Uniform intersecting families with covering number restrictions

Peter Frankl
CNRS, ER 175 Combinatoire,
54 Bd Raspail, 75006 Paris, France

Katsuhiro Ota
Department of Mathematics, Keio Univ.,
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223 Japan

Norihide Tokushige
Department of Computer Science, Meiji Univ.,
1-1-1 Higashimita, Tama-ku, Kawasaki, 214 Japan

June 16, 1994

Abstract

It is known that any k-uniform family with covering number t has at most k^t t-covers. In this paper, we give better upper bounds for the number of t-covers in k-uniform intersecting families with covering number t.

1 Introduction

Let X be a finite set. The family of all k-element subsets of X is denoted by $\binom{X}{k}$. We always assume that $|X|$ is sufficiently large with respect to k. A family $\mathcal{F} \subset \binom{X}{k}$ is called k-uniform. The vertex set of $\mathcal{F}$, denoted by $V(\mathcal{F})$, is defined to be $\bigcup_{F\in\mathcal{F}} F$, which is a subset of X in general. An element of $\mathcal{F}$ is called an edge of $\mathcal{F}$. A family $\mathcal{F} \subset \binom{X}{k}$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every $F, G \in \mathcal{F}$. A set $C \subset X$ is called a *cover* of $\mathcal{F}$ if it intersects every edge of $\mathcal{F}$, i.e., $C \cap F \neq \emptyset$ holds for all $F \in \mathcal{F}$. A cover C is also called t-cover if $|C| = t$. The *covering number* $\tau(\mathcal{F})$ of $\mathcal{F}$ is the minimum cardinality of the covers of $\mathcal{F}$. The degree of a vertex x if $\mathcal{F}$ is the number of edges in $\mathcal{F}$ containing x, and is denoted by $\deg_{\mathcal{F}}(x)$.

For a family $\mathcal{F} \subset \binom{X}{k}$ and an integer $t \geq 1$, define

$$\mathcal{C}_t(\mathcal{F}) = \{C \in \binom{X}{t} : C \cap F \neq \emptyset \text{ holds for all } F \in \mathcal{F}\}.$$

Note that $\mathcal{C}_t(\mathcal{F}) = \emptyset$ for $t < \tau(\mathcal{F})$. Define

$$p_t(k) = \max\{|\mathcal{C}_t(\mathcal{F})| : \mathcal{F} \subset \binom{X}{k} \text{ is intersecting and } \tau(\mathcal{F}) \geq t\}.$$

Note that $|\mathcal{C}_t(\mathcal{F})| \leq k^t$ was proved by Gyárfás[6] without the assumption of $\mathcal{F}$ being intersecting. In that inequality, equality is attained only if $\mathcal{F}$ consists of t pairwise disjoint sets, in particular, for $t \geq 2$ if $\mathcal{F}$ is non-intersecting. The aim of the present paper is to attain better bounds for $p_t(k)$.

It is shown in [2] (see also [3] and [5]) that the maximum size of k-uniform intersecting families with covering number t is $(p_{t-1}(k) + o(1))\binom{n}{k-t}$ as the number of vertices n tends to infinity. So, it is greatly important to determine the value $p_t(k)$. See [1], [2], [3] and [7] for the results on the maximum size of k-uniform intersecting families with covering number restrictions.

One can easily see that $p_1(k) = k$. For $t = 2$ and 3, the value $p_t(k)$ is determined in [2], [3] and [4].

Theorem A ([2]) *For $k \geq 2$, $p_2(k) = k^2 - k + 1$.* □

Theorem B ([3], [4]) *For $k = 3$ and $k \geq 9$, $p_3(k) = k^3 - 3k^2 + 6k - 4$.* □

For a fixed t, the following conjecture is found in [4].

Conjecture 1 ([4]) $p_t(k) = k^t - \binom{t}{2}k^{t-1} + O(k^{t-2})$. □

The coefficient of k^{t-1} in this conjecture is best possible if it is true.

Example 1 *Let T be any tournament with its vertex set $\{1, 2, \ldots, t\}$, and let α_i be the outdegree of the vertex i of T. Preparing t sets of vertices X_1, X_2, ..., X_t such that $|X_i| = k - \alpha_i$ for $(1 \leq i \leq t)$, we define a family $\mathcal{F}_i$ for each i $(1 \leq i \leq t)$ as follows:*

$$\mathcal{F}_i = \{X_i \cup A \ : \ |A| = \alpha_i, \ |A \cap X_j| = 1 \textit{ if and only if } i \textit{ dominates } j\}$$

