ON CROSS t-INTERSECTING FAMILIES OF SETS

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ABSTRACT. For all p,t with $0 < p < 0.11$ and $1 \le t \le 1/(2p)$, there exists n_0 such that for all n,k with $n > n_0$ and $k/n = p$ the following holds: if $\mathscr{A}$ and $\mathscr{B}$ are k-uniform families on n vertices, and $|A \cap B| \ge t$ holds for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$, then $|\mathscr{A}||\mathscr{B}| \le \binom{n-t}{k-t}^2$.

1. INTRODUCTION

Let n,k and t be integers, and let $[n] = \{1,2,\ldots,n\}$. Two families $\mathscr{F},\mathscr{G} \subset 2^{[n]}$ are called cross t-intersecting if $|F \cap G| \ge t$ holds for all $F \in \mathscr{F}, G \in \mathscr{G}$. Pyber [11] generalized the Erdős–Ko–Rado theorem [5] to cross 1-intersecting families, and the result was slightly refined by Matsumoto and Tokushige [9] and Bey [2] as follows.

Theorem 1. *Let $n \ge \max\{2k_1, 2k_2\}$. If $\mathscr{A}_1 \subset \binom{[n]}{k_1}$ and $\mathscr{A}_2 \subset \binom{[n]}{k_2}$ are cross 1-intersecting families, then $|\mathscr{A}_1||\mathscr{A}_2| \le \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}$.*

For a real $p \in (0,1)$ and a family $\mathscr{G} \subset 2^{[n]}$ we define the p-weight of $\mathscr{G}$, denoted by $w_p(\mathscr{G})$, as follows:

$$w_p(\mathscr{G}) = \sum_{G \in \mathscr{G}} p^{|G|}(1-p)^{n-|G|}.$$

Our first result is the following p-weight version of Theorem 1.

Theorem 2. *Let $p_1, p_2 \in (0,1/2)$. If $\mathscr{G}_1 \subset 2^{[n]}$ and $\mathscr{G}_2 \subset 2^{[n]}$ are cross 1-intersecting families, then $w_{p_1}(\mathscr{G}_1)w_{p_2}(\mathscr{G}_2) \le p_1 p_2$.*

Next we consider the p-weight of cross t-intersecting families for $t \ge 1$, cf. [1, 3, 4, 12, 8].

Theorem 3. *Let p be a real with $0 < p < 0.114$, and let t and n be integers with $1 \le t \le 1/(2p)$, $n \ge t$. Suppose that two families $\mathscr{G}_1 \subset 2^{[n]}$ and $\mathscr{G}_2 \subset 2^{[n]}$ are cross t-intersecting. Then we have $w_p(\mathscr{G}_1)w_p(\mathscr{G}_2) \le p^{2t}$ with equality holding iff $\mathscr{G}_1 = \mathscr{G}_2 = \{G \subset [n] : [t] \subset G\}$ (up to isomorphism).*

We conjecture that Theorem 3 is true for $0 < p \le 1/2$ and $1 \le t \le (1/p) - 1$. If $p > 1/2$, then we have $\lim_{n\to\infty} w_p(\mathscr{G}_1)w_p(\mathscr{G}_2) = 1$ for $\mathscr{G}_1 = \mathscr{G}_2 = \{G \subset [n] : 2|G| \ge n+t\}$. For $t > (1/p) - 1$, we have $w_p(\mathscr{G}_1)w_p(\mathscr{G}_2) = ((t+r)p^{t+r-1}q + p^{t+r})^2 > p^{2t}$ by taking $\mathscr{G}_1 = \mathscr{G}_2 = \{G \subset [n] : |G \cap [t+2]| \ge t+1\}$. See [10] for the case $p = 1/2$ and $t \ge 2$.

Finally we will deduce the following k-uniform version from Theorem 3.

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Theorem 4. *Let p be a real with $0<p<0.114$, and let t be an integer with $1\le t\le 1/(2p)$. For fixed p and t there exist positive constants ε, n_1 such that for all integers n,k with $n>n_1$ and $|\frac{k}{n}-p|<\varepsilon$, the following is true: if two families $\mathscr{A}_1\subset\binom{[n]}{k}$ and $\mathscr{A}_2\subset\binom{[n]}{k}$ are cross t-intersecting, then*

$$|\mathscr{A}_1||\mathscr{A}_2|\le\binom{n-t}{k-t}^2$$

with equality holding iff $\mathscr{F}_1=\mathscr{F}_2=\{F\in\binom{[n]}{k}:[t]\subset F\}$ (up to isomorphism).

Let $\mathscr{A}=\{A\in\binom{[n]}{k}:|A\cap[t+2]|\ge t+1\}$. Then $\mathscr{A}$ and $\mathscr{A}$ are cross t-intersecting and $|\mathscr{A}|>\binom{n-t}{k-t}$ iff $t+1>n/(k-t+1)$. Thus we cannot replace the condition $t\le 1/(2p)$ in Theorem 4 with $t\le 1/p$.

For the proof of our results, we will use the random walk method developed by Frankl in [6, 7], and a technique translating results about p-weight version to k-uniform version, cf. [13]. We will also include stability type results, see Theorems 5 and 6 at the ends of the following sections.

2. Proof of Theorem 2

For $i=1,2$ choose $0<\varepsilon_i<p_i$ such that $p_i+\varepsilon_i<1/2$, and let $q_i=1-p_i$, $I_i=((p_i-\varepsilon_i)n,(p_i+\varepsilon_i)n)\cap\mathbb{N}$. As the binomial distribution $B(n,p_i)$ is concentrated around p_in, we have

$$\lim_{n\to\infty}\sum_{k\in I_i}\binom{n-1}{k-1}p_i^kq_i^{n-k}=p_i,\ \text{and}\ \lim_{n\to\infty}\sum_{k\notin I_i}\binom{n}{k}p_i^kq_i^{n-k}=0.$$

