A FROG'S RANDOM JUMP AND THE PÓLYA IDENTITY

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ABSTRACT. A frog jumps along the lattice points on the x-axis. Starting from $x = x_0$, he jumps ℓ steps to the left with probability p, or he jumps r steps to the right with probability $1-p$ at each time. What is the probability that he ever lands on the origin? We answer this question by using a closed formula for $\sum_{k\geq 0}\binom{ck+s}{dk+t}z^k$, which is an extension of the Pólya identity. We also include a combinatorial proof of the Pólya identity.

1. A FROG PROBLEM AND THE PÓLYA IDENTITY

In this paper we consider the following problem (cf. section 10.6 of [3]).

Problem 1. *A frog lives on the line $\mathbb{Z}$. Starting from $x = x_0$, he jumps ℓ steps to the left (from x to $x-\ell$) with probability p, or he jumps r steps to the right (from x to $x+r$) with probability $q = 1-p$ at each time $t = 1, 2, \ldots$. What is the probability that we can catch him by setting a trap at the origin?*

More formally we consider random variables $X_1, X_2, \ldots$ with $\text{Prob}(X_i = -\ell) = p$ and $\text{Prob}(X_i = r) = q$ for all $i \geq 1$. Let f_k be the probability that the frog lands on the origin after k jumps for the first time, i.e.,

$$f_k = \text{Prob}(\textstyle\sum_{i=1}^{k} X_i = -x_0 \text{ and } \sum_{i=1}^{\ell} X_i \neq -x_0 \text{ for all } \ell < k).$$

Then what is $\sum_{i=1}^{\infty} f_k$? This definition of the probability is valid for all starting position $x_0 \in \mathbb{Z}$, but we exceptionally define the probability for the case $x_0 = 0$ (the case starting from the origin) to be 1 just for a technical reason.

Another way to state the problem is as follows.

Problem 2. *A frog lives in $\mathbb{Z}^2$. Starting from the origin, he jumps one unit up with probability p, or he jumps one unit right with probability $q = 1-p$ at each time. Then what is the probability that the frog ever lands on the line $rx - \ell y + x_0 = 0$?*

An l steps jump to the left (resp. r steps jump to the right) in Problem 1 is corresponding to one unit jump upwards (resp. one unit jump to the right) in Problem 2.

Problem 2 is naturally arisen when one deals with multiply intersecting families in extremal set theory, which was one of the motivations of this paper. In fact, the answer to the problem (and its variations) plays an important role in [1] and [8]. Also the problem is related to some interesting identities appeared in enumerative combinatorics. Among others, we give a closed formula for $\sum_{k\geq 0}\binom{ck+s}{dk+t}$, which is an extension of the Pólya identity.

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This paper is organized as follows. In this section, we solve a special case of the problem and give a heuristic proof of the Pólya identity. In section 2, we extend the Pólya identity and get some identities which we will use in the later sections. In section 3, we answer to the problem and show some concrete computations about the probability. A variation of the problem is considered in section 4.

To warm up let us solve Problem 2 for the case $x_0 = 1, \ell = 1, r = 2$. Define $\alpha = \alpha(p)$ as follows.

$$\alpha = \begin{cases} \frac{1}{2}(\sqrt{\frac{1+3p}{1-p}} - 1) & \text{if } 0 \le p \le 2/3, \\ 1 & \text{if } 2/3 \le p \le 1. \end{cases} \tag{1}$$

Note that α is a root of the equation $X = p + qX^3$.

Fact 1. Prob(*the frog reaches* $y = 2x+1) = \alpha$.

Proof. Let

$a_k :=$ Prob(the frog reaches $y = 2x+1$ at $(k, 2k+1)$ for the first time),
$b_k :=$ Prob(the frog reaches $y = 2x+2$ at $(k, 2k+2)$ for the first time),
$c_k :=$ Prob(the frog reaches $y = 2x+3$ at $(k, 2k+3)$ for the first time).