Then, $\mathcal{F} = \bigcup_{i=1}^{t} \mathcal{F}_i$ is a k-uniform intersecting family and $\tau(\mathcal{F}) = t$ if $k \geq t$. Now, we can get a t-cover of $\mathcal{F}$ by choosing any one vertex from each X_i $(1 \leq i \leq t)$. Hence,

$$\begin{aligned} |\mathcal{C}_t(\mathcal{F})| &\geq \prod_{i=1}^{t} |X_i| = \prod_{i=1}^{t}(k - \alpha_i) = k^t - \left(\sum_{i=1}^{t} \alpha_i\right) k^{t-1} + O(k^{t-2}) \\ &= k^t - \binom{t}{2} k^{t-1} + O(k^{t-2}). \end{aligned}$$ □

In view of this example, we give the following conjecture.

Conjecture 2 *Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = t$. Suppose that $\mathcal{F}$ is partitioned into t classes of edges $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$, and that for each i, every edge $F \in \mathcal{F}_i$ contains X_i, where $X_1, X_2, \ldots, X_t$ are pairwise disjoint subsets of X. Then, $\sum_{i=1}^{t}(k - |X_i|) \geq \binom{t}{2}$.* $\square$

Obviously, Conjecture 1 implies Conjecture 2. One of the main results in this paper is the other implication. In fact, we prove the following theorem, in which the function $b(t)$ is defined to be the minimum value of $\sum_{i=1}^{t}(k - |X_i|)$ among the families satisfying the assumption of Conjecture 2. Note that $b(t) \leq \binom{t}{2}$ holds, by Example 1.

Theorem 1 $p_t(k) = k^t - b(t)k^{t-1} + O(k^{t-2})$. $\square$

Moreover, in Section 4, determining the exact value for $b(4)$ and $b(5)$, we show that Conjecture 2, and hence Conjecture 1, is true for $t \leq 5$.

For a general t, we prove the following theorem in Section 3.

Theorem 2 $b(t) \geq \frac{1}{\sqrt{2}}\lfloor\frac{t-1}{2}\rfloor^{\frac{3}{2}}$. $\square$

Corollary 3 $p_t(k) \leq k^t - \frac{1}{\sqrt{2}}\lfloor\frac{t-1}{2}\rfloor^{\frac{3}{2}}k^{t-1} + O(k^{t-2})$. $\square$

In the subsequent argument, we use the following propositions without explicit reference.

Proposition 1 ([6]) $p_t(k) \leq k^t$. $\square$

For a family $\mathcal{A} \subset 2^X$ and vertices $x, y \in X$, we define

$$\begin{aligned}
\mathcal{A}(x) &= \{A \in \mathcal{A} \,:\, x \in A\}, \\
\mathcal{A}(\overline{x}) &= \{A \in \mathcal{A} \,:\, x \notin A\}, \\
\mathcal{A}(xy) &= \{A \in \mathcal{A} \,:\, x \in A,\ y \in A\}, \\
\mathcal{A}(x\overline{y}) &= \{A \in \mathcal{A} \,:\, x \in A,\ y \notin A\}, \textit{ etc.},
\end{aligned}$$

and for $Y \subset X$,

$$\begin{aligned}
\mathcal{A}(Y) &= \{A \in \mathcal{A} \,:\, Y \subset A\}, \\
\mathcal{A}(\overline{Y}) &= \{A \in \mathcal{A} \,:\, Y \cap A = \emptyset\}.
\end{aligned}$$

Proposition 2 ([4]) *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathcal{F}) = t$. Let $\mathcal{C} = \mathcal{C}_t(\mathcal{F})$. Then, for any subset A of X with $|A| < t$, $|\mathcal{C}(A)| \leq p_{t-|A|}(k)$ holds.* $\square$

2 Proof of Theorem 1

Throughout this section, we assume that t is a fixed positive integer, k is sufficiently larger than t, and that $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathcal{F}) = t$ such that $|\mathcal{C}_t(\mathcal{F})| \geq k^t - \binom{t}{2}k^{t-1}$. We write simply $\mathcal{C}$ for $\mathcal{C}_t(\mathcal{F})$.

For $A \in \mathcal{F}$ and $x \in A$, define

$$\begin{aligned} \gamma_i(x, A) &= \#\{C \in \mathcal{C}(x) \ : \ |C \cap A| = i\}, \\ c(x, A) &= \sum_{i=1}^{t} \frac{1}{i}\gamma_i(x, A). \end{aligned}$$

We call $c(x, A)$ the *contribution* of $x \in A$ for $|\mathcal{C}|$, because it is easy to see that $|\mathcal{C}| = \sum_{x \in A} c(x, A)$. Moreover, by the definition, $|\mathcal{C}(x)| = \sum_{i=1}^{t} \gamma_i(x, A)$ holds.