Thus, considering the case $n\to\infty$, we have

$$\begin{aligned}
w_{p_1}(\mathscr{G}_1)w_{p_2}(\mathscr{G}_2) &\le \prod_{1\le i\le 2}\Big(\sum_{k\in I_i}|\mathscr{G}_i\cap\binom{[n]}{k}|p_i^kq_i^{n-k}+\sum_{k\notin I_i}\binom{n}{k}p_i^kq_i^{n-k}\Big)\\
&= \Big(\sum_{k_1\in I_1}|\mathscr{G}_1\cap\binom{[n]}{k_1}|p_1^{k_1}q_1^{n-k_1}\Big)\Big(\sum_{k_2\in I_2}|\mathscr{G}_2\cap\binom{[n]}{k_2}|p_2^{k_2}q_2^{n-k_2}\Big)+o(1)\\
&\underset{\text{Thm 1}}{\le} \sum_{k_1\in I_1}\sum_{k_2\in I_2}\binom{n-1}{k_1-1}\binom{n-1}{k_2-1}p_1^{k_1}q_1^{n-k_1}p_2^{k_2}q_2^{n-k_2}+o(1)\\
&= \Big(\sum_{k_1\in I_1}\binom{n-1}{k_1-1}p_1^{k_1}q_1^{n-k_1}\Big)\Big(\sum_{k_2\in I_2}\binom{n-1}{k_2-1}p_2^{k_2}q_2^{n-k_2}\Big)+o(1)\\
&= p_1p_2+o(1). \qquad (1)
\end{aligned}$$

Now suppose that for some n there exist cross 1-intersecting families $\mathscr{G}_1,\mathscr{G}_2\subset 2^{[n]}$ with $w_{p_1}(\mathscr{G}_1)w_{p_2}(\mathscr{G}_2)>p_1p_2$. Set $\mathscr{G}_i'=\mathscr{G}_i\cup\{G\cup\{n+1\}:G\in\mathscr{G}_i\}$ for $i=1,2$ then $\mathscr{G}_1'$ and $\mathscr{G}_2'$ are cross 1-intersecting. Since $w_{p_i}(\mathscr{G}_i')=w_{p_i}(\mathscr{G}_i)(p_i+q_i)=w_{p_i}(\mathscr{G}_i)$ we have $w_{p_1}(\mathscr{G}_1')w_{p_2}(\mathscr{G}_2')>p_1p_2$, which contradicts (1). ☐

3. Proof of Theorem 3

Let us recall some basic facts about shifting from [7]. Let $p\in(0,1)$, and $\mathscr{F},\mathscr{G}\subset 2^{[n]}$. For integers $1\le i<j\le n$, we define the (i,j)-shift S_{ij} of $\mathscr{F}$ as follows:

$$S_{ij}(\mathscr{F})=\{S_{ij}(F):F\in\mathscr{F}\},$$

where

$$S_{ij}(F) = \begin{cases} (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F,\ (F - \{j\}) \cup \{i\} \notin \mathscr{F}, \\ F & \text{otherwise.} \end{cases}$$

Then $\mathscr{F}$ is called shifted if $S_{ij}(\mathscr{F}) = \mathscr{F}$ for all $1 \le i < j \le n$. One can easy to show that if $\mathscr{F}$ and $\mathscr{G}$ are cross t-intersecting, then so are $S_{ij}(\mathscr{F})$ and $S_{ij}(\mathscr{G})$. By repeating this process, one can eventually get shifted cross t-intersecting families $\mathscr{F}'$ and $\mathscr{G}'$ on the same vertex set without changing profile vectors (and therefore $w_p(\mathscr{F}) = w_p(\mathscr{F}')$, $w_p(\mathscr{G}) = w_p(\mathscr{G}')$ and $|\mathscr{F}| = |\mathscr{F}'|$, $|\mathscr{G}| = |\mathscr{G}'|$).

For $F \subset [n]$ we define the corresponding n-step walk on $\mathbb{Z}^2$, denoted by walk(F), as follows. The walk is from $(0,0)$ to $(|F|, n - |F|)$, and the i-th step is one unit up ($\uparrow$) if $i \in F$, or one unit to the right ($\rightarrow$) if $i \notin F$. Let $\lambda(\mathscr{F})$ be the maximum $u \in \mathbb{N}$ such that walk(F) touches the line $y = x + u$ for all $F \in \mathscr{F}$. Frankl [7] observed the following.

Lemma 1. *If $\mathscr{F}$ and $\mathscr{G}$ are shifted cross t-intersecting families, then $\lambda(\mathscr{F}) + \lambda(\mathscr{G}) \ge 2t$.*

Consider the infinite random walk in $\mathbb{Z}^2$ starting from $(x_0, y_0) \in \mathbb{Z}^2$, taking $\uparrow$ with probability p, and $\rightarrow$ with probability $q = 1 - p$ at each step independently. The random walk method is based on the following simple observation.

Lemma 2. *Let $\mathscr{A} \subset 2^{[n]}$ be a set of subsets A such that* walk(A) *satisfies some given property* P. *Then, the p-weight $w_p(\mathscr{A})$ is bounded from above by the probability that the (infinite) random walk satisfies* P *in the first n steps.*

Let us see an important example of Lemma 2. Suppose that $\mathscr{F}$ and $\mathscr{G}$ are cross t-intersecting. If $s \in \mathbb{Z}$ and $y_0 \le x_0 + s$, then the random walk (starting from (x_0, y_0)) hits the line $y = x + s$ with probability $\alpha^{s + x_0 - y_0}$, where $\alpha = p/q$ (see [7, 13]). Applying this to the case "$x_0 = y_0 = 0$ and $s = \lambda(\mathscr{F})$," we have $w_p(\mathscr{F}) \le \alpha^{\lambda(\mathscr{F})}$, because $w_p(\mathscr{F})$ is bounded from above by the probability that the random walk (starting from the origin) hits the line $y = x + \lambda(\mathscr{F})$ within the first n steps. Similarly we have $w_p(\mathscr{G}) \le \alpha^{\lambda(\mathscr{G})}$. Then Lemma 1 gives $w_p(\mathscr{F}) w_p(\mathscr{G}) \le \alpha^{\lambda(\mathscr{F}) + \lambda(\mathscr{G})} \le \alpha^{2t}$. This gives already a good upper bound for the product of p-weights, but the bound α^{2t} can be replaced with p^{2t} as we will show below.