(Note that $a_k = f_{3k+1}$, i.e., this is the probability that the frog hits the trap after $3k+1$ jumps. Similarly b_k corresponds to f_{3k+2} with different starting position $x_0 = 2$.) Let $A(x) := \sum_{k\ge 0} a_k x^k$, $B(x) := \sum_{k\ge 0} b_k x^k$, $C(x) := \sum_{k\ge 0} c_k x^k$ be generating functions. Suppose that the frog reaches $y = 2x+2$ at $(k, 2k+2)$ for the first time. Then he needs to reach $y = 2x+1$ at some point, say, at $(i, 2i+1)$ for the first time, which happens with probability a_i. Then during the journey from $(i, 2i+1)$ to $(k, 2k+2)$, he touches $y = 2x+2$ only at $(k, 2k+2)$ and this happens with probability a_{k-i}. Therefore we have $b_k = \sum_{i=0}^{k} a_i a_{k-i}$, which implies

$$B(x) = A(x)^2.$$

Similarly we have $c_k = \sum_{i=0}^{k} a_i b_{k-i}$, which implies

$$C(x) = A(x)B(x) = A(x)^3.$$

Since $a_0 = p$ and $a_i = qc_{i-1}$ for $i \ge 1$ we have

$$A(x) = p + qxC(x) = p + qxA(x)^3.$$

Substituting $x = 1$ and setting $X = A(1)$ we have

$$X = p + qX^3.$$

We have to choose the solution $X = y$ so that y becomes 0 for $p = 0$, and y becomes 1 for $p = 1$. Moreover y has to be a continuous function of p. Therefore we have $y(p) \equiv \alpha(p)$. □

The proof also gives

$$\text{Prob}(\text{the frog reaches } y = 2x+2) = B(1) = \alpha^2,$$
$$\text{Prob}(\text{the frog reaches } y = 2x+3) = C(1) = \alpha^3.$$

Then for $s \in \mathbb{N} := \{0,1,2,\dots\}$ it follows from induction that

$$\text{Prob}(\text{the frog reaches } y = 2x+s) = \alpha^s. \tag{2}$$

Let

$$u_k := \text{Prob}(\text{the frog arrives at } (k,2k)) = \binom{3k}{k} p^{2k}q^k,$$
$$v_k := \text{Prob}(\text{the frog arrives at } (k,2k+1)) = \binom{3k+1}{k} p^{2k+1}q^k.$$

Let $U(x) := \sum_{k\geq 0} u_k x^k$, $V(x) := \sum_{k\geq 0} v_k x^k$ be generating functions. Since $v_k = \sum_{i=0}^k a_i u_{k-i}$ we have $V(x) = A(x)U(x)$. Substituting $x = 1$, we have

$$\sum_{k\geq 0} \binom{3k+1}{k} p^{2k+1}q^k = \alpha u,$$

where $u := U(1) = \sum_{k\geq 0} \binom{3k}{k} p^{2k}q^k$. To generalize the above identity, let

$$w_k^{(j)} := \text{Prob}(\text{the frog arrives at } (k,2k+j)) = \binom{3k+j}{k} p^{2k+j}q^k$$

for $j \in \mathbb{Z}$, and let $W^{(j)}(x) := \sum_{k\geq 0} w_k^{(j)} x^k$. Since $w_k^{(j)} = \sum_{i=0}^k a_i w_{k-i}^{(j-1)}$ we have $W^{(j)}(x) = A(x)W^{(j-1)}(x)$. Substituting $x = 1$ and noting $W^{(0)}(1) = U(1) = u$, we have $W^{(s)}(1) = \alpha^s u$ for $s \in \mathbb{Z}$, namely,

$$\sum_{k\geq 0} \binom{3k+s}{k} p^{2k+s}q^k = \alpha^s u. \tag{3}$$

Fact 2. $u = (1-3q\alpha^2)^{-1}$.

