THE EIGENVALUE METHOD FOR CROSS t-INTERSECTING FAMILIES

NORIHIDE TOKUSHIGE

ABSTRACT. We show that the Erdős–Ko–Rado inequality for t-intersecting families of k-element subsets of an n-element set can be easily extended to an inequality for cross t-intersecting families by using the eigenvalue method if n is relatively large depending on k and t. The same method applies to the case of t-intersecting families of k-dimensional subspaces of an n-dimensional vector space over a finite field.

1. INTRODUCTION

The eigenvalue method is one of the proof techniques to get Erdős–Ko–Rado [5] type inequalities for t-intersecting families. Such examples include a proof for families of k-subsets by Wilson [13], a proof for families of k-subspaces by Frankl and Wilson [8], and a recent seminal proof for families of permutations by Ellis, Friedgut, and Pilpel [4]. The last one contains a stronger inequality for cross t-intersecting families which follows from a variant of the Hoffman–Delsarte bound. In this note we remark that one can also get the corresponding cross t-intersecting version of the first two results about k-subsets (Theorem 1) and k-subspaces (Theorem 2) in the same way quite easily. These results are new but the tools used in the proof are not new. The point of this note is to show that these tools have a wider application than previously thought.

Let $X_n = \{1,2,\dots,n\}$ be an n-element set. Two families of k-subsets $\mathscr{A},\mathscr{B} \subset \binom{X_n}{k}$ are called cross t-intersecting if $|A\cap B| \geq t$ holds for all $A \in \mathscr{A}, B \in \mathscr{B}$.

Theorem 1. *Let $k \geq t \geq 1$ and $\frac{k}{n} < 1-\frac{1}{\sqrt[t]{2}}$. Suppose that two families $\mathscr{A},\mathscr{B} \subset \binom{X_n}{k}$ are cross t-intersecting. Then we have*

$$|\mathscr{A}||\mathscr{B}| \leq \binom{n-t}{k-t}^2.$$

If $|\mathscr{A}||\mathscr{B}| = \binom{n-t}{k-t}^2$, then

$$\mathscr{A} = \mathscr{B} = \{F \in \binom{X_n}{k} : T \subset F\}$$

for some $T \in \binom{X_n}{t}$.

We notice that $\frac{\log 2}{t+1} < 1-\frac{1}{\sqrt[t]{2}} < \frac{\log 2}{t}$ and so the conclusion of Theorem 1 holds for $n > \frac{(t+1)k}{\log 2} \approx 1.44(t+1)k$. On the other hand, it is known from [13] that if $n > (t+1)(k-t+1)$,

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and $\mathscr{A}$ is t-intersecting (namely, $\mathscr{A}$ and $\mathscr{A}$ are cross t-intersecting), then $|\mathscr{A}| \le \binom{n-t}{k-t}$. Moreover equality holds only if $\mathscr{A}$ fixes some t-subset, that is, $\mathscr{A} = \{F \in \binom{X_n}{k} : T \subset F\}$ holds for some t-subset $T \subset X_n$. (In fact this result was first proved by Frankl [6] for the case $t \ge 15$ using a combinatorial method.) If $n = (t+1)(k-t+1)$, then we have another t-intersecting family $\mathscr{A} = \{A \in \binom{X_n}{k} : |A \cap [t+2]| \ge t+1\}$ with $|\mathscr{A}| = \binom{n-t}{k-t}$. See also [1] for the case $n < (t+1)(k-t+1)$.

Conjecture 1. *Let $k \ge t \ge 1$ and $n \ge (t+1)(k-t+1)$. Suppose that two families $\mathscr{A}, \mathscr{B} \subset \binom{X_n}{k}$ are cross t-intersecting. Then we have $|\mathscr{A}||\mathscr{B}| \le \binom{n-t}{k-t}^2$. If $n > (t+1)(k-t+1)$ and $|\mathscr{A}||\mathscr{B}| = \binom{n-t}{k-t}^2$, then $\mathscr{A} = \mathscr{B} = \{F \in \binom{X_n}{k} : T \subset F\}$ for some $T \in \binom{X_n}{t}$.*

Let V_n be an n-dimensional vector space over the q-element field. Let $\left[{V_n \atop k}\right]$ denote the set of all k-subspaces (k-dimensional subspaces) of V_n, and let $\left[{n \atop k}\right] = \#\left[{V_n \atop k}\right] = \prod_{i=0}^{k-1}(q^{n-i} - 1)/(q^{k-i} - 1)$. Two families of k-subspaces $\mathscr{A}, \mathscr{B} \subset \left[{V_n \atop k}\right]$ are called cross t-intersecting if $\dim(A \cap B) \ge t$ holds for all $A \in \mathscr{A}, B \in \mathscr{B}$.