Lemma 1 *For any pair of edges A and B in $\mathcal{F}$, $|A \cap B| < t^2$ or $|A \cap B| > k - t^2$ holds.*

Proof Define $a = |A \cap B|$. We assume that $t^2 \leq a \leq k - t^2$. We estimate the contribution of each vertex $x \in A$ for $|\mathcal{C}|$.

If $x \in A - B$, then every t-cover $C \in \mathcal{C}$ with $C \cap A = \{x\}$ must contain some vertex $y \in B - A$. For fixing $y \in B - A$, we have $|\mathcal{C}(xy)| \leq p_{t-2}(k) \leq k^{t-2}$. Hence,

$$\gamma_1(x, A) \leq |B - A|\, k^{t-2} = (k - a)k^{t-2}.$$

Thus,

$$\begin{aligned} c(x, A) &\leq \gamma_1(x, A) + \frac{1}{2}(|\mathcal{C}(x)| - \gamma_1(x, A)) \\ &= \frac{1}{2}(\gamma_1(x, A) + |\mathcal{C}(x)|) \\ &\leq \frac{1}{2}((k - a)k^{t-2} + k^{t-1}) \\ &= k^{t-1} - \frac{a}{2}k^{t-2}. \end{aligned}$$

If $x \in A \cap B$, then we have $c(x, A) \leq |\mathcal{C}(x)| \leq p_{t-1}(k) \leq k^{t-1}$. By summing up all contributions of $x \in A$, we get

$$\begin{aligned} |\mathcal{C}| = \sum_{x \in A} c(x, A) &\leq (k - a)(k^{t-1} - \frac{a}{2}k^{t-2}) + ak^{t-1} \\ &= k^t - \frac{a}{2}k^{t-1} + \frac{a^2}{2}k^{t-2}. \end{aligned}$$

Since $t^2 \leq a \leq k - t^2$, the RHS of the above inequality attains its maximum when $a = t^2$. So, $|\mathcal{C}| \leq k^t - \frac{t^2}{2}k^{t-1} + \frac{t^4}{2}k^{t-2}$, which contradicts the assumption that $|\mathcal{C}| \geq k^t - \binom{t}{2}k^{t-1}$ and $k \gg t$. ∎

The result of Lemma 1 implies that the set of edges in $\mathcal{F}$ is partitioned into the equivalence classes $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r$, where $|A \cap B| > k - t^2$ if and only if A and B are in the same class $\mathcal{F}_i$.

Lemma 2 *For each i $(1 \le i \le r)$, $|\bigcap_{F\in\mathcal{F}_i} F| > k - t^2$ holds.*

Proof Fix i and $A \in \mathcal{F}_i$. Let $X_i = \bigcap_{F\in\mathcal{F}_i} F$ and $a = |X_i|$. We assume that $a \le k - t^2$. If $x \in A - X_i$, then there exists an edge $B \in \mathcal{F}_i$ such that $x \notin B$. Note that $|A \cap B| > k - t^2$ and hence $|B - A| < t^2$. By the same argument used in Lemma 1, we have $\gamma_1(x, A) \le |B - A|\, k^{t-2} < t^2 k^{t-2}$. Therefore,

$$\begin{aligned} c(x, A) &\le \frac{1}{2}(\gamma_1(x, A) + |\mathcal{C}(x)|) \\ &< \frac{1}{2}(t^2 k^{t-2} + k^{t-1}). \end{aligned}$$

If $x \in X_i$, then $c(x, A) \le |\mathcal{C}(x)| \le k^{t-1}$. Thus,

$$\begin{aligned} |\mathcal{C}| = \sum_{x\in A} c(x, A) &< (k-a)\frac{1}{2}(t^2 k^{t-2} + k^{t-1}) + a k^{t-1} \\ &= \frac{1}{2}(k^t + t^2 k^{t-1} + a(k^{t-1} - t^2 k^{t-2})) \\ &\le \frac{1}{2}(k^t + t^2 k^{t-1} + (k - t^2)(k^{t-1} - t^2 k^{t-2})) \\ &= k^t - \frac{t^2}{2} k^{t-1} + \frac{t^4}{2} k^{t-2}. \end{aligned}$$

This is a contradiction. ∎

Remark 1 *By Lemma 2, $\tau(\mathcal{F}_i) = 1$ holds for each i $(1 \le i \le r)$. And hence, $r \ge t$ must hold, since $\tau(\mathcal{F}) = t$.* □