Proof of Theorem 3. Let p be given and let $1 \le t \le \frac{1}{2p}$. Let $\mathscr{F} = \mathscr{G}_1$ and $\mathscr{G} = \mathscr{G}_2$ be cross t-intersecting families on $[n]$. We may assume that both families are shifted and p-weight maximal. (By the p-weight maximality, we notice that $F \in \mathscr{F}$ and $F \subset F'$ imply $F' \in \mathscr{F}$.) Let $q = 1 - p$, $\alpha = p/q$, $u = \lambda(\mathscr{F})$ and $v = \lambda(\mathscr{G})$. Then, as mentioned above, we have $w_p(\mathscr{F}) \le \alpha^u$, $w_p(\mathscr{G}) \le \alpha^v$, and $u + v \ge 2t$. We will show that $w_p(\mathscr{F}) w_p(\mathscr{G}) \le p^{2t}$ for $p < 0.114$ by case-wise analysis, and we will try to find better condition for p (than 0.114) in each case.

If $u + v \ge 2t + 1$, then we have $w_p(\mathscr{F}) w_p(\mathscr{G}) \le \alpha^{u+v} \le \alpha^{2t+1}$. Since $f(t) := \alpha(\alpha/p)^{2t} = (p/q) q^{-2t}$ is an increasing function of t, we have $f(t) \le f(\frac{1}{2p})$. Then a simple computation shows $f(\frac{1}{2p}) < 1$ for $p < 0.241$. Namely, we have

$$w_p(\mathscr{F}) w_p(\mathscr{G}) \le \alpha^{2t+1} = p^{2t} f(t) < p^{2t} \tag{2}$$

for $p < 0.241$ and $t \le \frac{1}{2p}$. Thus we may assume that $u + v = 2t$, and $1 \le u \le t \le v$.

Let us define families $\mathscr{H}_0^u, \mathscr{H}_1^u \subset 2^{[n]}$ by

$$\begin{aligned}\mathscr{H}_0^u &= \{H \subset [n] : [u] \subset H\},\\ \mathscr{H}_1^u &= \{H \in 2^{[n]} \setminus \mathscr{H}_0^u : |H \cap [u+2]| = u+1\}.\end{aligned}$$

In other words, if $H \in \mathscr{H}_0^u$ then walk(H) hits $(0,u)$, and if $H \in \mathscr{H}_1^u$ then walk(H) hits $(1,u+1)$ without hitting $(0,u)$. We define $\mathscr{H}_0^v$ and $\mathscr{H}_1^v$ similarly. For $i \geq 0$ we define special subsets $A_i^u \in \mathscr{H}_0^u$ and $B_i^u \in \mathscr{H}_1^u$ as in Figure 1 by

$$\begin{aligned}A_i^u &= ([u] \cup \{u+i+2j+1 : j \geq 1\}) \cap [n],\\ B_i^u &= ([u-1] \cup \{u+1, u+2\} \cup \{u+i+2j+3 : j \geq 1\}) \cap [n].\end{aligned}$$

Set $A^u = A_0^u = [n] \setminus \{u+1, u+2, u+4, u+6, \ldots\}$, $B^u = B_0^u = [n] \setminus \{u, u+3, u+4, u+6, \ldots\}$. Consider a walk which satisfies that

(i) it does not cross the line $y = x+u$, and
(ii) it hits the line only at $(0,u)$.

Then, walk(A^u) is the maximal walk with these properties, namely, if walk(A) satisfies (i) and (ii), then we can find an $A' \subset A^u$ such that A' is obtained from A by shifting. (In fact, if $|A| = u+\ell$, then A' is uniquely determined by $A' = [u] \cup \{n+2j+1 : 1 \leq j \leq \ell\}$.) Similarly, walk$(B^u)$ is the maximal walk which does not cross the line $y = x+u$, and hits the line only at $(1,u+1)$. We will look at the structure of $\mathscr{F}$ and $\mathscr{G}$ using A^u and B^u. Let $\mathscr{F}_\ell^u = \mathscr{F} \cap \mathscr{H}_\ell^u$ and $\mathscr{G}_\ell^v = \mathscr{G} \cap \mathscr{H}_\ell^v$ for $\ell = 1,2$.

Case 1. *$A^u \notin \mathscr{F}$ and $B^u \notin \mathscr{F}$.*

First let $F \in \mathscr{F}_0^u$. Then $F \supset [u]$, and walk(F) must reach $(0,u)$. The next step goes to $(0,u+1)$ or $(1,u)$. If walk(F) reaches $(1,u)$, then the walk will hit the line $y = x+u$ after passing $(1,u)$. (Otherwise $A' = [u] \cup \{n+2j+1 : 1 \leq j \leq |F|-u\} \subset A^u$ can be obtained from F by shifting, but then it follows from the p-weight maximality that $A^u \in \mathscr{F}$, a contradiction.) In summary, walk(F) has one of the following two possibilities:

walk(F) reaches $(0,u+1)$,

or

walk(F) reaches $(1,u)$ and then it hits the line $y = x+u$.

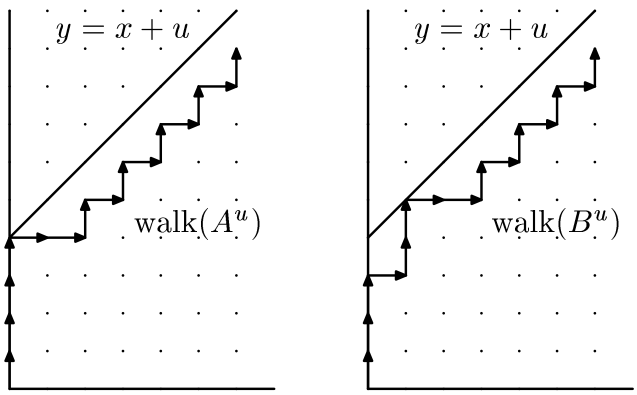

FIGURE 1.

The former occurs with probability p^{u+1}. The latter occurs with probability at most $p^u q\alpha$, where $p^u q$ is the probability that the random walk reaches $(1,u)$ after passing through $(0,u)$, and α is the upper bound for the probability that the random walk starting from $(1,u)$ hits the line $y=x+u$, or equivalently, the random walk starting from the origin hits the line $y=x+1$. Thus we have

$$w_p(\mathscr{F}_0^u) \le p^{u+1} + p^u q\alpha = 2p^{u+1}. \tag{3}$$

Next let $F \in \mathscr{F}_1^u$. Since $B^u \notin \mathscr{F}$, we find that

walk(F) reaches $(1,u+2)$,

or

walk(F) reaches $(2,u+1)$ and then it hits the line $y=x+u$.