Proof. First note that

$$\begin{aligned}
\sum_{k\geq 0} \binom{3k+3}{k+1} p^{2k+2}q^{k+1} &= \sum_{k\geq 0} \binom{3(k+1)}{k+1} p^{2(k+1)}q^{k+1} \\
&= \sum_{k\geq 1} \binom{3k}{k} p^{2k}q^k \\
&= \sum_{k\geq 0} \binom{3k}{k} p^{2k}q^k - 1 = u-1.
\end{aligned}$$

On the other hand, using $\binom{3k+3}{k+1} = \frac{3k+3}{k+1}\binom{3k+2}{k} = 3\binom{3k+2}{k}$, we have

$$\begin{aligned}
\sum_{k\geq 0} \binom{3k+3}{k+1} p^{2k+2}q^{k+1} &= \sum_{k\geq 0} 3\binom{3k+2}{k} p^{2k+2}q^{k+1} \\
&= 3q\sum_{k\geq 0} \binom{3k+2}{k} p^{2k+2}q^k \\
&= 3q\alpha^2 u. \quad \text{(by (3))}
\end{aligned}$$

Thus we have $u-1 = 3q\alpha^2 u$, or equivalently, $u = (1-3q\alpha^2)^{-1}$. □

For example, by setting $p=q=1/2$, Fact 2 gives

$$u=\sum_{k\geq 0}\binom{3k}{k}\left(\frac{1}{2}\right)^{3k}=1+\frac{3}{\sqrt{5}}. \tag{4}$$

Now we extend Fact 2. By (3) and Fact 2 we have

$$\sum_{k\geq 0}\binom{3k+s}{k}p^{2k+s}q^k=\alpha^s(1-3q\alpha^2)^{-1}.$$

In the same way, by considering the case $\ell=1$ and $r=m-1$, one can show

$$\sum_{k\geq 0}\binom{mk+s}{k}p^{(m-1)k+s}q^k=\alpha^s(1-mq\alpha^{m-1})^{-1}$$

for $m\in\mathbb{N}-\{0\}$ and $s\in\mathbb{Z}$, where α is the root of the equation $X=p+qX^m$ which becomes 0 for $p=0$ if $0<p<\frac{m-1}{m}$, and $\alpha=1$ if $\frac{m-1}{m}<p<1$. Since $\alpha=p+q\alpha^m$, or $1=\frac{p}{\alpha}+q\alpha^{m-1}$, we can rewrite the identity as

$$\sum_{k\geq 0}\binom{mk+s}{k}(p^{m-1}q)^k=\left(\frac{\alpha}{p}\right)^s\left(1-m(1-\frac{p}{\alpha})\right)^{-1}.$$

By setting $z=p^{m-1}q$ and $\beta=\frac{\alpha}{p}$, the above identity is equivalent to **the Pólya identity**[5]

$$\sum_{k\geq 0}\binom{mk+s}{k}z^k=\frac{\beta^{s+1}}{(1-m)\beta+m}, \tag{5}$$

where β is the root of the equation $X=1+zX^m$ which becomes 1 for $z=0$. In fact (5) holds for $m,s\in\mathbb{C}$. This can be checked by using Lagrange inversion with routine calculation (cf. Problem 216 of [6]). In the same assumptions, it follows also from Lagrange inversion that

$$\sum_{k\geq 0}\frac{s}{mk+s}\binom{mk+s}{k}z^k=\beta^s \tag{6}$$

(cf. Problem 212 of [6]), which is equivalent to

$$\sum_{k\geq 0}\frac{s}{mk+s}\binom{mk+s}{k}p^{(m-1)k+s}q^k=\alpha^s. \tag{7}$$

It is also possible to get (5) by differentiating (6). See [7] 6.2.6 and 6.2.7.

On the other hand, it follows that

$$\text{Prob}(\text{the frog reaches } y=(m-1)x+s)=\alpha^s \tag{8}$$

for $m,s\in\mathbb{N}-\{0\}$ in the same way as we get (2). Comparing (7) and (8) we find that

$$a_k=\frac{s}{mk+s}\binom{mk+s}{k}p^{(m-1)k+s}q^k$$

where a_k is the probability that the frog reaches the line $y=(m-1)x+s$ at $(k,(m-1)k+s)$ for the first time. In other words, we have the following.