Theorem 2. *Let $k \ge t \ge 1$ and $n \ge 2k$. Suppose that two families $\mathscr{A}, \mathscr{B} \subset \left[{V_n \atop k}\right]$ are cross t-intersecting. Then we have*

$$|\mathscr{A}||\mathscr{B}| \le \begin{bmatrix} n-t \\ k-t \end{bmatrix}^2.$$

If $n > 2k$ and $|\mathscr{A}||\mathscr{B}| = \left[{n-t \atop k-t}\right]^2$, then

$$\mathscr{A} = \mathscr{B} = \{F \in \begin{bmatrix} V_n \\ k \end{bmatrix} : T \subset F\}$$

for some $T \in \left[{V_n \atop t}\right]$.

In particular, if $n > 2k$ and $\mathscr{A} \subset \left[{V_n \atop k}\right]$ itself is t-intersecting with $|\mathscr{A}| = \left[{n-t \atop k-t}\right]$, then $\mathscr{A}$ fixes some t-subspace. This result has been claimed several times, but (except the case $t = 1$) it seems that the only correct proof appeared in the literature is due to Tanaka [14]. His proof heavily relies on the theory of association schemes, and has much wider application. See also §5.3 of [12] or [3] for historical details. We include an elementary proof of this result for completeness.

If $n \le 2k - t$ then $\left[{V_n \atop k}\right]$ itself is t-intersecting. Theorem 2 also tells us what happens for $2k - t < n \le 2k$ just by taking the orthogonal complements.

Theorem 3. *Let $k \ge t \ge 1$ and $2k - t < n \le 2k$. Suppose that two families $\mathscr{A}, \mathscr{B} \subset \left[{V_n \atop k}\right]$ are cross t-intersecting. Then we have*

$$|\mathscr{A}||\mathscr{B}| \le \begin{bmatrix} 2k-t \\ k \end{bmatrix}^2.$$

If $2k - t < n < 2k$ and $|\mathscr{A}||\mathscr{B}| = \left[{2k-t \atop k}\right]^2$, then $\mathscr{A} = \mathscr{B} = \left[{Y \atop k}\right]$ for some $Y \in \left[{V_n \atop 2k-t}\right]$.

We can apply both Theorem 2 and Theorem 3 to the case $n = 2k$, and we see that there are at least two different extremal configurations satisfying $|\mathscr{A}||\mathscr{B}| = \left[{n-t \atop k-t}\right]^2 = \left[{2k-t \atop k}\right]^2$.

Tanaka [14] proved that if $\mathscr{A}=\mathscr{B}$ then there are no other configuration having this maximum product. We extend this result as follows.

Theorem 4. *Let $k\geq t\geq 1$, and $n=2k$. Suppose that two families $\mathscr{A},\mathscr{B}\subset \begin{bmatrix}V_n\\k\end{bmatrix}$ are cross t-intersecting with $|\mathscr{A}||\mathscr{B}|=\begin{bmatrix}n-t\\k-t\end{bmatrix}^2=\begin{bmatrix}2k-t\\k\end{bmatrix}^2$. Then $\mathscr{A}=\mathscr{B}=\{F\in\begin{bmatrix}V_n\\k\end{bmatrix}:T\subset F\}$ for some $T\in\begin{bmatrix}V_n\\t\end{bmatrix}$ or $\mathscr{A}=\mathscr{B}=\begin{bmatrix}Y\\k\end{bmatrix}$ for some $Y\in\begin{bmatrix}V_n\\2k-t\end{bmatrix}$.*

Finally we mention some related results and problems. In [11] it is proved that if $n\geq\max\{2k,2\ell\}$ and two families $\mathscr{A}\subset\binom{X_n}{k}$ and $\mathscr{B}\subset\binom{X_n}{\ell}$ are cross 1-intersecting, then $|\mathscr{A}||\mathscr{B}|\leq\binom{n-1}{k-1}\binom{n-1}{\ell-1}$. It would be nice to have an algebraic proof of this result or a result for cross t-intersecting families with different uniformities.