The former occurs with probability $up^{u+2}q$, because there are u ways of walks from the origin to $(1,u+2)$ without hitting $(0,u)$. The latter occurs with probability at most $up^{u+1}q^2\alpha$, where $up^{u+1}q^2$ is the probability that the random walk reaches $(2,u+1)$ passing through $(1,u+1)$ without hitting $(0,u)$, and α is the upper bound for the probability that the random walk starting from $(2,u+1)$ hits the line $y=x+u$. Thus we have

$$w_p(\mathscr{F}_1^u) \le up^{u+2}q + up^{u+1}q^2\alpha = 2up^{u+2}q.$$

Finally let $F \in \mathscr{F} \setminus (\mathscr{F}_0^u \cup \mathscr{F}_1^u)$. Then walk$(F)$ hits the line $y=x+u$ without hitting $(0,u)$ nor $(1,u+1)$, and this occurs with probability at most $\alpha^u - (p^u + up^{u+1}q)$. Therefore we have

$$\begin{aligned} w_p(\mathscr{F}) &= w_p(\mathscr{F} \setminus (\mathscr{F}_0^u \cup \mathscr{F}_1^u)) + w_p(\mathscr{F}_0^u) + w_p(\mathscr{F}_1^u) \\ &\le \alpha^u - (p^u + up^{u+1}q) + 2p^{u+1} + 2up^{u+2}q \\ &= p^u(q^{-u} - (1-2p)(1+upq)). \end{aligned}$$

For $\mathscr{G}$ we use a trivial upper bound $w_p(\mathscr{G}) \le \alpha^v = p^{2t-u}q^{-2t+u}$. Consequently we have

$$w_p(\mathscr{F})w_p(\mathscr{G}) \le p^{2t}(q^{-u} - (1-2p)(1+upq))q^{-2t+u} := p^{2t} f(u,t).$$

Noting that $-\log q > p$, one can verify that $\frac{\partial}{\partial u} f(u,t) > 0$ and $\frac{\partial}{\partial t} f(t,t) > 0$. Thus we have $f(u,t) \le f(t,t) \le f(\frac{1}{2p}, \frac{1}{2p})$. Finally, for $p \le 0.1144$, we have $f(\frac{1}{2p}, \frac{1}{2p}) < 1$, which gives $w_p(\mathscr{F})w_p(\mathscr{G}) < p^{2t}$.

Case 2. *$A^u \notin \mathscr{F}$ and $B^u \in \mathscr{F}$.*

Using (3), we have

$$w_p(\mathscr{F}) = w_p(\mathscr{F} \setminus \mathscr{F}_0^u) + w_p(\mathscr{F}_0^u) \le (\alpha^u - p^u) + 2p^{u+1} = p^u(q^{-u} - (1-2p)). \tag{4}$$

Suppose that $B_i^u \in \mathscr{F}$ for some $i \ge 0$. We will find C with $|B_i^u \cap C| < t$. Then the cross t-intersecting property implies $C \notin \mathscr{G}$, which will give an upper bound for $w_p(\mathscr{G})$.

If $u = t-1$, then let

$$C = ([t+i+3] - \{t+1\}) \cup \{t+i+3+2j : j \ge 1\}.$$

Since $|B_i^u \cap C| = t-1$ we have $C \notin \mathscr{G}$. Let $G \in \mathscr{G}$. Since $C \notin \mathscr{G}$, we find that

walk(G) reaches $(0,t+1)$,

or

walk(G) does not reach $(0,t+1)$ and it hits the line $y=x+t+i+2$.

In the latter case, walk(G) must hit one of $(j,t+1-j)$, where $1\le j\le t+1$. The probability that the random walk starting from $(j,t+1-j)$ hits the line $y=x+t+i+2$ is at most α^{i+2j+1}. Thus the latter case occurs with probability at most

$$\sum_{j=1}^{t+1}\binom{t+1}{j}p^{t+1-j}q^j\alpha^{i+2j+1}=\alpha^{t+i+2}-p^{t+1}\alpha^{i+1},$$

where we used $\sum_{j=0}^{t+1}\binom{t+1}{j}p^{t+1-j}q^j\alpha^{i+2j+1}=(p/q)^{t+i+2}\sum_{j=0}^{t+1}\binom{t+1}{j}p^jq^{t+t-j}=\alpha^{t+i+2}$. Thus we have

$$\begin{aligned}w_p(\mathscr{G})&\le p^{t+1}+(\alpha^{t+i+2}-p^{t+1}\alpha^{i+1})=p^{t+1}(1+\alpha^{i+1}(q^{-t-1}-1))\\&\le p^{t+1}(1+\alpha(q^{-t-1}-1)).\end{aligned}\tag{5}$$

By (4) and (5), we have

$$w_p(\mathscr{F})w_p(\mathscr{G})\le p^{2t}(q^{-t+1}-(1-2p))(1+\alpha(q^{-t-1}-1))=:p^{2t}f(t).$$

Then a direct computation shows $f(t)\le f(\frac{1}{2p})<1$ for $p\le 0.188$.

If $u<t-1$, then let

$$C=[2t-u+i]\cup\{2t-u+i+2j:j\ge 1\}.$$

Since $|B_i^u\cap C|<t$ we have $C\notin\mathscr{G}$. So, for $G\in\mathscr{G}$, walk(G) hits the line $y=x+2t-u+i+1$. Thus we have $w_p(\mathscr{G})\le\alpha^{2t-u+i+1}\le\alpha^{2t-u+1}$. This together with (4) gives

$$w_p(\mathscr{F})w_p(\mathscr{G})\le p^u(q^{-u}-(1-2p))\alpha^{2t-u+1}=p^{2t}(q^{-u}-(1-2p))pq^{-2t+u-1}:=p^{2t}f(u,t).$$

Then a computation shows $f(u,t)\le f(t-2,t)\le f(\frac{1}{2p}-2,\frac{1}{2p})<1$ for $p\le 0.333$.

If $u=v=t$, then let

$$C=([t+i+4]\setminus\{t+1,t+2\})\cup\{t+i+4+2j:j\ge 1\}.$$

Let $G\in\mathscr{G}$. Since $C\notin\mathscr{G}$ we find that

walk(G) reaches $(0,t+1)$ or $(1,t+1)$,

or

walk(G) hits the line $y=x+t+i+1$ in $x\ge 2$.