Fact 3. *Let $m, s \in \mathbb{N} - \{0\}$ and $k \in \mathbb{N}$. Then the number of walks from $(0,0)$ to $(k, (m-1)k+s)$ which hit $y = (m-1)x+s$ only at $(k, (m-1)k+s)$ is given by $\frac{s}{mk+s}\binom{mk+s}{k}$.*

By counting the number of walks from $(0,0)$ to $(n, (m-1)n+s)$ we get the following identity (cf. (5.62) of [2] or Example 8.1.1 in [4]):

$$\binom{mn+s}{n} = \sum_{k=0}^{n} \frac{s}{mk+s}\binom{mk+s}{k}\binom{m(n-k)}{n-k}.$$

2. Extending the Pólya identity

We extend the Pólya identity.

Theorem 1. *For $m, s \in \mathbb{C}$ and $t \in \mathbb{N}$ we have*

$$\sum_{k\geq 0}\binom{mk+s}{k+t}z^k = \frac{\beta^{s+1}}{(\beta-1)^t((1-m)\beta+m)} - \sum_{j=1}^{t}\binom{s-mj}{t-j}z^{-j}$$

where β is the root of the equation $X = 1+zX^m$ which becomes 1 for $z=0$.

For example, it gives

$$\sum_{k\geq 0}\binom{3k+2}{k+1}(p^2q)^k = \frac{\beta^3}{(\beta-1)(3-2\beta)} - \frac{1}{p^2q}, \tag{9}$$

where $q = 1-p, \beta = \alpha/p$ and α is defined by (1). In the case $p = q = \frac{1}{2}$, the sum is $\frac{16}{\sqrt{5}}$.

Proof of Theorem 1. We deduce the identity from the Pólya identity. Rewrite the LHS as

$$\begin{aligned}\sum_{k\geq 0}\binom{mk+s}{k+t}z^k &= \sum_{\ell\geq t}\binom{m(\ell-t)+s}{\ell}z^{\ell-t} \qquad \text{(setting } \ell = k+t) \\ &= z^{-t}\sum_{\ell\geq 0}\binom{m\ell-mt+s}{\ell}z^{\ell} - \sum_{\ell=0}^{t-1}\binom{m\ell-mt+s}{\ell}z^{\ell-t}.\end{aligned}$$

For the first term, we use (5) and then we use the fact $\beta = 1+z\beta^m$ and $z^{-t}\beta^{-mt} = (z\beta^m)^{-t} = (\beta-1)^{-t}$. So we have

$$z^{-t}\sum_{\ell\geq 0}\binom{m\ell-mt+s}{\ell}z^{\ell} = \frac{z^{-t}\beta^{-mt+s+1}}{(1-m)\beta+1} = \frac{\beta^{s+1}}{(\beta-1)^t((1-m)\beta+1)}.$$

For the second term, by setting $j = t-\ell$, we have

$$\sum_{\ell=0}^{t-1}\binom{m\ell-mt+s}{\ell}z^{\ell-t} = \sum_{j=1}^{t}\binom{m(t-j)-mt+s}{t-j}z^{-j} = \sum_{j=1}^{t}\binom{s-mj}{t-j}z^{-j}.$$

This completes the proof of Theorem 1. □

We continue to extend Theorem 1 and find a closed formula for $\sum_{k\geq 0}\binom{ck+s}{dk+t}z^k$. Let $e_i := \exp(2\pi i\sqrt{-1}/d)$ be a d-th root of unity. Then using the following property

$$\frac{1}{d}\sum_{i=0}^{d-1} e_i^k = \begin{cases} 1 & \text{if } d \mid k \\ 0 & \text{if } d \nmid k, \end{cases}$$

we have

$$\begin{aligned}\sum_{d|k}\binom{mk+s}{k+t}w^k &= \sum_{k\geq 0}\left(\frac{1}{d}\sum_{i=0}^{d-1} e_i^k\right)\binom{mk+s}{k+t}w^k \\ &= \frac{1}{d}\sum_{i=0}^{d-1}\sum_{k\geq 0}\binom{mk+s}{k+t}(e_i w)^k.\end{aligned}$$