Conjecture 2. *Let $k\geq\ell\geq t\geq 1$ and $n\geq(t+1)(k-t+1)$. Suppose that two families $\mathscr{A}\subset\binom{X_n}{k}$ and $\mathscr{B}\subset\binom{X_n}{\ell}$ are cross t-intersecting. Then we have $|\mathscr{A}||\mathscr{B}|\leq\binom{n-t}{k-t}\binom{n-t}{\ell-t}$.*

The above conjecture fails if $t=1$, $\ell=k-1$ and $n=2k-1$, see [11] for a counterexample.

In [7] it is proved that if $(r-1)n\geq rk$ and r families $\mathscr{F}_1,\dots,\mathscr{F}_r\subset\binom{X_n}{k}$ are r-cross 1-intersecting, namely, $|F_1\cap\cdots\cap F_r|\geq 1$ holds for all $F_i\in\mathscr{F}_i$ $(1\leq i\leq r)$, then $\prod_{i=1}^r|\mathscr{F}_i|\leq\binom{n-1}{k-1}^r$. Is it possible to give an algebraic proof of this result? How about r-cross t-intersecting families?

Conjecture 3. *Let $n\geq k\geq t$, $0<p\leq(r-1)/r$ and $1\leq t\leq(p^{1-r}-p)/(1-p)-r$ where $p=k/n$. Suppose that r families $\mathscr{F}_1,\dots,\mathscr{F}_r\subset\binom{X_n}{k}$ are r-cross t-intersecting, namely, $|F_1\cap\cdots\cap F_r|\geq t$ holds for all $F_i\in\mathscr{F}_i$ $(1\leq i\leq r)$. Then we have $\prod_{i=1}^r|\mathscr{F}_i|\leq\binom{n-t}{k-t}^r$.*

In [15] some partial results were obtained by using a combinatorial technique with help of probabilistic methods, which verify the above conjecture for some special cases.

Gromov [10] showed that such inequalities concerning cross intersecting structures can be equivalently reformulated in terms of monomial subsets in the N-torus, and for example he obtained a homological separation inequality for pairs of disjoint subsets in N-torus from an inequality obtained in [11] mentioned above.

2. Tools

We introduce our main tools for the proof of the theorems. Let G be an N-vertex graph. A real symmetric $N\times N$ matrix $A=(a_{ij})$ is called a pseudo adjacency matrix of G if

- $a_{ij}=0$ whenever $\{i,j\}\notin E(G)$, and
- $\mathbf{1}$ (all 1 column vector in $\mathbb{R}^N$) is an eigenvector of A with a positive eigenvalue.

Let Λ be the set of eigenvalues of A. Let $\mu_1(A)\in\Lambda$ be the positive eigenvalue corresponding to $\mathbf{1}$, and let $\mu_2(A)$ be the eigenvalue having the largest absolute value in $\Lambda\setminus\{\mu_1(A)\}$. (In our applications below, μ_1 will be the largest eigenvalue and μ_2 will be the smallest eigenvalue.)

Ellis, Friedgut, and Pilpel used the following variant of the Hoffman–Delsarte bound to get a cross t-intersecting version of an EKR inequality for permutations in [4]. They attribute one of the origins of this result to [2].

Lemma 1. *Let G be an N-vertex graph with a pseudo adjacency matrix A, and let $U_1, U_2 \subset V(G)$. Suppose that there are no edges between U_1 and U_2. Then we have*

$$\sqrt{|U_1||U_2|} \leq \frac{|\mu_2(A)|}{\mu_1(A)+|\mu_2(A)|}N.$$

As we will see this result is very useful. We recall a quick proof for later use. (We will need Lemma 2 (b) which follows from the proof of Lemma 1 easily, but this is not mentioned in [4].) We start with the following simple inequality.