Thus we have

$$w_p(\mathscr{G})\le p^{t+1}+(t+1)p^{t+1}q+\alpha^{t+i+1}<p^{t+1}(1+(t+1)q+\alpha q^{-t-1}).$$

This together with (4) implies

$$w_p(\mathscr{F})w_p(\mathscr{G})<p^{2t}(q^{-t}-(1-2p))p(1+(t+1)q+\alpha q^{-t-1})=:p^{2t}f(t).$$

Then we have $f(t)\le f(\frac{1}{2p})<1$ for $p<0.2$.

Case 3. *$A^u\in\mathscr{F}$ and $u<v$.*

Suppose that $A_i^u \in \mathscr{F}$ for some $i \geq 0$. Let

$$C = [2t-u+i] \cup \{2t-u+i+2j : j \geq 1\}.$$

Then we find $|A_i^u \cap C| < t$ and $C \notin \mathscr{G}$. Thus, for $G \in \mathscr{G}$, walk(G) hits the line $y = x+2t-u+i+1$ and

$$w_p(\mathscr{G}) \leq \alpha^{2t-u+i+1} \leq \alpha^{2t-u+1}.$$

Thus we have $w_p(\mathscr{F})w_p(\mathscr{G}) \leq \alpha^u \alpha^{2t-u+1} = \alpha^{2t+1} < p^{2t}$ for $p < 0.241$ by (2).

Case 4. *$u = v = t$ and $A^t \in \mathscr{F}$, $A^t \notin \mathscr{G}$.*

Since $A^t \notin \mathscr{G}$, it follows from (3) that $w_p(\mathscr{G}_0^t) \leq 2p^{u+1} = p^t(2p)$.

First suppose that $[t] \in \mathscr{F}$. Then, the cross t-intersecting property implies that $[t] \subset G$ for all $G \in \mathscr{G}$, and so $\mathscr{G} \setminus \mathscr{G}_0^t = \emptyset$.

Next suppose that $A_i^t \in \mathscr{F}$ and $A_{i+1}^t \notin \mathscr{F}$ for some $i \geq 0$. Let

$$C = ([t+i+2] \setminus \{t\}) \cup \{t+i+2+2\ell : \ell \geq 1\}.$$

Then we have $|A_i^t \cap C| < t$. Since $A_i^t \in \mathscr{F}$ we have $C \notin \mathscr{G}$. Thus, for $G \in \mathscr{G} \setminus \mathscr{G}_0^t$, walk$(G)$ hits the line $y = x+t+i+1$ in $x \geq 1$, and

$$w_p(\mathscr{G} \setminus \mathscr{G}_0^t) \leq \alpha^{t+i+1} - p^t \alpha^{i+1} = p^t \alpha^{i+1}(q^{-t}-1) \leq p^t \alpha (q^{-t}-1).$$

Thus, in both cases, we have $w_p(\mathscr{G}) = w_p(\mathscr{G}_0^t) + w_p(\mathscr{G} \setminus \mathscr{G}_0^t) \leq p^t(2p + \alpha(q^{-t}-1))$ and

$$w_p(\mathscr{F})w_p(\mathscr{G}) \leq \alpha^t p^t (2p+\alpha(q^{-t}-1)) = p^{2t} q^{-t}(2p+\alpha(q^{-t}-1)) =: p^{2t} f(t).$$

Then a computation shows $f(t) \leq f(\frac{1}{2p}) < 1$ for $p < 0.195$.

Case 5. *$u = v = t$ and $A^t \in \mathscr{F}$, $A^t \in \mathscr{G}$.*

First suppose that $[t] \notin \mathscr{F}$ and $[t] \notin \mathscr{G}$. Then we can choose $i, j \geq 0$ so that $A_i^t \in \mathscr{F}$, $A_{i+1}^t \notin \mathscr{F}$, $A_j^t \in \mathscr{G}$ and $A_{j+1}^t \notin \mathscr{G}$. Let $F \in \mathscr{F}_0^t$. If walk(F) reaches $(i+2, t)$, then, using $A_{i+1}^t \notin \mathscr{F}$, we find that this walk hits the line $y = x+t-i-1$ in $x \geq i+2$. This gives

$$w_p(\mathscr{F}_0^t) \leq (p^t - p^t q^{i+2}) + p^t q^{i+2} \alpha = p^t(1-q^{i+1}(1-2p)). \tag{6}$$

Let

$$C = ([t+j+2] \setminus \{t\}) \cup \{t+j+2+2\ell : \ell \geq 1\}.$$

Then we have $|A_j^t \cap C| < t$. Since $A_j^t \in \mathscr{G}$ we have $C \notin \mathscr{F}$. Thus, for $F \in \mathscr{F} \setminus \mathscr{F}_0^t$, walk$(F)$ hits the line $y = x+t+j+1$ in $x \geq 1$, and

$$w_p(\mathscr{F} \setminus \mathscr{F}_0^t) \leq \alpha^{t+j+1} - p^t \alpha^{j+1} = p^t \alpha^{j+1}(q^{-t}-1). \tag{7}$$

Therefore we have

$$w_p(\mathscr{F}) = w_p(\mathscr{F}_0^t) + w_p(\mathscr{F} \setminus \mathscr{F}_0^t) \leq p^t(1-q^{i+1}(1-2p) + \alpha^{j+1}(q^{-t}-1)). \tag{8}$$

We use

$$q^{-t} \leq q^{1/(2p)} \leq 2 \tag{9}$$

for $0 \leq p \leq 1/2$. Then, for $p \leq 1/4$, the RHS of (8) is less than $p^t c_{i,j}$, where $c_{i,j} = 1 - q^{i+1}/2 + \alpha^{j+1}$. In the same way, we also have $w_p(\mathscr{G}) \leq p^t c_{j,i}$. Now it suffices to show $c_{i,j}c_{j,i} < 1$, or equivalently, $\log c_{i,j} + \log c_{j,i} < 0$. Using $\log(1+x) < x$, we have $\log c_{i,j} + \log c_{j,i} < (c_{i,j}-1) + (c_{j,i}-1) = -q^{i+1}/2 + \alpha^{i+1} - q^{j+1}/2 + \alpha^{j+1}$. By symmetry it suffices to show $-q^{i+1}/2+\alpha^{i+1}<0$, or equivalently, $2<(q^2/p)^{i+1}$. For $p\le 1/4$ we certainly have $2<q^2/p<(q^2/p)^{i+1}$.