On the other hand, setting $m = \frac{c}{d}$ and $w^d = z$ we have

$$\sum_{d|k}\binom{mk+s}{k+t}w^k = \sum_{k\geq 0}\binom{ck+s}{dk+t}z^k$$

and then

$$\sum_{k\geq 0}\binom{ck+s}{dk+t}z^k = \frac{1}{d}\sum_{i=0}^{d-1}\sum_{k\geq 0}\binom{mk+s}{k+t}(e_i w)^k.$$

Let β_i be the root of the equation $X = 1+(e_i w)X^m$ (i.e., $(X-1)^d = w^d X^c$) which becomes 1 for $w = 0$. Then by Theorem 1 we have

$$\begin{aligned}\sum_{k\geq 0}\binom{ck+s}{dk+t}z^k &= \frac{1}{d}\sum_{i=0}^{d-1}\left(\frac{\beta_i^{s+1}}{(\beta_i-1)^t((1-m)\beta_i+m)} - \sum_{j=1}^{t}\binom{s-mj}{t-j}(e_i w)^{-j}\right) \\ &= \sum_{i=0}^{d-1}\frac{\beta_i^{s+1}}{(\beta_i-1)^t((d-c)\beta_i+c)} - \sum_{j=1}^{t}\left(\frac{1}{d}\sum_{i=0}^{d-1}e_i^{-j}\right)\binom{s-mj}{t-j}w^{-j}.\end{aligned}$$

The second term can be simplified as

$$\begin{aligned}\sum_{j=1}^{t}\left(\frac{1}{d}\sum_{i=0}^{d-1}e_i^{-j}\right)\binom{s-mj}{t-j}w^{-j} &= \sum_{1\leq j\leq t,\, d|j}\binom{s-mj}{t-j}w^{-j} \\ &= \sum_{u=1}^{\lfloor t/d\rfloor}\binom{s-(c/d)du}{t-du}w^{-du} \quad (\text{setting } j = du) \\ &= \sum_{u=1}^{\lfloor t/d\rfloor}\binom{s-cu}{t-du}z^{-u}.\end{aligned}$$

Finally we have the following.

Theorem 2. *Let $c, s \in \mathbb{C}$, $d, t \in \mathbb{N}$ and let $\beta_0, \ldots, \beta_{d-1}$ be the roots of the equation $(X-1)^d = zX^c$ which become 1 for $z = 0$. Then we have*

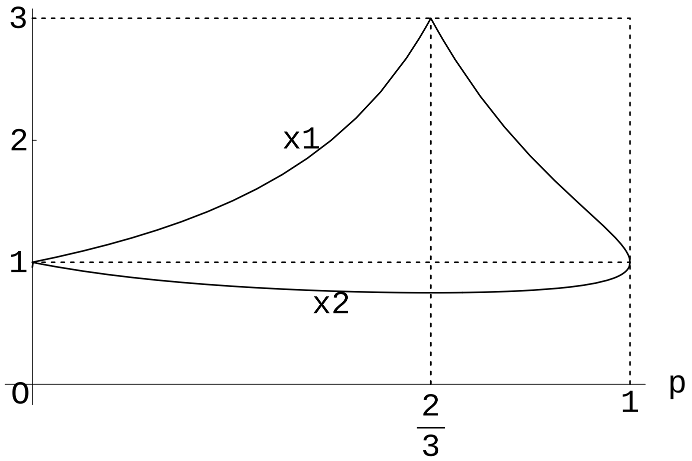
$$\sum_{k\geq 0}\binom{ck+s}{dk+t}z^k = \sum_{i=0}^{d-1}\frac{\beta_i^{s+1}}{(\beta_i-1)^t((d-c)\beta_i+c)} - \sum_{u=1}^{\lfloor t/d\rfloor}\binom{s-cu}{t-du}z^{-u}.$$

Example 1. Let us see examples for the case $c=3, d=2$, which we will use in the next section. Let x_1, x_2 be the roots of the equation $(X-1)^2 = zX^3$ which become 1 for $z=0$. Then for $z = p^2 q = p^2(1-p)$ we find that

$$\begin{aligned} x_1 &= \begin{cases} \xi & \text{if } 0 \le p \le \frac{2}{3} \\ \gamma_+ & \text{if } \frac{2}{3} \le p \le 1, \end{cases} \\ x_2 &= \gamma_- \quad \text{for } 0 \le p \le 1, \end{aligned} \tag{10}$$

where

$$\xi := \frac{1}{q}, \quad \gamma_\pm := \frac{1+p \pm \sqrt{1+2p-3p^2}}{2p^2}.$$