Claim 1. *If $c, d \in [0,1]$ then $\sqrt{1-c}\sqrt{1-d} \leq 1-\sqrt{cd}$.*

Proof. We apply the inequality of arithmetic and geometric means twice:

$$\sqrt{1-c}\sqrt{1-d} \leq \frac{(1-c)+(1-d)}{2} = 1-\frac{c+d}{2} \leq 1-\sqrt{cd}.$$

□

Proof of Lemma 1. Let $\mu_i = \mu_i(A)$ for $i = 1, 2$. Since $A = (a_{ij})$ is a symmetric matrix we can choose orthogonal eigenvectors $v_1 = \mathbf{1}, v_2, \ldots, v_N \in \mathbb{R}^N$ corresponding to the eigenvalues $\mu_1, \mu_2, \ldots, \mu_N$ so that $v_i^T v_j = N\delta_{ij}$. Let $f = (f_1, \ldots, f_N)$ be the characteristic row vector of U_1, that is, $f_i = 1$ if $i \in U_1$ and $f_i = 0$ if $i \notin U_1$. We can write it as a linear combination of eigenvectors as $f = \sum_{i=1}^N c_i v_i^T$. Also we can write the characteristic row vector of U_2 as $g = \sum_{i=1}^N d_i v_i^T$. Then we have

$$|U_1| = f\mathbf{1} = (\sum_{i=1}^N c_i v_i^T)v_1 = c_1 v_1^T v_1 = c_1 N,$$

and

$$|U_1| = ff^T = (\sum_{i=1}^N c_i v_i^T)(\sum_{j=1}^N c_j v_j) = \sum_{i=1}^N c_i^2 v_i^T v_i = \sum_{i=1}^N c_i^2 N.$$

Thus we have

$$[0,1] \ni |U_1|/N = c_1 = \sum_{i=1}^N c_i^2.$$

Similarly, from $|U_2| = g\mathbf{1} = gg^T$, we also get

$$[0,1] \ni |U_2|/N = d_1 = \sum_{i=1}^N d_i^2.$$

Now we compute fAg^T in two ways. On one hand we have

$$fAg^T = \sum_{ij\in E(G)} a_{ij} f_i g_j + \sum_{ij\notin E(G)} a_{ij} f_i g_j = 0,$$

because $f_i g_j = 0$ if $ij \in E(G)$ and $a_{ij} = 0$ if $ij \notin E(G)$. On the other hand, we have

$$fAg^T = (\sum_{i=1}^N c_i v_i^T)A(\sum_{j=1}^N d_j v_j) = (\sum_{i=1}^N c_i v_i^T)(\sum_{j=1}^N d_j \mu_j v_j) = \sum_{i=1}^N c_i d_i \mu_i N.$$

Thus we have $0=\sum_{i=1}^{N} c_i d_i \mu_i$ and

$$c_1 d_1 \mu_1 = \left|\sum_{i=2}^{N} c_i d_i \mu_i\right| \le \sum_{i=2}^{N} |c_i d_i \mu_i| \overset{\text{(i)}}{\le} |\mu_2| \sum_{i=2}^{N} |c_i d_i| \overset{\text{(ii)}}{\le} |\mu_2| \sqrt{\sum_{i=2}^{N} c_i^2}\sqrt{\sum_{i=2}^{N} d_i^2}$$

$$= |\mu_2|\sqrt{c_1 - c_1^2}\sqrt{d_1 - d_1^2} \overset{\text{(iii)}}{\le} |\mu_2|\sqrt{c_1 d_1}(1-\sqrt{c_1 d_1}),$$

where we used the Cauchy–Schwarz inequality at (ii) and the claim at (iii). This gives $\frac{\mu_1}{|\mu_2|} \le \frac{1}{\sqrt{c_1 d_1}} - 1$, or equivalently,

$$\sqrt{\frac{|U_1|}{N}\frac{|U_2|}{N}} = \sqrt{c_1 d_1} \le \left(\frac{\mu_1}{|\mu_2|}+1\right)^{-1} = \frac{|\mu_2|}{\mu_1 + |\mu_2|},$$

as desired. □

Lemma 2. *Suppose that equality holds in Lemma 1. Then we have the following.*

(a) *Both of the characteristic vectors of U_1 and U_2 are contained in the subspace spanned by eigenvectors corresponding to $\mu_1(A)$ and $\lambda \in \Lambda$ with $|\lambda| = |\mu_2(A)|$.*

