Classification of the congruent embeddings of a tetrahedron into a triangular prism

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Abstract

Let $\mathbf{P}(t)$ denote an infinitely long right triangular prism whose base is an equilateral triangle of edge length t. Let $\mathcal{F}(t)$ be the family of those subsets of $\mathbf{P}(t)$ that are congruent to a regular tetrahedron of unit edge. We present complete classification of the members of $\mathcal{F}(t)$ modulo rigid motions within the prism $\mathbf{P}(t)$, for every $t > 0$.

1 Introduction

Problems related to embedding or inscribing simplices into circular cylinders are considered by many authors, mostly to study the outer j-radii of simplices, or to compute the cylinders through the vertices of a simplex. See, e.g., Brandenberg, et al. [2, 3], Devillers, et al. [4], Pukhov [8], Schömer, et al. [9]. Maehara [7] treats embedding itself, and proved that all embeddings of a regular tetrahedron in a circular cylinder are equivalent modulo rigid motions within the cylinder.

In this paper, we classify the congruent embeddings of a regular tetrahedron in a right prism whose base is an equilateral triangle. This study arouse from the investigation [1] of the minimum size of an equilateral triangular hole in a plane through which a regular tetrahedron of unit edge can pass.

A regular tetrahedron with unit edge is simply called a *unit tetrahedron*. A right triangular prism $\mathbf{P} = \Delta \times \mathbb{R}$ with equilateral triangular base Δ is called simply a *prism*. The *size* of a prism $\mathbf{P}$, size($\mathbf{P}$), is the length of the edge of Δ. A prism of size t is denoted by $\mathbf{P}(t)$. An *embedding* of a unit tetrahedron in $\mathbf{P}$ means such a subset of $\mathbf{P}$ that is congruent to a unit tetrahedron. Two embeddings $T_1, T_2 \subset \mathbf{P}$ of a unit tetrahedron in $\mathbf{P}$ are said to be *equivalent* (written as $T_1 \sim T_2$ in $\mathbf{P}$) if it is possible to superpose T_1 on T_2 by a continuous rigid motion of T_1 within $\mathbf{P}$. More precisely, $T_1 \sim T_2$ in $\mathbf{P}$ if there is a continuous map $F : T_1 \times [0,1] \to \mathbf{P}$ such that

(1) for every $t \in [0,1]$, the map $f_t : T_1 \to \mathbf{P}$ defined by $f_t(x) = F(x,t)$ gives an isometry from T_1 to $f_t(T_1)$, and

(2) f_0 is the inclusion map, and $f_1(T_1) = T_2$.

The relation $\sim$ in $\mathbf{P}$ is clearly an equivalence relation. Let $\nu(t)$ denote the maximum number of mutually non-equivalent embeddings of T in $\mathbf{P}(t)$. We prove the following.

Theorem 1.1.

$$\nu(t) = \begin{cases} 0 & \textit{for} \quad t < t_0 := \frac{1+\sqrt{2}}{\sqrt{6}} \\ 6 & \textit{for} \quad t_0 \le t < t_1 := \frac{\sqrt{3}+3\sqrt{2}}{6} \\ 18 & \textit{for} \quad t_1 \le t < 1 \\ 1 & \textit{for} \quad 1 \le t. \end{cases}$$

Thus, a unit tetrahedron can be embedded in $\mathbf{P}(t)$ if and only if $t \ge \frac{1+\sqrt{2}}{\sqrt{6}}$. This fact is used in [1] to prove that a unit tetrahedron can pass through an equilateral triangular hole in a plane if and only if the edge length of the triangular hole is at least $\frac{1+\sqrt{2}}{\sqrt{6}}$.

Let $\nu_\circ(t)$ denote the number of equivalence classes of the embeddings of a unit tetrahedron into an infinite circular cylinder of diameter t modulo rigid motions within the cylinder. The number $\nu_\circ(t)$ is determined in [7]: $\nu_\circ(t) = 0$ for $r < 1$, and $\nu_\circ(t) = 1$ for $r \ge 1$. Let $\nu_\Box(t)$ be the number of equivalence classes of all embeddings of a unit tetrahedron into a square prism whose base is a square with diameter t, modulo rigid motions within the prism. Since a square of diagonal t can be inscribed in a circle of diameter t, $\nu_\circ(t) = 0$ for $r < 1$ implies that $\nu_\Box(t) = 0$ for $t < 1$, see also Itoh, et al. [5].

Problem. Determine $\nu_\Box(t)$ for $t \ge 1$.

Throughout this paper, prisms are assumed to be vertically placed in $\mathbb{R}^3$, that is, their generators are parallel to the z-axis. Hence the intersection of a prism $\mathbf{P}$ and the xy-plane is an equilateral triangle.

2 A cross embedding and a tangential embedding

Lemma 2.1. *Let $t_0 = (1+\sqrt{2})/\sqrt{6}$. Then $\mathbf{P}(t_0)$ contains a unit tetrahedron.*

Proof. Put $h = t_0/2 = (1+\sqrt{2})/\sqrt{24}$, and let Δ be the triangle on the xy-plane with vertices $(\pm h, 0, 0)$, $(0, \sqrt{3}h, 0)$. Then Δ is an equilateral triangle of edge length t_0, as easily verified. Put $k = (\sqrt{2}-1)/\sqrt{24}$, $\ell = 1/\sqrt{2}$, and define four points A, B, C, D by

$$A = (k, \ell, -h),\ B = (-h, 0, -k),\ C = (h, 0, k),\ D = (-k, \ell, h).$$

These four points span a unit tetrahedron, and their orthogonal projections on the xy-plane lie on Δ, see Figure 1. Thus the unit tetrahedron $ABCD$ is contained in $\mathbf{P}(t_0) = \Delta \times \mathbb{R}$. $\Box$