Lemma 3 *$r = t$.*

Proof Suppose that $r \ge t+1$. Choose one edge F_i from each $\mathcal{F}_i$, $1 \le i \le t+1$, and define $\mathcal{H} = \{F_1, F_2, \ldots, F_{t+1}\}$. Let Y be the set of vertices of which the degree in $\mathcal{H}$ is at least two. Note that $|F_i \cap F_j| < t^2$ if $i \ne j$, and hence $|Y| < \binom{t+1}{2} t^2$. On the other hand, every t-cover of $\mathcal{F}$ must contain some vertex in Y. Thus,

$$\begin{aligned} |\mathcal{C}| &\le \sum_{y\in Y} |\mathcal{C}(y)| \le |Y|\, p_{t-1}(k) \\ &< \binom{t+1}{2} t^2 k^{t-1}. \end{aligned}$$

This is a contradiction. ∎

For each i $(1 \le i \le t)$, define $X_i = \bigcap_{F\in\mathcal{F}_i} F$ and $\alpha_i = k - |X_i|$. By Lemma 2, we have $\alpha_i < t^2$.

Remark 2 *The vertex sets X_1, X_2, ..., X_t are pairwise disjoint, for otherwise $\mathcal{F}$ can be covered by at most $t-1$ vertices.* $\square$

Lemma 4 $|\mathcal{C}| = k^t - \left(\sum_{i=1}^t \alpha_i\right) k^{t-1} + O(k^{t-2})$.

Proof Define

$$\mathcal{C}' = \{C \in \binom{X}{t} \ : \ |C \cap X_i| = 1 \text{ for all } i, \ 1 \le i \le t\}.$$

Obviously, $\mathcal{C}' \subset \mathcal{C} = \mathcal{C}_t(\mathcal{F})$, and

$$|\mathcal{C}'| = \prod_{i=1}^{t} |X_i| = \prod_{i=1}^{t} (k - \alpha_i) = k^t - \left(\sum_{i=1}^{t} \alpha_i\right) k^{t-1} + O(k^{t-2}).$$

Hence, in order to prove the lemma, it suffices to show that $|\mathcal{C} - \mathcal{C}'| = O(k^{t-2})$.

For each i $(1 \le i \le t)$, let $\mathcal{C}_i$ be the set of t-covers C of $\mathcal{F}$ such that $C \cap X_i = \emptyset$. Fix i and $A \in \mathcal{F}_i$. Since every t-cover $C \in \mathcal{C}_i$ contains some vertex in $A - X_i$, there exists a vertex $x \in A - X_i$ such that $|\mathcal{C}_i(x)| \ge \frac{1}{\alpha_i}|\mathcal{C}_i|$. Now, there exists an edge $B \in \mathcal{F}_i$ such that $x \notin B$. Since every cover $C \in \mathcal{C}_i(x)$ must contain some vertex in $B - X_i$, there exists a vertex $y \in B - X_i$ such that $|\mathcal{C}_i(xy)| \ge \frac{1}{\alpha_i}|\mathcal{C}_i(x)| \ge \frac{1}{\alpha_i^2}|\mathcal{C}_i|$.

On the other hand, $|\mathcal{C}_i(xy)| \le |\mathcal{C}(xy)| \le p_{t-2}(k) \le k^{t-2}$. The last two inequalities imply $|\mathcal{C}_i| \le \alpha_i^2 k^{t-2} < t^4 k^{t-2}$. Thus,

$$|\mathcal{C} - \mathcal{C}'| \le \sum_{i=1}^{t} |\mathcal{C}_i| < t^5 k^{t-2} = O(k^{t-2}).$$

This completes the proof of Lemma 4. $\blacksquare$

Now, we can easily prove Theorem 1. Suppose that k is sufficiently large with respect to t. Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = t$ such that $|\mathcal{C}_t(\mathcal{F})| = p_t(k)$. Because we know the fact that $b(t) \le \binom{t}{2}$ (cf. Example 1),

$$|\mathcal{C}_t(\mathcal{F})| \ge k^t - b(t)k^{t-1} \ge k^t - \binom{t}{2} k^{t-1}.$$

Then, by Lemma 4,

$$|\mathcal{C}_t(\mathcal{F})| \le k^t - b(t)k^{t-1} + O(k^{t-2}).$$

This completes the proof of Theorem 1. $\blacksquare$

3 Proof of Theorem 2

We assume that $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = t$ such that $\mathcal{F}$ is partitioned into t classes of edges $\mathcal{F}_1$, $\mathcal{F}_2$, ..., $\mathcal{F}_t$, and that for each i, every edge $F \in \mathcal{F}_i$ contains X_i, where X_1, X_2, ..., X_t are pairwise disjoint subset of X. Let $|X_i| = k - \alpha_i$ for $1 \leq i \leq t$.