(b) *If $\mu_2(A) < 0$ and $-\mu_2(A) \notin \Lambda$ then $U_1 = U_2$. Namely, if $\mu_2(A)$ is the smallest eigenvalue which has the largest absolute value in $\Lambda \setminus \{\mu_1(A)\}$, then $U_1 (= U_2)$ is an independent set itself.*

Proof. We reuse the proof of Lemma 1. First we notice that

$$c_1 = d_1 \text{ and } |c_i| = |d_i| \text{ for } 2 \le i \le N. \tag{1}$$

In fact we have equality at (iii), which gives $c_1 = d_1$, and we have equality at (ii), which gives $|c_i| = |d_i|$ for $2 \le i \le N$. We also need equality at (i), so for $2 \le i \le N$ we have $c_i = d_i = 0$ or $|\mu_i| = |\mu_2|$. This gives (a).

Next suppose that $\mu_2(A) < 0$ and $-\mu_2(A) \notin \Lambda$. Since we have

$$c_1 d_1 \mu_1 = -\sum_{i=2}^{N} c_i d_i \mu_i = (-\mu_2)\sum_{i=2}^{N} c_i d_i = |\mu_2| \sum_{i=2}^{N} |c_i d_i|,$$

we get $c_i d_i = |c_i d_i|$ for all $2 \le i \le N$. This together with (1) implies that $c_i = d_i$ for all i. This means $f = g$, or equivalently, $U_1 = U_2$, which completes the proof of (b). □

3. Application

For the proof of Theorem 1 we modify a result due to Wilson slightly as follows.

Lemma 3. *Let $\frac{k}{n} < 1 - \frac{1}{\sqrt[t]{2}}$. Let G be a graph with $V(G) = \binom{X_n}{k}$ and two vertices $F, F' \in V(G)$ are adjacent iff $|F \cap F'| < t$. Then there is a pseudo adjacency matrix A of G with $\mu_1(A) = \binom{n}{k}\binom{n-t}{k-t}^{-1} - 1$, $\mu_2(A) = -1$, and $-\mu_2(A) \notin \Lambda$.*

Here we summarize some known facts about the pseudo adjacency matrix used in Lemma 3 from [13]. (We will need them to state the next lemma precisely.) The matrix A above is defined as follows:

$$A := \sum_{i=0}^{t-1} (-1)^{t-1-i} \binom{k-1-i}{k-t} \binom{n-k-t+i}{k-t}^{-1} B_{k-i},$$

where B_{k-i} is an $\binom{n}{k} \times \binom{n}{k}$ matrix indexed by $\binom{X_n}{k} \times \binom{X_n}{k}$ whose (F,F')-entry is given by

$$\#\{J \in \binom{X_n}{k-i} : J \cap F = \emptyset \text{ and } J \subset F'\} = \binom{|F' \setminus F|}{k-i}.$$

Equivalently we can write $B_j = (\bar{W}_{jk})^T W_{jk}$, where W_{jk} (resp. $\bar{W}_{jk}$) is an $\binom{n}{j} \times \binom{n}{k}$ matrix whose (J,K)-entry is 1 if $J \subset K$ (resp. $J \cap K = \emptyset$) and 0 otherwise. Let U_j be the row space of W_{jk}. Then we have $U_0 \subset U_1 \subset \cdots \subset U_k$ and U_k is the entire vector space E spanned by all characteristic vectors of the k-subsets of X_n. (This follows from $W_{ij}W_{jk} = \binom{k-i}{j-i} W_{ik}$ for $i \le j \le k$.) Let $V_0 = U_0$ and for $j \ge 1$ let V_j be the orthogonal complement of U_{j-1} in U_j, so that $U_j = V_j \oplus U_{j-1}$. Then we have an orthogonal decomposition $E = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, and after some computation we have the corresponding eigenvalues $\theta_0, \theta_1, \dots, \theta_k$ of the matrix A such that $Av = \theta_j v$ for all $v \in V_j$. (Recall that A is real symmetric, and so all these eigenvalues are real.) Moreover Wilson obtained the following result to prove the EKR inequality for t-intersecting families of k-subsets in [13].