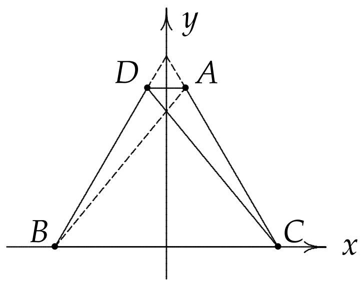


Figure 1: Top view of the tetrahedron in the triangular prism

This embedding is referred to as a *cross embedding*. Similarly, we can construct six different cross embeddings (modulo translations) by changing the face σ of $\mathbf{P}(t_0)$ containing the edge BC, and by changing the crossing type ϵ (which can be done by changing the signs of the z-coordinates of A, B, C, D). These six different cross embeddings are denoted by

$$\alpha(\sigma, \epsilon) \quad (\sigma = \sigma_1, \sigma_2, \sigma_3, \quad \epsilon = \epsilon_1, \epsilon_2),$$

where $\sigma_1, \sigma_2, \sigma_3$ denotes the the faces of $\mathbf{P}(t_0)$, and $\epsilon_1 =$ , $\epsilon_2 =$ .

A *tangential embedding* $T \subset \mathbf{P}$ is an embedding such that some three vertices of T lie on one and the same face of $\mathbf{P}$.

Lemma 2.2. *Let* $t_1 := (\sqrt{3} + 3\sqrt{2})/6 \approx 0.99578$. *Then* $\mathbf{P}(t_1)$ *contains a tangential embedding of a unit tetrahedron.*

Proof. Let Δ_1 be the triangle on the xy-plane with vertices

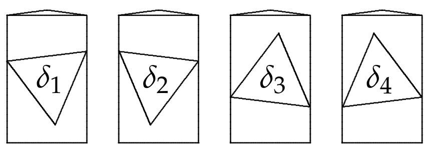
$$\bar{A} = (\tfrac{\sqrt{2}}{3}, 0, 0),\ \bar{B} = (-\tfrac{\sqrt{3}+\sqrt{2}}{6}, 0, 0),\ E = (-\tfrac{\sqrt{3}-\sqrt{2}}{12}, \tfrac{\sqrt{6}+1}{4}, 0).$$

A straightforward calculation shows that Δ_1 is an equilateral triangle with edge length t_1. Let $T_1 = ABCD$ be the tetrahedron with vertices

$$A = (\tfrac{\sqrt{2}}{3}, 0, \tfrac{1}{3}),\ B = (-\tfrac{\sqrt{3}+\sqrt{2}}{6}, 0, \tfrac{\sqrt{6}-1}{6}),\ C = (\tfrac{\sqrt{3}-\sqrt{2}}{6}, 0, -\tfrac{\sqrt{6}+1}{6}),$$
$$D = (0, \tfrac{\sqrt{6}}{3}, 0).$$

Figure 2 shows how the face ABC is embedded in a face of $\mathbf{P}(t_1)$, see also Figure 5 in Section 4. The vertex D lies on another face of $\mathbf{P}(t_1)$. Then T_1 is a tangential embedding of T in $\mathbf{P}(t_1)$. $\square$

Similarly, we can construct different tangential embeddings by changing the face σ of $\mathbf{P}(t_1)$ that contains ABC, and changing the embedding type δ of ABC in σ in the following four different ways:



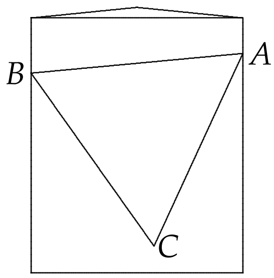


Figure 2: Face ABC in a face of $\mathbf{P}(t_1)$

Thus, there are 12 different tangential embeddings modulo translations in $\mathbf{P}(t_1)$. They are denoted by

$$\beta(\sigma, \delta) \ \ (\sigma = \sigma_1, \sigma_2, \sigma_3, \quad \delta = \delta_1, \delta_2, \delta_3, \delta_4).$$

From now on, we assume that the prisms $\mathbf{P}(t)$, $t \in \mathbb{R}_+$ are *nested* in such a way that they have the same center axis and parallel faces. Thus the sections of some two prisms by a horizontal plane look like . Then an embedding $T \subset \mathbf{P}(s)$ is naturally regarded as an embedding $T \subset \mathbf{P}(t)$ for $s < t$. Thus the embedding $\alpha(\sigma, \epsilon) \subset \mathbf{P}(t_0)$ is an embedding in $\mathbf{P}(t)$ for $t \geq t_0$, and $\beta(\sigma, \delta) \subset \mathbf{P}(t_1)$ is an embedding in $\mathbf{P}(t)$ for $t \geq t_1$.

3 Conditions to reduce the containment size

An *interior vertex* of $T \subset \mathbf{P}$ is a vertex of T lying in the interior $\mathbf{P}^\circ$ of $\mathbf{P}$. A *corner vertex* of $T \subset \mathbf{P}$ is a vertex lying on the corner line of $\mathbf{P}$. For every point $P \in \mathbb{R}^3$, let $z(P)$ denote the z-coordinate of P, and $\bar{P}$ denote the orthogonal projection of P on the xy-plane.

Lemma 3.1. *Let $T \subset \mathbf{P}$ be an embedding. If*

(1) *T has an interior vertex, or*

(2) *T has at most one corner vertex,*

then T can be congruently moved into $\mathbf{P}^\circ$.

Proof. Let $T = ABCD$. A face of $\mathbf{P}$ that contains no vertex of T is called an *empty face*. Note that if $\mathbf{P}$ has an empty face σ, then we can push T slightly toward σ so that T goes into $\mathbf{P}^\circ$.

(1) First, note that if T has two interior vertices, say, A, B, and $\mathbf{P}$ has no empty face, then C, D must be corner vertices. In this case, a small rotation of T around the line through the midpoint of CD and perpendicular to the face containing CD makes two faces of $\mathbf{P}$ empty.