Define $s = \lfloor \frac{t-1}{2} \rfloor$. Let F_1, F_2, ..., F_s be the edges of $\mathcal{F}$ such that F_i and F_j are in the different classes of $\mathcal{F}_1$, $\mathcal{F}_2$, ..., $\mathcal{F}_t$ if $i \neq j$. Define

$$\mathcal{H} = \{F_1, F_2, \ldots, F_s\}.$$

We assume that F_1, F_2, ..., F_s are chosen so that $\sum_{x \in V(\mathcal{H})}(\deg_{\mathcal{H}}(x) - 1)$ is maximum possible. We may also assume that $F_i \in \mathcal{F}_i$ for each i $(1 \leq i \leq s)$. Let x_1, x_2, ..., x_s be s vertices of $\mathcal{H}$ with degrees in $\mathcal{H}$ being largest possible. Define $d = \min_{1 \leq i \leq s} \deg_{\mathcal{H}}(x_i)$. Now,

$$\begin{aligned} \deg_{\mathcal{H}}(x_i) \geq d & \quad \text{for each } i \ (1 \leq i \leq s), \text{ and} \\ \deg_{\mathcal{H}}(y) \leq d & \quad \text{for each } y \in V(\mathcal{H}) - \{x_1, x_2, \ldots, x_s\}. \end{aligned}$$

Case 1. $d \geq \sqrt{s/2}$.
Since $\deg_{\mathcal{H}}(x_i) \geq d$ for each i $(1 \leq i \leq s)$,

$$\sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) \geq s(d-1) \geq \frac{1}{\sqrt{2}} s^{\frac{3}{2}} - s.$$

On the other hand,

$$\sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) = ks - |V(\mathcal{H})| \leq ks - \sum_{i=1}^{s} |X_i| = \sum_{i=1}^{s} \alpha_i.$$

Hence, we have $\sum_{i=1}^{s} \alpha_i \geq \frac{1}{\sqrt{2}} s^{\frac{3}{2}} - s$. Moreover, since $\mathcal{F}$ is intersecting, at most one of α_{s+1}, ..., α_t is 0. Thus,

$$\begin{aligned} \sum_{i=1}^{t} \alpha_i & \geq \sum_{i=1}^{s} \alpha_i + (t - s - 1) \\ & \geq (\frac{1}{\sqrt{2}} s^{\frac{3}{2}} - s) + s \ = \ \frac{1}{\sqrt{2}} s^{\frac{3}{2}}. \end{aligned}$$

Case 2. $d < \sqrt{s/2}$.
For each i $(1 \leq i \leq s)$, choose one vertex $y_i \in X_i$. Since $\tau(\mathcal{F}) = t > 2s$, there exists an edge $G \in \mathcal{F}$ such that $G \cap \{x_1, \ldots, x_s, y_1, \ldots, y_s\} = \emptyset$. We may assume that $G \in \mathcal{F}_{s+1}$. We will find an edge $F_l \in \mathcal{H}$ such that $(\mathcal{H} - \{F_l\}) \cup \{G\}$ contradicts the maximality of $\sum_{x \in V(\mathcal{H})}(\deg_{\mathcal{H}}(x) - 1)$.

Let Y be the set of vertices y in $V(\mathcal{H})$ such that $\deg_{\mathcal{H}}(y) \geq 2$, and define $a_i = |F_i \cap Y|$ for $1 \leq i \leq s$. Then,

$$\sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) = \sum_{y \in Y} (\deg_{\mathcal{H}}(y) - 1) = \sum_{i=1}^{s} a_i - |Y|.$$

Obviously, $|Y| \leq \sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1)$ holds, and hence,

$$\sum_{i=1}^{s} a_i = \sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) + |Y| \leq 2 \sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1).$$

If $\sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) \geq s(\sqrt{s/2} - 1)$, then by the same argument used in Case 1, we are done. Hence, we may assume that $\sum_{i=1}^{s} a_i < 2s(\sqrt{s/2} - 1)$. Therefore, there exists some l $(1 \leq l \leq s)$ such that $a_l < 2(\sqrt{s/2} - 1) = \sqrt{2s} - 2$.