Next suppose that $[t]\in\mathscr{F}$ and $[t]\notin\mathscr{G}$. Choose $j\ge 0$ so that $A^t_j\in\mathscr{G}$ and $A^t_{j+1}\notin\mathscr{G}$. Since $[t]\in\mathscr{F}$, we have $\mathscr{G}\setminus\mathscr{G}^t_0=\emptyset$. Then, using the same reasoning as we get (6), we have

$$w_p(\mathscr{G})=w_p(\mathscr{G}^t_0)+w_p(\mathscr{G}\setminus\mathscr{G}^t_0)=w_p(\mathscr{G}_0)\le p^t(1-q^{j+1}(1-2p)). \tag{10}$$

Using a trivial bound $w_p(\mathscr{F}^t_0)\le p^t$ and (7) with (9), we have

$$w_p(\mathscr{F})=w_p(\mathscr{F}^t_0)+w_p(\mathscr{F}\setminus\mathscr{F}^t_0)\le p^t(1+\alpha^{j+1}(q^{-t}-1))\le p^t(1+\alpha^{j+1}). \tag{11}$$

We will show $w_p(\mathscr{F})w_p(\mathscr{G})<p^{2t}$, or $(1-q^{j+1}(1-2p))(1+\alpha^{j+1})<1$. For this, it suffices to show $\alpha^{j+1}<q^{j+1}(1-2p)$, or equivalently, $1/(1-2p)<(q^2/p)^{j+1}$. For $p\le 1/4$, we have $1/(1-2p)\le 2<(q^2/p)^{j+1}$, as desired.

Finally suppose that $[t]\in\mathscr{F}$ and $[t]\in\mathscr{G}$. Then we have $\mathscr{F}=\mathscr{F}^t_0$ and $\mathscr{G}=\mathscr{G}^t_0$. Thus we have $w_p(\mathscr{F})w_p(\mathscr{G})\le p^{2t}$ with equality holding iff $\mathscr{F}=\mathscr{G}=\{F\subset[n]:[t]\subset F\}$. For later use, we notice that this is the only case we have equality in our target inequality. This completes the proof of Theorem 3. $\square$

For the proof of Theorem 3 we only needed to show $w_p(\mathscr{F})w_p(\mathscr{G})\le p^{2t}$, but actually we have proved slightly more. Namely, in Cases 1–4, our proof shows

$$w_p(\mathscr{F})w_p(\mathscr{G})<(0.999p^t)^2, \tag{12}$$

which we will use to prove Theorem 4.

On the other hand, in Case 5, we see that if $[t]\in\mathscr{F}$ then $w_p(\mathscr{G}\setminus\mathscr{G}^t_0)=0$; if $[t]\notin\mathscr{F}$ then there is some i such that $A^t_i\in\mathscr{F}$ and $A^t_{i+1}\notin\mathscr{F}$, which implies

$$w_p(\mathscr{F}^t_0)\le p^t(1-q^{i+1}(1-2p))\text{ and }w_p(\mathscr{G}\setminus\mathscr{G}^t_0)\le p^t\alpha^{i+1}(q^{-t}-1)<p^t\alpha^{i+1}, \tag{13}$$

cf. (6), (7). So, if $w_p(\mathscr{F}^t_0)$ is large, then i needs to be large, and $w_p(\mathscr{G}\setminus\mathscr{G}^t_0)$ is small. In fact, for every $\varepsilon>0$ we can find some $\delta=\delta(\varepsilon)>\varepsilon$ such that if $w_p(\mathscr{F}^t_0)>(1-\delta)p^t$ then $w_p(\mathscr{G}\setminus\mathscr{G}^t_0)<\varepsilon p^t$. To see this, let $\delta=q^{\log\varepsilon/\log\alpha}(1-2p)$ and suppose, on the contrary, that $\varepsilon p^t\le w_p(\mathscr{G}\setminus\mathscr{G}^t_0)<\alpha^{i+1}p^t$. Then, by (13), we have $w_p(\mathscr{F}^t_0)\le(1-\delta(\alpha^{i+1}))p^t<(1-\delta(\varepsilon))p^t$. We can summarize this observation as the following stability type statement.

Theorem 5. *Let p be a real with $0<p<0.114$, and let t and n be integers with $1\le t\le 1/(2p)$, $n\ge t$. Suppose that $\mathscr{G}_1\subset 2^{[n]}$ and $\mathscr{G}_2\subset 2^{[n]}$ are shifted cross t-intersecting families. Then, for any $\varepsilon\in(0,p]$ there exists $\gamma>0$ such that if $w_p(\mathscr{G}_1)w_p(\mathscr{G}_2)>(1-\gamma)p^{2t}$, then $w_p(\mathscr{G}_1\setminus\mathscr{G}'_1)w_p(\mathscr{G}_2\setminus\mathscr{G}'_2)<\varepsilon p^{2t}$, where $\mathscr{G}'_i=\{G\in\mathscr{G}_i:[t]\subset G\}$ for $i=1,2$.*

Sketch of proof. Let ε be given. Choose γ so that $1-\gamma=\max_{\varepsilon\le\alpha\beta\le p}(1-\delta(\alpha)+\alpha)(1-\delta(\beta)+\beta)$. Let $w_p(\mathscr{G}_i\setminus\mathscr{G}'_i)=\varepsilon_i p^t$ for $i=1,2$. Suppose, on the contrary, that $\varepsilon_1\varepsilon_2\ge\varepsilon$. Then, by the remark after (13), we have $w_p(\mathscr{G}_i)\le(1-\delta(\varepsilon_i)+\varepsilon_i)p^t$, and $w_p(\mathscr{G}_1)w_p(\mathscr{G}_2)\le(1-\gamma)p^{2t}$. $\square$

4. Proof of Theorem 4

Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be k-uniform shifted cross t-intersecting families on $[n]$. Let $q=1-p$, $\alpha=p/q$, $u=\lambda(\mathscr{A}_1)$ and $v=\lambda(\mathscr{A}_2)$. We start with the case corresponding to Case 5 in the proof of Theorem 3 and we borrow notation used there. In this part we will just translate verbatim what we did for p-weight version to k-uniform version. For $A = \{a_1, a_2, \dots, a_k, \dots\}$ with $a_1 < a_2 < \cdots$, let $\mathrm{first}(A) = \{a_1, a_2, \dots, a_k\}$ be consisting of the first k elements of A. Let $K^t = \mathrm{first}(A^t)$ and $T = \mathrm{first}(A^t_{k-t-1})$, which will play a role of A^t and $[t]$ in the p-weight version, respectively. We consider the case that

$$u = v = t \text{ and } K^t \in \mathscr{A}_1, K^t \in \mathscr{A}_2. \tag{14}$$