Now, suppose that A is an interior vertex. If one of B, C, D, say, D, is not a corner vertex, then a small rotation of T around the line BC makes A, D interior vertices. Suppose that B, C, D are all corner vertices. Then no two of them lie on the same corner line, because the dihedral angle of a unit tetrahedron is greater than $\pi/6$. Therefore, B, C, D lie in different corners, the equilateral triangle BCD must be horizontal, and hence $\operatorname{size}(\mathbf{P}) = 1$. In this case, a small rotation around the line BC makes A, D interior vertices.

(2) Let Δ be the section of $\mathbf{P}$ by the xy-plane. We may suppose that none of A, B, C, D is an interior vertex, and $\mathbf{P}$ has no empty face.

If T has no corner vertex, then there is a face σ of $\mathbf{P}$ that contains two vertices of T. Let ℓ be the line perpendicular to σ and passing through the midpoint of the other two vertices. Then an appropriate rotation of T around ℓ sends the two vertices not lying on σ into $\mathbf{P}^\circ$.

Suppose that T has only one corner vertex, say, D. Let σ be the face opposite to D. Then one of A, B, C does not lie on σ. To see this, suppose that A, B, C lie on σ. Let G be the barycenter of ABC. Then DG is horizontal. Suppose that $\theta := \angle GD\bar{A} \geq \angle GD\bar{B} \geq \angle GD\bar{C}$. Then θ attains its minimum when $\bar{C}$ is the midpoint of $\bar{A}\bar{B}$ (i.e., when $\bar{C} = G$). In this case, noting that $|D\bar{C}| = \sqrt{2/3}$ and $|\bar{A}\bar{C}| = 1/2$, we have $\tan\theta = |\bar{A}\bar{C}|/|D\bar{C}| = \sqrt{3/8} > \sqrt{1/3} = \tan(\pi/6)$, and thus $\theta > \pi/6$. If D is a corner vertex, then it follows from $\sigma \perp GD$ that $\theta \leq \pi/6$, a contradiction. Thus σ contains at most two of A, B, C.

If σ contains two vertices of T, then a rotation of T around the line through D and perpendicular to σ sends the remaining vertex into $\mathbf{P}^\circ$, and we are done. So, we may assume that σ contains only one vertex of T, say, C. If A, B lie on the same face, say τ, then A, B, D lie on τ. Let DX be a line segment obtained by cutting τ horizontally, and let M be the midpoint of DX. Then a small rotation around the line through M and perpendicular to τ sends C into $\mathbf{P}^\circ$.

Thus, we may assume that A, B lie on different faces. In this case, A, B are both lower (or both higher) than D, for otherwise, $\angle ADB$ would be greater than $\pi/3$. So, we may suppose that $z(A) < z(B) < z(D)$. Let F be the midpoint of AB. Then $z(A) < z(F) < z(B) < z(D)$, $\bar{A}\bar{D} < \bar{B}\bar{D}$ and $\angle \bar{D}\bar{F}\bar{A} < \angle \bar{D}\bar{F}\bar{B}$.

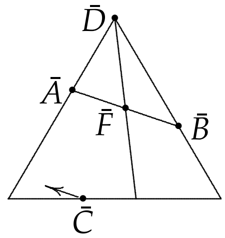


Figure 3: Just one corner vertex

To show that $\bar{A}$ and $\bar{C}$ lie in the same side of the line $\bar{D}\bar{F}$ in the xy-plane, suppose, on the contrary, that $\bar{B}$ and $\bar{C}$ lie on the same side. In this case, using

$\angle \bar{D}\bar{F}\bar{B} > \pi/2$ we have $\angle \bar{B}\bar{F}\bar{C} < \pi/2$. Noting that $\angle BFC = \pi/2$, we have $(z(B) >) z(F) > z(C)$. Similarly, using $\angle \bar{D}\bar{F}\bar{C} > \angle \bar{D}\bar{F}\bar{B} > \pi/2$ and $\angle DFC < \pi/2$, we have $(z(D) >) z(F) < z(C)$, a contradiction. Thus, $\bar{A}$ and $\bar{C}$ must lie on the same side, see Figure 3.

Let us verify that $z(A) < z(C)$. If $\angle \bar{A}\bar{F}\bar{C} < \pi/2$, then this follows from $\angle AFC = \pi/2$ and $z(A) < z(F)$. Otherwise we have $\angle \bar{D}\bar{F}\bar{C} > \angle \bar{A}\bar{F}\bar{C} > \pi/2$. Then $\angle DFC < \pi/2$ and $z(D) > z(F)$ imply $z(F) < z(C)$, and thus $z(A) < z(F) < z(C)$.

Thus, $z(A) < z(B)$, $z(A) < z(C)$, and $AB \perp$ (the plane DFC). Now, if we rotate T around the line DF so that the inclination of AB becomes steeper (B goes up, A goes down in the z-direction), then A and B moves inward $\mathbf{P}$. In this case

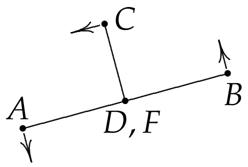


Figure 4: View in the direction from F to D

the vertex C moves in the direction $\overrightarrow{BA}$, see Figure 4, and thus, $\bar{C}$ moves in the direction $\overrightarrow{\bar{B}\bar{A}}$. Namely, $\bar{C}$ moves into the interior Δ° of Δ, because $|\bar{A}\bar{D}| < |\bar{B}\bar{D}|$, see Figure 3. Therefore, C moves inward $\mathbf{P}$. Hence all A, B, C become interior points of $\mathbf{P}$. $\square$

4 Minimal containment size of a unit tetrahedron

Lemma 4.1. *Let $T = ABCD \subset \mathbf{P}$ be an embedding such that T has at least two corner vertices and has no interior vertex. Then the following holds.*

(1) *If T is a tangential embedding, then size$(\mathbf{P}) = t_1$, and the embedding is equivalent to one of $\beta(\sigma, \delta)$.*

(2) *If T is not a tangential embedding, then size$(\mathbf{P}) = t_0$ and the embedding is equivalent to one of $\alpha(\sigma, \epsilon)$.*

Proof. Since two corner vertices cannot lie on the same corner line (because the dihedral angle of T is greater than $\pi/3$), T cannot have three corner vertices, for otherwise, size$(\mathbf{P})$ would be 1 and one vertex would be an interior vertex. Hence T has exactly two corner vertices.