Lemma 4. *Let $n > (t+1)(k-t+1)$. Then $\theta_0 = \binom{n}{k}\binom{n-t}{k-t}^{-1} - 1$, $\theta_1 = \theta_2 = \cdots = \theta_t = -1$, $0 < \theta_{t+1} < \theta_0$, and $1 > |\theta_{t+2}| > |\theta_{t+3}| > \cdots > |\theta_k|$.*

If $|\theta_{t+1}| < 1$ then Lemma 3 follows from Lemma 4 by setting $\mu_1 = \theta_0$ and $\mu_2 = \theta_1$. Namely, it suffices to show the following claim for the proof of Lemma 3.

Claim 2. *Let $\frac{k}{n} < 1 - \frac{1}{\sqrt[t]{2}}$. Then $0 < \theta_{t+1} < 1$.*

Proof. We start with the expression $\theta_{t+1} = t\delta_{t-1}(k-1, n-k+1; t, 1)$ using (4.4) and (4.5) of [13], which reads as follows:

$$\theta_{t+1} = t \sum_{i=0}^{t-1} \frac{1}{i+1} \binom{t-1}{i} \binom{(k-1)-(t-1)}{t-(t-1)+i} \binom{(n-k-t)+i}{i+1}^{-1}.$$

Rewriting this, we get

$$\begin{aligned}
\theta_{t+1} &= \sum_{i=0}^{t-1} \binom{t}{i+1} \frac{(k-t)\cdots(k-t-i)}{(n-k-t+i)\cdots(n-k-t)} \\
&\le \sum_{i=0}^{t-1} \binom{t}{i+1} \left(\frac{k-t}{n-k-t}\right)^{i+1} < \sum_{i=0}^{t-1} \binom{t}{i+1} \left(\frac{k}{n-k}\right)^{i+1} \\
&= \sum_{j=1}^{t} \binom{t}{j} \left(\frac{k}{n-k}\right)^{j} = \sum_{j=0}^{t} \left(\binom{t}{j} \left(\frac{k}{n-k}\right)^{j} 1^{t-j}\right) - 1 \\
&= \left(\frac{k}{n-k} + 1\right)^{t} - 1 = \left(1 - \frac{k}{n}\right)^{-t} - 1 < 1.
\end{aligned}$$

We used $\frac{k-t}{n-k-t} < \frac{k}{n-k}$ in the second inequality, which follows from $\frac{k}{n} < 1 - \frac{1}{\sqrt[t]{2}} \le \frac{1}{2}$. We also needed $\frac{k}{n} < 1 - \frac{1}{\sqrt[t]{2}}$ for the last inequality. □

Proof of Theorem 1. We construct a graph G and a pseudo adjacency matrix A with $\mu_1 = \binom{n}{k}\binom{n-t}{k-t}^{-1} - 1$ and $\mu_2 = -1$ as above. Then $\mathscr{A}, \mathscr{B} \subset V(G)$, and there are no edges between

$\mathscr{A}$ and $\mathscr{B}$ because of the cross t-intersecting property. By applying Lemma 1 with $N = \binom{n}{k}$ we get

$$\sqrt{|\mathscr{A}||\mathscr{B}|} \le \frac{|\mu_2|}{\mu_1 + |\mu_2|} N = \frac{1}{(\binom{n}{k}\binom{n-t}{k-t}^{-1} - 1) + 1} \binom{n}{k} = \binom{n-t}{k-t}.$$

It is known from [13] that a t-intersecting family $\mathscr{F} \subset \binom{X_n}{k}$ with $|\mathscr{F}| = \binom{n-t}{k-t}$ fixes some t-subset. Thus we get the desired extremal configuration by Lemma 2 (b). We can also obtain a simpler proof of this part by following the proof of Theorem 2 below, because the subspace proof can be modified to apply to subsets as well. □

For the proof of Theorem 2 we need the following result due to Frankl and Wilson [8].