Thus by the above theorem we have

$$\sum_{k \ge 0} \binom{3k+i}{2k} (p^2 q)^k = \frac{x_1^{i+1}}{3-x_1} + \frac{x_2^{i+1}}{3-x_2}, \tag{11}$$

$$\sum_{k \ge 0} \binom{3k+i+1}{2k+1} (p^2 q)^k = \frac{x_1^{i+2}}{(x_1-1)(3-x_1)} + \frac{x_2^{i+2}}{(x_2-1)(3-x_2)}. \tag{12}$$

Let x_3 be the remaining root of the equation $(X-1)^2 = p^2 q X^3$, namely, $x_3 = \gamma_+$ for $0 \le p \le 2/3$ and $x_3 = \xi$ for $2/3 \le p \le 1$. Then, since $\frac{x_1}{3-x_1} + \frac{x_2}{3-x_2} = \frac{x_3}{x_3-3}$, it follows from (11) that

$$\sum_{k \ge 0} \binom{3k}{2k} (p^2 q)^k = \frac{x_3}{x_3-3}. \tag{13}$$

Noting that $x_3 = \frac{\alpha}{\alpha-p}$ (α is defined by (1)), the above identity confirms Fact 2. □

As an application of Theorem 2, we have the following identity.

Theorem 3. *For $s \in \mathbb{C}$ and $c, d, t \in \mathbb{N}$ with $c > d$, we have*

$$\sum_{u=u_0}^{\lfloor t/d \rfloor} \binom{s-cu}{t-du} z^{-u} = \sum_{i=0}^{c-1} \frac{\gamma_i^{s+1}}{(\gamma_i - 1)^{s-t}(d\gamma_i - c)},$$

where $u_0 = \max\{1, \lfloor \frac{s-t}{c-d} \rfloor\}$, *and* $\gamma_0, \ldots, \gamma_{c-1}$ *are the roots of the equation* $(Y-1)^{c-d} = zY^c$.

Proof. First we note that if β is a root of the equation $(X-1)^d = zX^c$ then $\gamma := \frac{\beta}{\beta-1}$ is a root of the equation $(Y-1)^{c-d} = zY^c$, and it follows that

$$\frac{\gamma^{s+1}}{(\gamma-1)^{s-t}(d\gamma-c)} = \frac{\beta^{s+1}}{(\beta-1)^t((d-c)\beta+c)}.$$

Moreover if β becomes 1 for $z=0$ then γ does not become 1 for $z=0$, and if γ becomes 1 for $z=0$ then β does not. Let $\beta_0, \ldots, \beta_{c-1}$ be the roots of $(X-1)^d = zX^c$ such that $\beta_0, \ldots, \beta_{d-1}$ become 1 for $z=0$. Then by setting $\gamma_i := \frac{\beta_i}{\beta_i-1}$ we find that $\gamma_0, \ldots, \gamma_{c-1}$ are the roots of $(Y-1)^{c-d} = zY^c$ such that $\gamma_d, \ldots, \gamma_{c-1}$ become 1 for $z=0$.

Now we calculate $\sum \binom{ck+s}{dk+t} z^k = \sum \binom{ck+s}{(c-d)k+s-t} z^k$ separately using Theorem 2. From the LHS we have

$$\begin{aligned}
\sum_{k\geq 0} \binom{ck+s}{dk+t} z^k &= \sum_{i=0}^{d-1} \frac{\beta_i^{s+1}}{(\beta_i-1)^t((d-c)\beta_i+c)} - \sum_{u=1}^{\lfloor t/d \rfloor} \binom{s-cu}{t-du} z^{-u} \\
&= \sum_{i=0}^{d-1} \frac{\gamma_i^{s+1}}{(\gamma_i-1)^{s-t}(d\gamma_i-c)} - \sum_{u=1}^{\lfloor t/d \rfloor} \binom{s-cu}{t-du} z^{-u}.
\end{aligned}$$