(1) First suppose that T is a tangential embedding. Let A, B be the two corner vertices of T, and let σ be the face of $\mathbf{P}$ that contains the edge AB. Then C or D lies on σ. This can be seen as follows: Suppose that none of C, D lies on σ. Then, since T is a tangential embedding, C, D and one of A, B, say, B lie on the

same face of **P**. Let Z be the barycenter of BCD. We may suppose that AZ lies on the xy-plane. Now, when we rotate T around AZ, then the minimum value θ of $\max\{\angle ZA\bar{B}, \angle ZA\bar{C}, \angle ZA\bar{D}\}$ is attained in the case that one of $\bar{B}, \bar{C}, \bar{D}$, say $\bar{D}$ coincides with Z. In this case, since $|\bar{B}Z| = 1/2$ and $|AZ| = \sqrt{2/3}$, we have $\tan\angle ZA\bar{B} = (1/2)/\sqrt{2/3} = \sqrt{3/8} > \sqrt{1/3} = \tan(\pi/6)$. Therefore, $\theta > \pi/6$. This implies that if A is a corner vertex, and BCD lie on the plane determined by the opposite face of the corner where A is lying, then $ABCD$ is never contained in the prism **P**. Thus, one of C, D, say C lies on σ.

Let G be the barycenter of ABC, and let τ be the face containing D. Then, GD is horizontal. We may suppose that GD lie on the xy-plane. Suppose that $A \in \tau \cap \sigma$, see Figure 5. Let us verify that this is $\beta(\sigma, \delta)$ for some δ given in the proof of Lemma 2.2, and $\mathrm{size}(\mathbf{P}_1) = \mathrm{t}_1$.

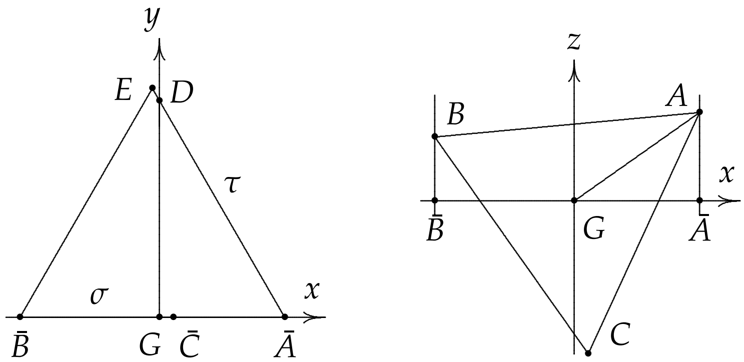


Figure 5: Top view of $\mathbf{P}_1$ and front view of σ

Let $\phi = \angle\bar{A}AG$. Then $\mathrm{size}(\mathbf{P}_1) = |\bar{\mathrm{A}}\bar{\mathrm{B}}| = |\mathrm{AB}|\sin(\angle\bar{\mathrm{A}}\mathrm{AB}) = \sin(\angle\bar{\mathrm{A}}\mathrm{AG} + \angle\mathrm{GAB}) = \sin(\phi + \pi/6)$. On the other hand, using $|GD| = \sqrt{2/3}$ and $\angle\bar{A}DG = \pi/6$, we have $|\bar{A}G| = \sqrt{2}/3$, and thus $\sin\phi = |\bar{A}G|/|AG| = \sqrt{2/3}$, $\cos\phi = 1/\sqrt{3}$. Hence

$$\bar{A}\bar{B} = \sin(\phi + \pi/6) = \sin\phi\cos(\pi/6) + \cos\phi\sin(\pi/6) = (3\sqrt{2} + \sqrt{3})/6,$$

namely, $\mathrm{size}(\mathbf{P}_1) = |\bar{\mathrm{A}}\bar{\mathrm{B}}| = \mathrm{t}_1$, which proves the tangential embedding case.

(2) Now we consider the non-tangential embedding case. Let B, C be the two corner vertices of T. Then, none of A, D lies on the face of **P** that conains the edge BC, and A, D lie on different faces of **P**.

Let us show that $\bar{A}\bar{D} \parallel \bar{B}\bar{C}$. Let σ be the face of **P** that contains BC. Let Π be the plane that perpendicularly bisects BC. Then A, D lie on Π. Let XYZ be the section of **P** by Π, YZ be the line segment $\Pi \cap \sigma$, X be the intersection point of Π and the corner line of **P** opposite to σ. Let M be the midpoint of BC (and hence the midpoint of YZ). Then the segment XM is horizontal. Thus XYZ is an isosceles triangle with base YZ, and A, D lie on $XY \cup XZ$. Since $|MX| < \sqrt{3}/2$

and $|BM| = |CM| = 1/2$, the locus γ of points on Π that are at unit distance apart from B (and C) is a circle with center M, radius $\sqrt{3}/2$. Since X lies inside the circle γ, XY intersects γ at a single point, and also XZ intersects γ at a single point. Thus $(XY \cup XZ) \cap \gamma$ consists of two points, and they must be A and D, see Figure 6. Since XYZ is an isosceles triangle with base YZ, we have $AD \parallel YZ$, and hence $\bar{A}\bar{D} \parallel \bar{Y}\bar{Z}$. Since the two lines $\bar{B}\bar{C}$ and $\bar{Y}\bar{Z}$ are the same line, we have $\bar{A}\bar{D} \parallel \bar{B}\bar{C}$.