Lemma 5. *Let $n \ge 2k$. Let G be a graph with $V(G) = \begin{bmatrix} V_n \\ k \end{bmatrix}$ and two vertices $F, F' \in V(G)$ are adjacent iff $\dim(F \cap F') < t$. Then there is a pseudo adjacency matrix A of G with $\mu_1(A) = \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-t \\ k-t \end{bmatrix}^{-1} - 1$, $\mu_2(A) = -1$, and $-\mu_2(A) \notin \Lambda$.*

The pseudo adjacency matrix used in Lemma 5 is as follows:

$$A := q^{-k^2+k+\binom{t}{2}} \sum_{i=0}^{t-1} (-1)^{t-1-i} \begin{bmatrix} k-1-i \\ k-t \end{bmatrix} \begin{bmatrix} n-k-t+i \\ k-t \end{bmatrix}^{-1} B_{k-i},$$

where B_{k-i} is an $\begin{bmatrix} n \\ k \end{bmatrix} \times \begin{bmatrix} n \\ k \end{bmatrix}$ matrix indexed by $\begin{bmatrix} V_n \\ k \end{bmatrix} \times \begin{bmatrix} V_n \\ k \end{bmatrix}$ whose (F, F')-entry is given by

$$\#\{J \in \begin{bmatrix} V_n \\ k-i \end{bmatrix} : J \cap F = \{\mathbf{0}\} \text{ and } J \subset F'\}.$$

Proof of Theorem 2. We construct a graph G and a pseudo adjacency matrix A with $\mu_1 = \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-t \\ k-t \end{bmatrix}^{-1} - 1$ and $\mu_2 = -1$ as in Lemma 5. Then $\mathscr{A}, \mathscr{B} \subset V(G)$, and there are no edges between $\mathscr{A}$ and $\mathscr{B}$ because of the cross t-intersecting property. By applying Lemma 1 with $N = \begin{bmatrix} n \\ k \end{bmatrix}$ we get

$$\sqrt{|\mathscr{A}||\mathscr{B}|} \le \frac{|\mu_2|}{\mu_1 + |\mu_2|} N = \frac{1}{(\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-t \\ k-t \end{bmatrix}^{-1} - 1) + 1} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-t \\ k-t \end{bmatrix}.$$

Now suppose that $n > 2k$ and we will determine the extremal configurations. The proof below is based on ideas due to Wilson [13] and Godsil and Newman [9]. We can apply Lemma 2 (b). So it suffices to show that if $\mathscr{F} \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ is t-intersecting with $|\mathscr{F}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}$, then $\mathscr{F}$ fixes some t-subspace.

Let $f \in \{0,1\}^{\begin{bmatrix} n \\ k \end{bmatrix}}$ be the characteristic row vector of $\mathscr{F}$. Let $W_{tk}(n)$ be an $\begin{bmatrix} n \\ t \end{bmatrix} \times \begin{bmatrix} n \\ k \end{bmatrix}$ matrix indexed by $\begin{bmatrix} V_n \\ t \end{bmatrix} \times \begin{bmatrix} V_n \\ k \end{bmatrix}$, whose (T, F)-entry is 1 if $T \subset F$, and 0 if $T \not\subset F$. In this case it is known from [8] that f is contained in the row space of $W_{tk}(n)$. Thus there is a row vector $h \in \mathbb{R}^{\begin{bmatrix} n \\ t \end{bmatrix}}$ such that $f = hW_{tk}(n)$. Fix $F_0 \in \mathscr{F}$. Let $d = n - k + (t-1)$ and choose $C \in \begin{bmatrix} V_n \\ d \end{bmatrix}$ such that $\dim(F_0 \cap C) = t - 1$ arbitrarily. Let W_C be the $\begin{bmatrix} n \\ t \end{bmatrix} \times \begin{bmatrix} d \\ k \end{bmatrix}$ submatrix of $W_{tk}(n)$ consisting of columns indexed by $\begin{bmatrix} C \\ k \end{bmatrix}$. Let f_C be the subvector of f consisting of entries indexed by $\begin{bmatrix} C \\ k \end{bmatrix}$. Then $f_C = hW_C$. By the t-intersecting property of $\mathscr{F}$, none of k-subspaces in $\begin{bmatrix} C \\ k \end{bmatrix}$ belong to $\mathscr{F}$, that is, $f_C = \mathbf{0}$. Thus we have $\mathbf{0} = f_C = hW_C$.