Now define $\mathcal{H}' = (\mathcal{H} - \{F_l\}) \cup \{G\}$. Let $Z = V(\mathcal{H} - \{F_l\}) \cap G$. Recall that G contains none of the vertices $x_1, \ldots, x_s$. So, the degree of every vertex of Z in $\mathcal{H}$ (and hence, in $\mathcal{H} - \{F_l\}$) is at most $d < \sqrt{s/2}$, while G must intersect with $s-1$ edges of $\mathcal{H} - \{F_l\}$. Therefore, $|Z| \geq \frac{s-1}{d} > \sqrt{2s} - \sqrt{2/s}$ holds. Thus,

$$\begin{aligned}
\sum_{x \in V(\mathcal{H}')} (\deg_{\mathcal{H}'}(x) - 1) &= \sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) - a_l + |Z| \\
&> \sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1) - (\sqrt{2s} - 2) + (\sqrt{2s} - \sqrt{2/s}) \\
&\geq \sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1).
\end{aligned}$$

This contradicts the maximality of $\sum_{x \in V(\mathcal{H})} (\deg_{\mathcal{H}}(x) - 1)$. ∎

4 $p_4(k)$ and $p_5(k)$

In this section, we show that Conjecture 2, and hence Conjecture 1, is true for $t = 4$ and $t = 5$.

Theorem 4 $p_4(k) = k^4 - 6k^3 + O(k^2)$. □

Theorem 5 $p_5(k) = k^5 - 10k^4 + O(k^3)$. □

Proof of Theorem 4 We will use the result of Theorem 1. Let $\mathcal{F} \subset \binom{X}{2}$ be an intersecting family with $\tau(\mathcal{F}) = 4$. Suppose that $\mathcal{F}$ is partitioned into four classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ such that for each i $(1 \leq i \leq 4)$, every edge $F \in \mathcal{F}_i$ contains X_i, where X_1, X_2, X_3 and X_4 are pairwise disjoint subsets of X. We may assume that $|X_1| \geq |X_2| \geq |X_3| \geq |X_4|$. We want to show that $\sum_{i=1}^{4} (k - |X_i|) \geq 6$.

We use the following notation. For $I \subset \{1,2,3,4\}$, define $\mathcal{F}_I = \bigcup_{i \in I} \mathcal{F}_i$ and $X_I = \bigcup_{i \in I} X_i$. If $I = \{i, j, \ldots\}$, then we write $\mathcal{F}_{ij\ldots}$ and $X_{ij\ldots}$ instead of $\mathcal{F}_{\{i,j,\ldots\}}$ and $X_{\{i,j,\ldots\}}$, respectively. Note that $\tau(\mathcal{F}_I) = |I|$, for otherwise, i.e., if $\tau(\mathcal{F}_I) < |I|$, then $\mathcal{F}$ can be covered by at most three vertices.

Case 1. $|X_1| = k$.
If $|X_2| \le k-2$, then $\sum_{i=1}^{4}(k - |X_i|) \ge 6$, and we are done. So, we may assume that $|X_2| = k-1$. In this case, for any $F \in \mathcal{F}_{12}$, $F \subset X_{12}$ holds, i.e., $F \cap X_{34} = \emptyset$. Since $\tau(\mathcal{F}_{12}) = 2$, every edge $G \in \mathcal{F}_{34}$ contains at least two vertices of X_{12}, in order to intersect with all edges in $\mathcal{F}_{12}$. Hence, we have $|X_3| \le k-2$. We may assume that $|X_3| = k-2$. Then, $V(\mathcal{F}_{123}) = X_{123}$. In particular, for every edge $F \in \mathcal{F}_{123}$, $F \cap X_4 = \emptyset$. Since $\tau(\mathcal{F}_{123}) = 3$, every edge $G \in \mathcal{F}_4$ must contain at least three vertices of X_{123}. Hence, $|X_4| \le k-3$. Thus, $\sum_{i=1}^{4}(k - |X_i|) \ge 6$ has been proved.

Case 2. $|X_1| \le k-1$.
We may assume that $|X_1| = |X_2| = |X_3| = k-1$ and that $|X_4| = k-1$ or $k-2$. Let $H \in \mathcal{F}_4$. Since $|H - X_4| \le 2$ and $\tau(\mathcal{F}_{123}) = 3$, $H - X_4$ does not cover $\mathcal{F}_{123}$. This implies that there exists an edge $F \in \mathcal{F}_{123}$ such that $F \cap H \subset X_4$. We may assume that $F \in \mathcal{F}_1$. In particular, $F \subset X_{14}$. Then, every edge $G \in \mathcal{F}_i$ $(i = 2,3)$ consists of X_i and some vertex in $F \subset X_{14}$. In this situation, it is easy to see that some edge $G \in \mathcal{F}_2$ and $G' \in \mathcal{F}_3$ do not intersect, or that $\tau(\mathcal{F}_{12})$ or $\tau(\mathcal{F}_{13})$ is one, a contradiction. ∎

Our proof of Theorem 5 is lengthy and tedious. So, we give only a part of the proof.