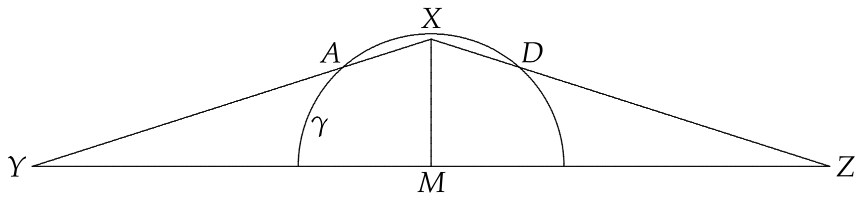


Figure 6: On the plane Π that bisects BC perpendicularly

Thus, $\bar{A}\bar{B}\bar{C}\bar{D}$ (or $\bar{D}\bar{B}\bar{C}\bar{A}$) is a trapezoid in Δ with all vertices on $\partial\Delta$, and $\bar{B}\bar{C}$ is an edge of Δ, just as shown in Figure 1. Let us find the edge length $t = |\bar{B}\bar{C}|$ of Δ. Since the height of the trapezoid is the distance between the opposite edges of $ABCD$, it is equal to $1/\sqrt{2}$. Then, by comparing the heights of the equilateral triangles $\bar{A}\bar{D}X$ and $\bar{B}\bar{C}X$, we have $|\bar{A}\bar{D}| : t = (\sqrt{3}t/2 - 1/\sqrt{2}) : \sqrt{3}t/2$, and thus $|\bar{A}\bar{D}| = t - \sqrt{2/3}$. Let θ be the angle of inclination of AD. Then, since $\bar{A}\bar{D} \parallel \bar{B}\bar{C}$, the angle of inclination of BC is $\pi/2 - \theta$. Hence $t - \sqrt{2/3} = |\bar{A}\bar{D}| = \cos\theta$ and $t = |\bar{B}\bar{C}| = \sin\theta$. Therefore, $1 = (t - \sqrt{2/3})^2 + t^2$, and solving this equation we have $t = (1 + \sqrt{2})/\sqrt{6}$. This proves that $\text{size}(\mathbf{P}) = t_0$ and T is equivalent to one of $\alpha(\sigma, \epsilon)$. □

Lemma 4.2. *For any embedding $T \subset \mathbf{P}(t)$, there is the minimum value s_0 such that T is equivalent to $T_0 \subset \mathbf{P}(s_0) \subset \mathbf{P}(t)$ in $\mathbf{P}(t)$. Moreover, $s_0 = t_0$ or $s_0 = t_1$.*

Proof. Let $s_0 = \inf\{s \leq t \mid \exists T' \subset \mathbf{P}(s) \text{ such that } T \sim T' \text{ in } \mathbf{P}(t)\}$. Then there is a sequence of points $(A_n, B_n, C_n, D_n) \in \mathbb{R}^{12}, n = 1, 2, 3, \ldots$, and a sequence $s_n \in \mathbb{R}_+$, $n = 1, 2, 3, \ldots$, such that for each n,

1. $T_n := A_nB_nC_nD_n$ is a unit tetrahedron contained in $\mathbf{P}(s_n) \cap [-2 \leq z \leq 2]$,
2. $T \sim T_n$ in $\mathbf{P}(t)$, and
3. $\lim s_n = s_0$,

where $[-2 \leq z \leq 2] := \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq z \leq 2\}$. Since $\mathbf{P}(t) \cap [-2 \leq z \leq 2]$ is compact, a convergent subsequence (A_m, B_m, C_m, D_m) exists and converges to (A_0, B_0, C_0, D_0). Then $T_0 := A_0B_0C_0D_0$ is a unit tetrahedron contained in $\mathbf{P}(s_0)$. Let $c_m = 1/2^2 + 1/2^3 + \cdots + 1/2^m$. Since $T_m \sim T_{m+1}$ in $\mathbf{P}(t)$, there is a motion $F_m : T_m \times [c_m, c_{m+1}] \to \mathbf{P}(t)$ of T_m that sends T_m to T_{m+1} and a motion $F : T \times$

$[0,1/2] \to \mathbf{P}(t)$ that send T to T_1. Connecting these motions, we have a motion $F : T \times [0,1) \to \mathbf{P}(t)$. This motion can be extended to $F : T \times [0,1] \to \mathbf{P}(t)$ by putting $F(A,1) = A_0, \ldots, F(D,1) = D_0$ and extending linearly for all $x \in T$. Then F is a continuous map and a motion of T to T_0. Since s_0 is the minimum containment size, T_0 satisfies neither (1) nor (2) of Lemma 3.2. Hence, by Lemma 4.1, we have $s_0 = t_0$ or $s_0 = t_1$. □

Corollary 4.1. *For $t_0 \leq s < t_1$, every embedding $T \subset \mathbf{P}(s)$ is equivalent to one of $\alpha(\sigma, \epsilon)$, and for $t_1 \leq t$, every embedding $T \subset \mathbf{P}(t)$ is equivalent to one of $\alpha(\sigma, \epsilon)$ or one of $\beta(\sigma, \delta)$.* □

5 Territories and borders in a prism

In a prism $\mathbf{P}$, the **territory** of a corner of $\mathbf{P}$ consists of those points of $\mathbf{P}$ that are nearer to the corner than to other corners. Each territory is a quadrilateral prism, and the three territories are mutually congruent. A **border** is the intersection of any two territories, see Figure 7.