We further divide W_C into two parts $\tilde{W}$ and O, where $\tilde{W}$ is a $\begin{bmatrix} d \\ t \end{bmatrix} \times \begin{bmatrix} d \\ k \end{bmatrix}$ matrix whose rows are indexed by $\begin{bmatrix} C \\ t \end{bmatrix}$. Then O is a zero matrix because each row of O is indexed by a t-subspace not contained by C while each columns is indexed by a k-subspace contained by C. On the other hand, it is known that $W_{tk}(m)$ has full row rank if $m \geq t+k$, see, e.g. [8]. Noting that $\tilde{W}$ and $W_{tk}(d)$ have the same rank and $d = n-k+t-1 \geq t+k$ (we need $n > 2k$ here), it follows that $\tilde{W}$ has full row rank.

By rearranging the rows of W_C we may assume that $\tilde{W}$ is the first $\begin{bmatrix} d \\ t \end{bmatrix}$ rows of W_C (and the remaining rows of W_C are all $\mathbf{0}$). We write $h = (h_C, h')$ accordingly, namely h_C is the first $\begin{bmatrix} d \\ t \end{bmatrix}$ entries corresponding to $\begin{bmatrix} C \\ t \end{bmatrix}$. Then it follows that

$$\mathbf{0} = hW_C = (h_C, h') \begin{pmatrix} \tilde{W} \\ O \end{pmatrix} = h_C\tilde{W} + h'O = h_C\tilde{W}.$$

Since $\tilde{W}$ has full row rank, we have $h_C = \mathbf{0}$.

For every $T \in \begin{bmatrix} V_n \\ t \end{bmatrix}$ such that $T \not\subset F_0$ we can find some $C' \in \begin{bmatrix} V_n \\ d \end{bmatrix}$ with $\dim(F_0 \cap C') = t-1$ and $T \subset C'$. Then $h_{C'} = \mathbf{0}$, in particular, the entry of h corresponding to T is 0. Therefore if there is a nonzero entry of h, then the corresponding t-subspace is contained in all $F \in \mathscr{F}$. Since $|\mathscr{F}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}$ the only possibility is that h is a $\{0,1\}$-vector having 1 at only one position. Then $f = hW_{tk}(n)$ coincides with one of the rows of $W_{tk}(n)$, say, a row corresponding to $T_0 \in \begin{bmatrix} V_n \\ t \end{bmatrix}$, which means $\mathscr{F} = \{F \in \begin{bmatrix} V_n \\ k \end{bmatrix} : T_0 \subset F\}$. □

Proof of Theorem 3. This follows from a well-known technique, and we only give a sketch here (see e.g., [8] for details). Consider any non-degenerate bilinear form (say, the standard inner product) $f : V_n \times V_n \to \mathbb{F}_q$, and for a subspace U in V_n let $U^\perp := \{v \in V_n : f(u,v) = 0 \text{ for all } u \in U\}$ be the orthogonal complement. For $\mathscr{F} \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ let $\mathscr{F}^\perp := \{F^\perp : F \in \mathscr{F}\}$. Then we have $\mathscr{F} \subset \begin{bmatrix} V_n \\ n-k \end{bmatrix}$ and $|\mathscr{F}^\perp| = |\mathscr{F}|$.

Suppose that $2k - t < n \leq 2k$, and let $\ell = n-k$ and $s = n-2k+t$. We notice that $n \geq 2\ell \geq 2s \geq 2$. Moreover it is not difficult to verify that $\mathscr{A}, \mathscr{B} \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ are cross t-intersecting if and only if $\mathscr{A}^\perp, \mathscr{B}^\perp \subset \begin{bmatrix} V_n \\ \ell \end{bmatrix}$ are cross s-intersecting. By applying Theorem 2 to $\mathscr{A}^\perp$ and $\mathscr{B}^\perp$, we get the desired conclusion. □

Proof of Theorem 4. This is a direct consequence of Lemma 2 (b) and Lemma 5. □

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College of Education, Ryukyu University, Nishihara, Okinawa 903-0213, Japan
E-mail address: hide@edu.u-ryukyu.ac.jp