Proof of Theorem 5 (A Sketch) As assumed in the proof of Theorem 4, let $\mathcal{F} \subset \binom{X}{2}$ be an intersecting family with $\tau(\mathcal{F}) = 5$. Suppose that $\mathcal{F}$ is partitioned into five classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ such that for each i $(1 \le i \le 5)$, every edge $F \in \mathcal{F}_i$ contains X_i, where X_1, X_2, X_3, X_4 and X_5 are pairwise disjoint subsets of X.
We use the same notation used in the proof of Theorem 4. Also, we use the following facts.

(1) *For $I \subset \{1,2,3,4,5\}$, $\tau(\mathcal{F}_I) = |I|$ holds.*

(2) *For $F \in \mathcal{F}_i$ and $G \in \mathcal{F}_j$ $(i \ne j)$, if $F \cap (G - X_j) = \emptyset$, then $F \cap X_j \ne \emptyset$.*

(3) *Let $I \subset \{1,2,3,4,5\}$. Suppose that $V(\mathcal{F}_I) \cap X_j = \emptyset$. Then, for every $F \in \mathcal{F}_j$, $F - X_j$ covers $\mathcal{F}_I$. In particular, $|F - X_j| = k - |X_j| \ge |I|$.*

We may assume that $|X_1| \ge |X_2| \ge |X_3| \ge |X_4| \ge |X_5|$. Now, we want to show that $\sum_{i=1}^{5}(k - |X_i|) \ge 10$. So, we may also assume that $|X_1| \ge k-1$. We distinguish the following five cases.

Case 1. $|X_1| = k$ and $|X_2| = k-1$.

Case 2. $|X_1| = k$ and $|X_2| \le k-2$.

Case 3. $|X_1| = |X_2| = |X_3| = k-1$.

Case 4. $|X_1| = |X_2| = k - 1$ and $|X_3| \le k - 2$.

Case 5. $|X_1| = k - 1$ and $|X_2| \le k - 2$.

Here, we consider only the last case (Case 5), that is the most complicated case in a sense.

Now, we may assume that $|X_1| = k - 1$ and $|X_2| = |X_3| = |X_4| = |X_5| = k - 2$.

Subcase 5.1. There exists an edge $A_1 \in \mathcal{F}_1$ such that $A_1 \cap X_{2345} \ne \emptyset$.
We may assume that $A_1 = X_1 \cup \{e_1\}$ with $e_1 \in X_5$. Let E_0 be an edge in $\mathcal{F}_5$. Note that $|(E_0 - X_5) \cup \{e_1\}| = 3$. So, $(E_0 - X_5) \cup \{e_1\}$ does not cover $\mathcal{F}_{2345}$. We may assume that there exists an edge $B_1 \in \mathcal{F}_2$ such that $B_1 \cap ((E_0 - X_5) \cup \{e_1\}) = \emptyset$. This edge B_1 must intersect with A_1 and E_0. Hence, B_1 must contain a vertex a_1 of X_1 and a vertex e_2 of X_5 ($e_2 \ne e_1$), i.e., $B_1 = X_2 \cup \{a_1, e_2\}$.

Consider the set $\{a_1, e_1, e_2\}$, which does not cover $\mathcal{F}_{1345}$. We may assume that there exists an edge $C_1 \in \mathcal{F}_3$ such that $C_1 \cap \{a_1, e_1, e_2\} = \emptyset$. Since C_1 must intersect with A_1 and B_1, we can put $C_1 = X_3 \cup \{a_2, b_1\}$ with $a_2 \in X_1 - \{a_1\}$ and $b_1 \in X_2$.

The set $\{a_2, b_1, e_1\}$ does not cover $\mathcal{F}_{1245}$. So, there exists an edge $D_1 \in \mathcal{F}_4$ such that $D_1 \cap \{a_2, b_1, e_1\} = \emptyset$. Since D_1 must intersect with A_1 and C_1, D_1 must contain some vertex in X_1 and some vertex in X_3. Also, D_1 intersects with B_1, and hence D_1 must contain a_1. Let $D_1 = X_4 \cup \{a_1, c_1\}$ with $c_1 \in X_3$.

The set $\{a_1, c_1, e_1\}$ does not cover $\mathcal{F}_{1235}$. So, there exists an edge $B_2 \in \mathcal{F}_2$ such that $B_2 \cap \{a_1, c_1, e_1\} = \emptyset$. Since B_2 must intersect with A_1 and D_1, we can put $B_2 = X_2 \cup \{a, d_1\}$ where $a \in X_1 - \{a_1\}$ and $d_1 \in X_4$.

