.1 Numerical data

The following is a table of the size of k-uniform intersecting families with covering number k, i.e., known lower bounds for $r(k)$.

k	Erdős–Lovász construction	Example 1, Example 2
2	3	3
3	10	10
4	41	42
5	206	228
6	1,237	1,639
7	8,660	13,264
8	69,281	128,469
9	623,530	1,327,677
10	6,235,301	15,962,373
11	68,588,312	202,391,317
12	823,059,745	2,942,955,330
13	10,699,776,686	44,744,668,113
14	149,796,873,605	770,458,315,037
15	2,246,953,104,076	13,752,147,069,844
16	35,951,249,665,217	274,736,003,372,155

4.2 $k = \tau = 3$

The maximum size of 3-uniform intersecting families with covering number 3 is 10, i.e., $r(3) = 10$. There are 7 configurations which attain the maximum. The following is the list of these extremal configurations.

```
(#1) 123           (#2) 123           (#3) 123           (#4) 123
     12 4               12 4               12 4               12 4
     12  5              12  5              12  5              12  5
       345                345                345                345
     1 34               1 3  6             1 3  6             1 34
     1 3 5              1  4 6             1  4 6             1  4 6
     1  45              1   56             1   56             1   56
      234                23  6             1 34                23 5
      23 5                2 4 6              23  6              23  6
      2 45                2  56              2 4 6              2 45
```

```
(#5) 123       (#6) 123       (#7) 123
     12 4           12 4           12 4
     12  5          12  5          12  5
       345            345            345
     1 34           1 34           1 34
     1 3 5          1 3  6         1 3  6
     1   56         1   56         1  4  7
      23 5           23 5           234
      2 45           23  6          23   7
      23  6          2 4 6          2 4 6
```

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