First suppose that $T \notin \mathscr{A}_1$ and $T \notin \mathscr{A}_2$. Let $\mathscr{A}_1' = \{A \in \mathscr{A}_1 : [t] \subset A\}$. Then in this k-uniform version, (6) reads as follows:

$$|\mathscr{A}_1'| \le \tbinom{n-t}{k-t} - \tbinom{n-t-i-2}{k-t} + \tbinom{n-t-i-2}{k-t-1}, \tag{15}$$

where we used the reflection principle to count the number of walks touching the line. Also, (7) reads

$$|\mathscr{A}_1 \setminus \mathscr{A}_1'| \le \tbinom{n}{k-t-j-1} - \tbinom{n-t}{k-t-j-1} \le \tbinom{n}{k-t-j-1}. \tag{16}$$

By (15) and (16) we have

$$|\mathscr{A}_1| / \tbinom{n-t}{k-t} \le 1 + \{ -\tbinom{n-t-i-2}{k-t} + \tbinom{n-t-i-2}{k-t-1} + \tbinom{n}{k-t-j-1} \} / \tbinom{n-t}{k-t} =: 1 + c_{i,j}.$$

In the same way, we have $|\mathscr{A}_2| / \tbinom{n-t}{k-t} \le 1 + c_{j,i}$. We need to show $(1 + c_{i,j})(1 + c_{j,i}) \le 1$, or $\log(1 + c_{i,j}) + \log(1 + c_{j,i}) \le 0$. Using $\log(1 + x) < x$, it suffices to show $c_{i,j} + c_{j,i} \le 0$, and by symmetry this follows from

$$\tbinom{n}{k-t-i-1} \le \tbinom{n-t-i-2}{k-t} - \tbinom{n-t-i-2}{k-t-1}, \tag{17}$$

which can be verified for $p = k/n \le 0.17$ and $t \le 1/(2p)$ by standard calculation. Frankl proved (17) in [7].

Next suppose that $T \in \mathscr{A}_1$ and $T \notin \mathscr{A}_2$. Notice that $T \in \mathscr{A}_1$ implies $\mathscr{A}_2 = \mathscr{A}_2'$. In this subcase, (10) and (11) read as

$$|\mathscr{A}_2| = |\mathscr{A}_2'| + |\mathscr{A}_2 \setminus \mathscr{A}_2'| = |\mathscr{A}_2'| \le \tbinom{n-t}{k-t} - \tbinom{n-t-j-2}{k-t} + \tbinom{n-t-j-2}{k-t-1},$$
$$|\mathscr{A}_1| = |\mathscr{A}_1'| + |\mathscr{A}_1 \setminus \mathscr{A}_1'| \le \tbinom{n-t}{k-t} + \tbinom{n}{k-t-j-1}.$$

Then, $|\mathscr{A}_1||\mathscr{A}_2| < \tbinom{n-t}{k-t}^2$ follows from (17).

Finally suppose that $T \in \mathscr{A}_1$ and $T \in \mathscr{A}_2$. Then we have $\mathscr{A}_1 = \mathscr{A}_1'$ and $\mathscr{A}_2 = \mathscr{A}_2'$. Thus we have $|\mathscr{A}_1||\mathscr{A}_2| \le \tbinom{n-t}{k-t}^2$ with equality holding iff $\mathscr{A}_1 = \mathscr{A}_2 = \{A \in \tbinom{[n]}{k} : [t] \subset A\}$. So far, this is the only case we have equality in our target inequality, and we will see that equality never holds in the remaining cases below.

Now we consider the situation corresponding to Cases 1–4. Namely we assume the negation of (14). For $s = 1, 2$ let $\mathscr{G}_s$ be the collection of all upper shadows of $\mathscr{A}_s$, that is, $\mathscr{G}_s = \bigcup_{k \le j \le n} (\nabla_j(\mathscr{A}_s))$, where $\nabla_j(\mathscr{A}_s) = \{H \in \tbinom{[n]}{j} : H \supset \exists F \in \mathscr{A}_s\}$. Then $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfy one of Cases 1–4, and we get (12). We only use the following weaker claim.

Claim 1. *Let $0 < p < 0.114$ and $1 \le t \le 1/(2p)$ be fixed. Let $n \ge t$ and let $\mathscr{G}_1, \mathscr{G}_2 \subset 2^{[n]}$ be cross t-intersecting families corresponding to Cases 1–4. Then there exist $\gamma, \varepsilon > 0$ such that $w_x(\mathscr{G}_1) w_x(\mathscr{G}_2) < (1-\gamma) x^{2t}$ holds for all x with $|x - p| < \varepsilon$.*

To complete the proof of Theorem 4, it suffices to show the following.

Claim 2. *Let $0 < p < 0.114$ and $1 \le t \le 1/(2p)$ be fixed. Then there exist $\gamma, \varepsilon > 0$ and n_0 such that the following holds for all $n,k \in \mathbb{N}$ with $n > n_0$ and $|\frac{k}{n} - p| < \varepsilon$: If $\mathscr{A}_1 \subset \binom{[n]}{k}$ and $\mathscr{A}_2 \subset \binom{[n]}{k}$ are cross t-intersecting families corresponding to Cases 1–4, then $|\mathscr{A}_1||\mathscr{A}_2| < (1-\gamma)\binom{n-t}{k-t}^2$.*

Proof of Claim 2. Assume the negation of Claim 2. Then the statement starts with

$$\underline{\exists p}\ \underline{\exists t}\ \forall\gamma\,\forall\varepsilon\,\forall n_0\ \underline{\exists n}\ \underline{\exists k}\ \cdots, \tag{18}$$

where the underlines will indicate the choice of parameters described below. We will construct a counterexample to Claim 1 using (18). Recall that Claim 1 starts with

$$\forall p\,\forall t\ \underline{\exists\gamma}\ \underline{\exists\varepsilon}\ \cdots. \tag{19}$$

First, assuming the negation of Claim 2, there exists some p and t (corresponding to the first and second underlines in (18)) such that the rest of Claim 2 does not hold. For this p and t, Claim 1 provides some $\gamma_0 = \gamma_0(p,t)$ and $\varepsilon_0 = \varepsilon_0(p,t)$ (corresponding to the third and fourth underlines in (19)) such that

$$w_x(\mathscr{G}_1)w_x(\mathscr{G}_2) < (1-\gamma_0)x^{2t} \tag{20}$$

holds for all x with $|x-p| < \varepsilon_0$.