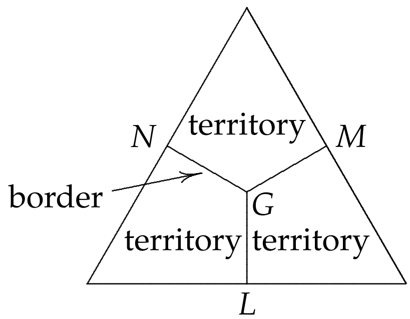


Figure 7: Territories and borders of $\mathbf{P}$, top view

Lemma 5.1. *Let $\mathbf{P}$ be a prism of size $t < 1$ and let $T \subset \mathbf{P}$ be a unit tetrahedron. Then no vertex of T lies on a border.*

Proof. Let $T = ABCD$ and suppose that A lies on a border. We may assume that $z(A) = 0 < z(B)$. Let Δ be the section of $\mathbf{P}$ by the plane $z = 0$, G be the barycenter of Δ, and L, M, N be the midpoints of the edges of Δ, see Figure 7. Then A lies on $GL \cup GM \cup GN$. We may suppose that A lies on GM. Let Ω be the intersection of $\mathbf{P}$ and the unit sphere with center A. This intersection Ω is the union of two connected surfaces, Ω^+ in the half space $z > 0$ and Ω^- in the half space $z < 0$. Figure 8 shows the upper surface Ω^+. The vertex B lies on Ω^+. Let P, Q, R be the corner point such that $|AP| = |AQ| = |AR| = 1$ and $z(P) = z(Q) > z(R) > 0$. Then Ω^+ intersects the faces of $\mathbf{P}$ in three circular arcs $\widehat{PQ}, \widehat{QR}, \widehat{RP}$. Let S be the corner point on the same corner line as R such that $z(S) = z(P)$. (If $A = G$, then R

coincides with S.) Let the arcs $\widehat{RQ}$ and $\widehat{RP}$ cross SQ and SP at U, V, respectively. If $A = G$ then $R = S = U = V$, and if $A = M$, then U is the midpoint of SQ and $z(R) = \sqrt{1 - 3t^2/4}$. Hence we have

$$|SU| = |SV| \le t/2, \;\; 1/2 < \sqrt{1 - 3t^2/4} \le z(R) \le z(P) \le \sqrt{1 - t^2/4} < 1. \quad (1)$$

Thus, Ω^+ is contained in the open half space $z > 1/2$. Similarly, Ω^- is contained in the open half space $z < -1/2$. Since B lies on Ω^+, the remaining vertices C, D must also lie on Ω^+.

From (1), we have $|RU| = |RV| < 1$, $|UV| < 1$, $|PQ| = |QS| = |SP| < 1$. Hence, we can deduce that

$$\max\{\angle PAQ, \angle PAS, \angle QAS, \angle UAV, \angle UAR, \angle VAR\} < \pi/3. \quad (2)$$

Now, we divide Ω^+ by the plane $z = z(P)$ into two surfaces; Ω_1^+ ,the upper part, and Ω_2^+, the lower part. Here, we note that if X, Y belong the tetrahedron $APQS$, then $\angle XAY \le \max\{\angle PAQ, \angle PAS, \angle QAS\}$. (Proof of this fact will be elementary.) From this fact it follows that for any points $X, Y \in \Omega_1^+$, $\angle XAY < \pi/3$. This implies that the diameter of Ω_1^+ is less than 1, and hence Ω_1^+ cannot contain more than one vertex of T. Similarly, the diameter of Ω_2^+ is less than 1, and it cannot contain more than one vertex of T. Therefore, Ω^+ cannot contain the three vertices B, C, D, which is a contradiction. □

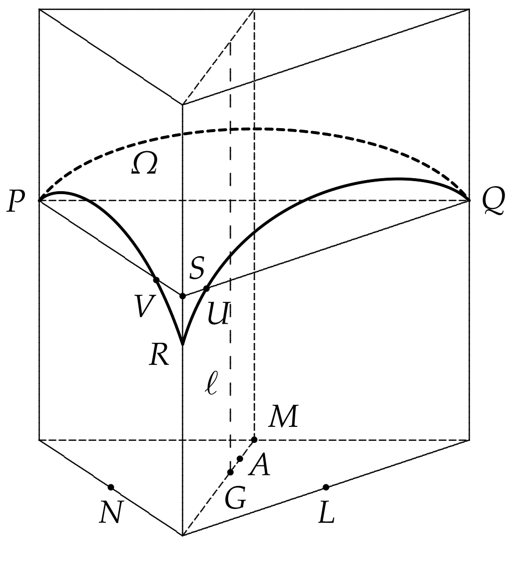


Figure 8: A section by the unit sphere with center A

Let $T \subset \mathbf{P}$ be an embedding into a prism of size $t < 1$. Then, since $\mathbf{P}$ cannot contain a horizontal line segment of length 1, we can label the vertices of T with

A_1, A_2, A_3, A_4 so that $z(A_1) < z(A_2) < z(A_3) < z(A_4)$. We call the vertex of label A_i the ith vertex of T. Notice that the labels of the vertices of T do not vary under any continuous motion of T within $\mathbf{P}$.