From the RHS we have

$$\begin{aligned}
&\sum_{k\geq 0} \binom{ck+s}{(c-d)k+s-t} z^k \\
&= \sum_{i=d}^{c-1} \frac{\gamma_i^{s+1}}{(\gamma_i-1)^{s-t}(-d\gamma_i+c)} - \sum_{v=1}^{\lfloor \frac{s-t}{c-d} \rfloor} \binom{s-cv}{(s-t)-(c-d)v} z^{-v} \\
&= -\sum_{i=d}^{c-1} \frac{\gamma_i^{s+1}}{(\gamma_i-1)^{s-t}(d\gamma_i-c)} - \sum_{v=1}^{\lfloor \frac{s-t}{c-d} \rfloor} \binom{s-cv}{t-dv} z^{-v}.
\end{aligned}$$

Combining the above two identities we get

$$\sum_{u=1}^{\lfloor t/d \rfloor} \binom{s-cu}{t-du} z^{-u} - \sum_{v=1}^{\lfloor \frac{s-t}{c-d} \rfloor} \binom{s-cv}{t-dv} z^{-v} = \sum_{i=1}^{c-1} \frac{\gamma_i^{s+1}}{(\gamma_i-1)^{s-t}(d\gamma_i-c)},$$

which is the desired identity. □

Setting $s=t=n$, we have the following identity as a special case.

$$\sum_{j=1}^{\lfloor n/d \rfloor} \binom{n-cj}{n-dj} z^{-j} = \sum_{i=0}^{c-1} \frac{\gamma_i^{n+1}}{d\gamma_i - c},$$

where $\gamma_0, \ldots, \gamma_{c-1}$ are the roots of $(Y-1)^{c-d} = zY^c$. For example, it follows that

$$\sum_{j=1}^{n} \binom{n-3j}{n-j} 8^j = \frac{(3+\sqrt{5})^{n+1}}{\sqrt{5}} - \frac{(3-\sqrt{5})^{n+1}}{\sqrt{5}} - 2^{n+1} = \left\lfloor \frac{(3+\sqrt{5})^{n+1}}{\sqrt{5}} \right\rfloor - 2^{n+1}.$$

3. Answer to the frog problem

Let us return to the Problem 2. We will find the probability P that the frog lands on the line $rx-\ell y+x_0=0$, where r and ℓ are coprime. Since the frog lands only on lattice points, we are interested in the set of non-negative lattice points on the line, that is, $L=\{(\ell k+i, rk+j) : k\in\mathbb{N}\}$, where $i\in\mathbb{N}$ is the least number such that $\ell \mid (ri+x_0)$ and $j:=(ri+x_0)/\ell$. Let

$a_k :=$ Prob(the frog lands on L at $(\ell k+i, rk+j)$ for the first time),
$u_k :=$ Prob(the frog lands on $(\ell k, rk)) = \binom{(\ell+r)k}{\ell k} p^{rk}q^{\ell k}$,
$v_k :=$ Prob(the frog lands on $(\ell k+i, rk+j)) = \binom{(\ell+r)k+i+j}{\ell k+i} p^{rk+j}q^{\ell k+i}$,

and let $A(x):=\sum_{k\geq 0} a_k x^k$, $U(x):=\sum_{k\geq 0} u_k x^k$, $V(x):=\sum_{k\geq 0} v_k x^k$ be generating functions. Since $v_k=\sum_{h=0}^{k} a_h u_{k-h}$ we have $V(x)=A(x)U(x)$. Substituting $x=1$ and noting that $P=A(1)$, we have

$$P=\frac{V(1)}{U(1)}, \tag{14}$$

where

$$U(1)=\sum_{k\geq 0}\binom{(\ell+r)k}{\ell k}(p^r q^\ell)^k,$$
$$V(1)=p^j q^i \sum_{k\geq 0}\binom{(\ell+r)k+i+j}{\ell k+i}(p^r q^\ell)^k.$$

We can calculate these values by using Theorem 2.