The set $\{a_2, b_1, c_1, d_1\}$ does not cover $\mathcal{F}$. So, there exists an edge $E_1 \in \mathcal{F}_5$ such that $E_1 \cap \{a_2, b_1, c_1, d_1\} = \emptyset$. Since E_1 must intersect with C_1, we can put $E_1 = X_5 \cup \{c_2, x\}$ with $c_2 \in X_3 - \{c_1\}$ and $x \notin \{a_2, b_1, c_1, d_1\}$. Now, $E_1 \cap D_1 \ne \emptyset$ and $E_1 \cap B_2 \ne \emptyset$, while $(D_1 \cup B_2) \cap (X_5 \cup \{c_2\}) = \emptyset$. Hence, $x \in D_1 \cap B_2 = \{d_1\}$. This is a contradiction.

Subcase 5.2. $V(\mathcal{F}_1) \cap X_{2345} = \emptyset$.
In this case, every edge in $\mathcal{F}_{2345}$ contains some vertex of $V(\mathcal{F}_1)$. Let $A_0 \in \mathcal{F}_1$ and $B_0 \in \mathcal{F}_2$. Since $|A_0 - X_1| = 1$ and $|B_0 - X_2| = 2$, $(A_0 - X_1) \cup (B_0 - X_2)$ does not cover $\mathcal{F}_{1345}$. So, we may assume that there exists an edge $C_1 = X_3 \cup \{a_1, b_1\} \in \mathcal{F}_3$ with $a_1 \in X_1$ and $b_1 \in X_2$. Next, the set $(A_0 - X_1) \cup \{a_1, b_1\}$ does not cover $\mathcal{F}_{1245}$. So, we may assume that there exists an edge $D_1 = X_4 \cup \{a_2, c_1\} \in \mathcal{F}_4$ with $a_2 \in X_1 - \{a_1\}$ and $c_1 \in X_3$.

The set $\{a_1, b_1, c_1\}$ does not cover $\mathcal{F}_{1235}$. So, we may assume that there exists an edge $E_1 = X_5 \cup \{c_2, x\} \in \mathcal{F}_5$ with $c_2 \in X_3 - \{c_1\}$ and $x \notin \{a_1, b_1, c_1\}$. Now, E_1 must intersect with A_0 and D_1, while $(A_0 \cup D_1) \cap (X_5 \cup \{c_2\}) = \emptyset$. Hence, $x \in A_0 \cap D_1 = \{a_2\}$, i.e., $x = a_2$. In particular, every edge $E \in \mathcal{F}_5(\overline{a_1 b_1 c_1})$ contains a_2.

Then, the set $\{a_1, a_2, b_1, c_1\}$ does not cover $\mathcal{F}$, but covers $\mathcal{F}_{1235}$. So, there exists an edge $D_2 \in \mathcal{F}_4$ such that $D_2 \cap \{a_1, a_2, b_1, c_1\} = \emptyset$. Since D_2 must intersect with A_0, C_1 and E_1, D_2 contains a vertex of A_0 and the vertex c_2. Let $D_2 = X_4 \cup \{a', c_2\}$ where $a' \in A_0 - \{a_1, a_2\}$ and $c_2 \in X_3 - \{c_1\}$. This argument implies that every edge $D \in \mathcal{F}_4(\overline{a_1 a_2 b_1})$ contains c_2.

Next, consider the set $\{a_1, a_2, b_1, c_2\}$, which does not cover $\mathcal{F}$. This set covers $\mathcal{F}_{123}$, and also by the result in the last paragraph, covers $\mathcal{F}_4$. So, there exists an edge $E_2 \in \mathcal{F}_5$ such that $E_2 \cap \{a_1, a_2, b_1, c_2\} = \emptyset$. This edge E_2 must contain the vertices a' and c_1.

Now, we can easily see that for every edge in $F \in \mathcal{F}_{45}(\overline{a_1 b_1})$ must contain one of the vertices c_1 and c_2. This implies that $\mathcal{F}$ is covered by $\{a_1, b_1, c_1, c_2\}$, a contradiction. ∎

References

[1] P. Erdős, C. Ko and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.

[2] P. Frankl. On intersecting families of finite sets. *Bull. Austral. Math. Soc.*, 21:363–372, 1980.

[3] P. Frankl, K. Ota, N. Tokushige. Uniform intersecting families with covering number four. preprint, 1992.

[4] P. Frankl, K. Ota, N. Tokushige. Covers in uniform intersecting families and a counterexample to a conjecture of Lovász. preprint, 1992.

[5] Z. Füredi. Matchings and covers in hypergraphs. *Graphs and Comb.*, 4:115–206, 1988.

[6] A. Gyárfás. Partition covers and blocking sets in hypergraphs (in Hungarian). *MTA SZTAKI Tanulmányok*, 71: 1977.

[7] A.J.W. Hilton and E.C.Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 18:369–384, 1967.