Lemma 5.2. *Let $T_1, T_2 \subset \mathbf{P}(t)$ $(t_0 \le t < 1)$ be two embeddings of a unit tetrahedron. If, for some $i = 1, 2, 3, 4$, the ith vertex of T_1 and the ith vertex of T_2 lie in different territories, then T_1 and T_2 are not equivalent.*

Proof. If $T_1$and T_2 are equivalent then there is a motion of T_1 in $\mathbf{P}(t)$ which sends the ith vertex of T_1 to the ith vertex of T_2. Since they belong different territories in the begining, the ith vertex of T_1 must cross a border in the midway, which is impossible by Lemma 5.1. □

Lemma 5.3. *If $T \subset \mathbf{P}$, size$(\mathbf{P}) = t < 1$, then each territory of $\mathbf{P}$ contains a vertex of T.*

Proof. Since the width of the union of two territories is $(\frac{\sqrt{3}}{4})t$ (see Figure 7) which is smaller than $1/\sqrt{2}$, the width of T (see [10] or [6]), the convex hull of two territories cannot contain T. Hence each territory contains a vertex of T. □

Lemma 5.4. *Let $\mathbf{P}$ be a prism of size $t < 1$ and $T \subset \mathbf{P}$ be a unit tetrahedron. Suppose that the vertices A_1, A_4 of T lie in the territory of a corner line ℓ. Then the line A_1A_4 is never parallel to (or never contained in) the plane that bisects the dihedral angle at ℓ.*

Proof. Suppose that A_1A_4 is parallel to the plane H that bisects the dihedral angle at ℓ. We may suppose that H is the xz-plane in $\mathbb{R}^3$. Let K be the plane that perpendicularly bisects the edge A_1A_4. Then, K intersects H orthogonally. Hence the section of $\mathbf{P}$ by K is an isosceles triangle XYZ with base YZ in the face of $\mathbf{P}$ opposite to ℓ. Then $|YZ| = t$ and $s := |XY| = |XZ| > t$. Let L, M, N be the midpoints of YZ, ZX, XY, respectively, and let G be the barycenter of XYZ as shown in Figure 9. Since the width of a unit tetrahedron is $1/\sqrt{2}$, we have $z(A_4) - z(A_1) \ge 1/\sqrt{2}$. Hence the angle between the line A_1A_4 and the xy-plane is at least $\pi/4$, and hence the angle between K and the xy-plane is at most $\pi/4$. Therefore, $|XL|$ is at most $\sqrt{2} \times (\frac{\sqrt{3}}{2})t$, and hence $s = |XZ| = |XY| \le (\frac{\sqrt{7}}{2})t$.

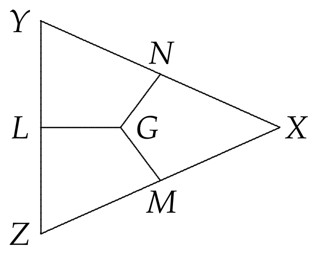


Figure 9: Section of $\mathbf{P}$ by the perpendicular bisector of AD

Since A_2, A_3 lie on the plane K, and hence lie on the isosceles triangle XYZ. Since A_1, A_4 are in the territory containing X, the vertices A_2, A_3 must lie in the pentagon $YZMGN$. On the other hand, by applying the parallelogram theorem, we have

$$\begin{aligned} |YM|^2 = |ZN|^2 &= \frac{1}{2}|ZY|^2 + \frac{1}{2}|ZX|^2 - |XN|^2 \\ &= \frac{1}{2}t^2 + \frac{1}{2}s^2 - \frac{1}{4}s^2 = \frac{1}{2}t^2 + \frac{1}{4}s^2 \\ &\leq \frac{1}{2}t^2 + \frac{1}{4}(\frac{\sqrt{7}}{2}t)^2 = \frac{15}{16}t^2 < 1. \end{aligned}$$

Hence the diamter of the pentagon $YZMGN$ is less than 1. This implies that the pentagon $YZMGN$ cannot contain $\{A_2, A_3\}$, a contradiction. □

6 Proof of the theorem

Lemma 6.1. *If $t_0 \leq t < 1$, the six $\alpha = \alpha(\sigma, \epsilon)$ are mutually non-equivalent in $\mathbf{P}(t)$.*

Proof. Let $\sigma_1, \sigma_2, \sigma_3$ be the faces of $\mathbf{P}(t)$ such that A_3 of $\alpha(\sigma_1, \epsilon_1)$ (the cross embedding constructed in the proof of Lemma 2.1) lies on the line $\sigma_1 \cap \sigma_2$. We prove that $\alpha(\sigma_i, \epsilon_j)$, $i = 1, 2, 3$, $j = 1, 2$ are all non-equivalent in $\mathbf{P}(t)$. Proof is given by the following table. Let us explain what means a number in a cell of the table. Look at, for instance, the cell in the low of $\alpha(\sigma_2, \epsilon_2)$ and the column of $\alpha(\sigma_3, \epsilon_1)$. The number in this cell is 1. This means that the first vertex of $\alpha(\sigma_2, \epsilon_2)$ and the first vertex of $\alpha(\sigma_3, \epsilon_1)$ lie in different territories. Then by Lemma 5.2, we have $\alpha(\sigma_2, \epsilon_2) \not\sim \alpha(\sigma_3, \epsilon_1)$ in $\mathbf{P}(t)$. Now it is easy to check that the entries in the cells are all correct.

	$\alpha(\sigma_1, \epsilon_1)$	$\alpha(\sigma_1, \epsilon_2)$	$\alpha(\sigma_2, \epsilon_1)$	$\alpha(\sigma_2, \epsilon_2)$	$\alpha(\sigma_3, \epsilon_1)$	$\alpha(\sigma_3, \epsilon_2)$
$\alpha(\sigma_1, \epsilon_1)$	-	2	1	1	1	1
$\alpha(\sigma_1, \epsilon_2)$	2	-	1	1	1	1
$\alpha(\sigma_2, \epsilon_1)$	1	1	-	2	1	1
$\alpha(\sigma_2, \epsilon_2)$	1	1	2	-	1	1
$\alpha(\sigma_3, \epsilon_1)$	1	1	1	1	-	2
$\alpha(\sigma_3, \epsilon_2)$	1	1	1	1	2	-

□

Lemma 6.2. *For $t_1 \leq t < 1$, the twelve $\beta(\sigma, \delta)$ are mutually non-equivalent in $\mathbf{P}(t)$.*

Proof. First we show that $\beta(\sigma, \delta_i)$, $i = 1, 2, 3, 4$ are all non-equivalent in $\mathbf{P}(t)$. Proof is given by the following table.