For reals $0 < \varepsilon \ll p$ we write $p \pm \varepsilon$ to mean the open interval $(p-\varepsilon, p+\varepsilon)$. Since we have fixed p and t, we note that $f(x) := x^{2t}$ is a uniformly continuous function of x on $p \pm \varepsilon_0$. Let $\varepsilon = \varepsilon_0/2$, $\gamma = \gamma_0/4$, and $X = p \pm \varepsilon$. Now we are going to define n_0. Choose $\varepsilon_1 \ll \varepsilon$ so that

$$(1-3\gamma)f(x) > (1-4\gamma)f(x+\delta) \tag{21}$$

holds for all $x \in X$ and all $0 < \delta \le \varepsilon_1$. As the binomial distribution $B(n,p)$ is concentrated around pn, we can choose n_1 so that

$$\sum_{j\in J}\binom{n}{j}y^j(1-y)^{n-j} > \sqrt{(1-3\gamma)/(1-2\gamma)} \tag{22}$$

holds for all $n > n_1$ and all $y \in Y := p \pm \frac{3\varepsilon}{2}$, where $J = ((y-\varepsilon_1)n, (y+\varepsilon_1)n) \cap \mathbb{N}$. A little computation shows that we can choose n_2 so that

$$(1-\gamma)\binom{n-t}{k-t}^2 > (1-2\gamma)f(\tfrac{k}{n})\binom{n}{k}^2 \tag{23}$$

holds for all $n > n_2$ and all k with $k/n \in X$. Finally set $n_0 = \max\{n_1, n_2\}$.

We plug these γ, ε and n_0 into (18). Then the negation of Claim 2 gives us some n,k and cross t-intersecting families $\mathscr{A}_1, \mathscr{A}_2 \subset \binom{[n]}{k}$ with

$$|\mathscr{A}_1||\mathscr{A}_2| \ge (1-\gamma)\binom{n-t}{k-t}^2, \tag{24}$$

where $n > n_0$ and $k/n \in X$. We fix these n,k and $\{\mathscr{A}_1, \mathscr{A}_2\}$, and set $x = k/n$. By (23) and (24) we have $|\mathscr{A}_1||\mathscr{A}_2| > (1-2\gamma)f(x)\binom{n}{k}^2$, or equivalently,

$$c_1c_2 > (1-2\gamma)f(x) \tag{25}$$

where $c_s = |\mathscr{A}_s|/\binom{n}{k}$ for $s = 1,2$. Fix $y := x + \varepsilon_1 \in Y$.

Claim 3. $|\nabla_j(\mathscr{A}_s)| \ge c_s\binom{n}{j}$ *for* $j \in J$.

Proof of Claim 3. Choose a real $z \le n$ so that $c_s\binom{n}{k} = \binom{z}{n-k}$. Since $|\mathscr{A}_s| = c_s\binom{n}{k} = \binom{z}{n-k}$ the Kruskal–Katona theorem implies that $|\nabla_j(\mathscr{A}_s)| \ge \binom{z}{n-j}$. Thus it suffices to show that $\binom{z}{n-j} \ge c_s\binom{n}{j}$, or equivalently,

$$\frac{\binom{z}{n-j}}{\binom{z}{n-k}} \ge \frac{c_s\binom{n}{j}}{c_s\binom{n}{k}}.$$

Using $j \ge k$ this is equivalent to $j \cdots (k+1) \ge (z-n+j)\cdots(z-n+k+1)$, which follows from $z \le n$. □

Let $\mathscr{G}_s = \bigcup_{k\le j\le n}(\nabla_j(\mathscr{A}_s))$ for $s = 1,2$. By Claim 3 we have

$$w_y(\mathscr{G}_s) \ge \sum_{j\in J} |\nabla_j(\mathscr{A}_s)| y^j (1-y)^{n-j} \ge c_s \sum_{j\in J} \binom{n}{j} y^j (1-y)^{n-j}. \tag{26}$$

Therefore we have

$$\begin{aligned} w_y(\mathscr{G}_1)w_y(\mathscr{G}_2) &\underset{(26)}{>} c_1c_2\Big(\sum_{j\in J}\binom{n}{j}y^j(1-y)^{n-j}\Big)^2 \\ &\underset{(25),(22)}{>} (1-2\gamma)f(x)\times(1-3\gamma)/(1-2\gamma) = (1-3\gamma)f(x) \\ &\underset{(21)}{>} (1-4\gamma)f(x+\varepsilon_1) = (1-\gamma_0)f(y), \end{aligned}$$

which contradicts (20) because $y \in Y = p \pm \frac{3\varepsilon}{2} = p \pm \frac{3\varepsilon_0}{4} \subset p \pm \varepsilon_0$. This completes the proof of Claim 2 and Theorem 4. □

Similarly to the proof of Theorem 5, we have the following stability type statement.

Theorem 6. *Let p be a rational number with $0 < p < 0.114$, and let t,n,k be integers with $1 \le t \le 1/2p$, $n \ge n_0(t,p)$, and $p = k/n$. Suppose that $\mathscr{A}_1 \subset \binom{[n]}{k}$ and $\mathscr{A}_2 \subset \binom{[n]}{k}$ are shifted cross t-intersecting families. Then, for any $\varepsilon \in (0,p]$ there exists $\gamma > 0$ such that if $|\mathscr{A}_1||\mathscr{A}_2| > (1-\gamma)\binom{n-t}{k-t}^2$, then $|\mathscr{A}_1 \setminus \mathscr{A}_1'||\mathscr{A}_2 \setminus \mathscr{A}_2'| < \varepsilon\binom{n-t}{k-t}^2$, where $\mathscr{A}_i' = \{A \in \mathscr{A}_i : [t] \subset A\}$ for $i = 1,2$.*

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