-	$\beta(\sigma, \delta_1)$	$\beta(\sigma, \delta_2)$	$\beta(\sigma, \delta_3)$	$\beta(\sigma, \delta_4)$
$\beta(\sigma, \delta_1)$	-	1	2	1
$\beta(\sigma, \delta_2)$	1	-	1	2
$\beta(\sigma, \delta_3)$	2	1	-	1
$\beta(\sigma, \delta_4)$	1	2	1	-

Now it will be sufficient to show that if $\sigma_1 \neq \sigma_2$, then $\beta(\sigma_1, \delta_i)$ and $\beta(\sigma_2, \delta_j)$ are not equivalent in $\mathbf{P}(t)$ for all i, j. To make the argument clear, we may suppose that $\sigma_1 \cap \sigma_2$ contains the vertex A_4 (the highest vertex) of $\beta(\sigma_1, \delta_1)$. Then, we have the following incomplete table with two blank cells.

-	$\beta(\sigma_2, \delta_1)$	$\beta(\sigma_2, \delta_2)$	$\beta(\sigma_2, \delta_3)$	$\beta(\sigma_2, \delta_4)$
$\beta(\sigma_1, \delta_1)$	1	2	1	
$\beta(\sigma_1, \delta_2)$	1	1	1	1
$\beta(\sigma_1, \delta_3)$	1		1	2
$\beta(\sigma_1, \delta_4)$	1	1	1	1

Let us show that $\beta(\sigma_1, \delta_1)$ and $\beta(\sigma_2, \delta_4)$ (corresponding to the upper-right blank cell) are not equivalent in $\mathbf{P}(t)$.

Note that in both $\beta(\sigma_1, \delta_1)$ and $\beta(\sigma_2, \delta_4)$, the vertices A_1, A_4 lie in the territory of the corner line $\ell := \sigma_1 \cap \sigma_2$. Let H be the plane that bisects the dihedral angle at the corner ℓ of $\mathbf{P}(t)$. Let $d(P, H)$ denote the distance from a point P to the plane H. Then, in $\beta(\sigma_1, \delta_1)$, we have $d(A_1, H) > 0, d(A_4, H) = 0$, whereas, in $\beta(\sigma_2, \delta_4)$, we have $d(A_1, H) = 0$, $d(A_4, H) > 0$. Therefore, if $\beta(\sigma_1, \delta_1) \sim \beta(\sigma_2, \delta_4)$, then on the way of the motion of $\beta(\sigma_1, \delta_1)$ from its original position to the position of $\beta(\sigma_2, \delta_4)$, there must be a moment $d(A_1, H) = d(A_4, H)$ holds. But this is impossible by Lemma 5.4. Hence $\beta(\sigma_1, \delta_1) \not\sim \beta(\sigma_2, \delta_4)$ in $\mathbf{P}(t)$. Similarly, it can be proved by applying Lemma 5.4 that $\beta(\sigma_1, \delta_3)$ and $\beta(\sigma_2, \delta_2)$ (the ones corresponding to the other blank cell) are not equivalent in $\mathbf{P}(t)$. Thus, all twelve $\beta(\sigma, \delta)$ are mutually non-equivalent in $\mathbf{P}(t)$. $\square$

Corollary 6.1. *Let $t_1 \leq t < 1$. Then no β is equivalent to an α in $\mathbf{P}(t)$.*

Proof. If some β is equivalent to some α in $\mathbf{P}(t)$, then every β would be equivalent to an α in $\mathbf{P}(t)$. However, mutually non-equivalent twelve βs cannot be equivalent to six αs. $\square$

Proof of Theorem 1.1.

By Lemma 4.2, $\nu(t) = 0$ for $t < t_0$, and by Corollary 4.1 and Lemma 6.1, we have $\nu(t) = 6$ for $t_0 \leq t < t_1$. By Corollary 4.1, Lemmas 6.1, 6.2 and Corollary 6.1, it follows that $\nu(t) = 6 + 12 = 18$ for $t_1 \leq t < 1$.

Now, suppose that $t = 1$. Then, every $T \subset \mathbf{P}(1)$ is equivalent to some α or some β by Corollary 4.1. Suppose that $T \subset \mathbf{P}(1)$ is equivalent to some α, say, to the cross embedding $ABCD \subset \mathbf{P}(t_0)$ given in the proof of Lemma 2.1. Then by applying a translation along the y-axis, we may suppose that the edge BC lies on a face, say σ of $\mathbf{P}(1)$, and the line L passing through the midpoint of BC and perpendicular to σ meets the corner line opposite to the face σ. Then, by rotating around the line L, we can move $ABCD$ within $\mathbf{P}(1)$ so that BC becomes horizontal and AB becomes vertical. Now, rotating the resulting tetrahedron around the horizontal line BC within $\mathbf{P}(1)$ so that a face of the tetrahedron becomes horizontal, and one vertex lies above the horizontal face.

Next, suppose that $T \subset \mathbf{P}(1)$ is equivalent to a β, say, to the tangential embedding $ABCD \subset \mathbf{P}(t_1)$ given in the proof of Lemma 2.2. We can translate $ABCD$ so that A comes to the corner line and ABC lie on a face σ of $\mathbf{P}(1)$. Now, by rotating $ABCD$ around the line passing through A and perpendicular to the face σ, we can make the line AB horizontal. Then, rotate around the edge AB, we can make one face of the tetrahedron horizontal, and one vertex lies above the face. Thus, every embedding $T \subset \mathbf{P}(1)$ is equivalent to an embedding in which one face is horizontal and one vertex is above the horizontal face. Therefore $\nu(1) = 1$, and hence $\nu(t) = 1$ for $t \geq 1$. This completes the proof. □

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