Example 2. Let us consider the case $\ell=2, r=3, x_0=2, p=q=1/2$. (In the language of Problem 1, the frog starts from $x=2$ and he jumps 2 steps to the left or 3 steps to the right with equal probabilities. Then P is the probability that we can catch the frog by the trap at the origin.) The non-negative lattice points on the line $3x-2y+2=0$ is

$$L=\{(2k,3k+1) : k\in\mathbb{N}\}.$$

This means $i=0$ and $j=1$. Then using (14) and Theorem 2 we have

$$\begin{aligned} P &= \left(\frac{1}{2}\sum_{k\geq 0}\binom{5k+1}{2k}\left(\frac{1}{2}\right)^{5k}\right)\Big/\left(\sum_{k\geq 0}\binom{5k}{2k}\left(\frac{1}{2}\right)^{5k}\right) \\ &= \frac{3\beta_-\beta_+(\beta_-+\beta_+)-5({\beta_-}^2+{\beta_+}^2)}{12\beta_-\beta_+-10(\beta_-+\beta_+)} \\ &\approx 0.67783, \end{aligned}$$

where $\beta_-\approx 1.43948$ and $\beta_+\approx 0.873822$ are the selected roots of $X=1\pm(X/2)^{5/2}$. One can also check that P is the only positive root of the equation

$$151X^6-441X^5+2335X^4-3465X^3+636X^2+1622X-757=0. \quad \square$$

Example 3. Here we consider the case $\ell=1, r=2$ again. For $0\neq n\in\mathbb{Z}$, let $P(n)$ be the probability that the frog reaches the line $2x-y+n=0$. Recall that we set $P(0)=1$. From (2), it follows that

$$P(n)=\alpha^n \tag{15}$$

for $n \in \mathbb{N}$, where α is defined in (1). We also note that

$$P(n) = pP(n-1) + qP(n+2) \tag{16}$$

holds for all $n \in \mathbb{Z} - \{0\}$, where $q = 1 - p$. Therefore if we know $P(-1)$ then we get all $P(n)$ for $n \leq -2$ by using (16), or using

$$P(n) = \frac{1}{p}P(n+1) - \frac{q}{p}P(n+3), \tag{17}$$

which is valid for $n \neq -1$.

The value $P(-1)$ can be computed by (14) and the fact $i = j = 1$ in this case, i.e., the corresponding lattice points are $L = \{(k+1, 2k+1) : k \in \mathbb{N}\}$. Then using (9) and Fact 2 we have

$$\begin{aligned} P(-1) &= \left(pq \sum_{k\geq 0} \binom{3k+2}{k+1} \left(p^2 q\right)^k \right) / \left(\sum_{k\geq 0} \binom{3k}{k} (p^2 q)^k \right) \\ &= \frac{\left(2\,\alpha^2 - 5\,a\,p + 3\,p^2 + \alpha^3\,p\,q\right)\,\left(1 - 3\,\alpha^2\,q\right)}{p\,\left(2\,\alpha^2 - 5\,\alpha\,p + 3\,p^2\right)}. \end{aligned} \tag{18}$$

In the case $p = q = \frac{1}{2}$, we have $\alpha = (\sqrt{5} - 1)/2$, and $P(-1) = 3 - \sqrt{5}$ by (18) (or we can use (4) and (9)). Using this together with (17) and (15), we find that the sequence $\{P(-n)\}_{n>0}$ satisfies the following Fibonacci-like property:

$$P(-n) = \frac{1}{2}(a_n - \sqrt{5}b_n)$$

where

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} + 5, \quad a_1 = 6, a_2 = 13, \\ b_n &= b_{n-1} + b_{n-2} + 3, \quad b_1 = 2, b_2 = 5. \end{aligned}$$

Alternatively, we can get an explicit formula for $P(-n)$ ($n > 0$) by using (14) and Theorem 2. Note that the corresponding lattice points are $L = \{(k+i, 2k+j) : k \in \mathbb{N}\}$ where

$$i := \lceil n/2 \rceil, \quad j := \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

So by (14) and (4) and using $\binom{3k+i+j}{k+i} = \binom{3k+i+j}{2k+j}$, we have

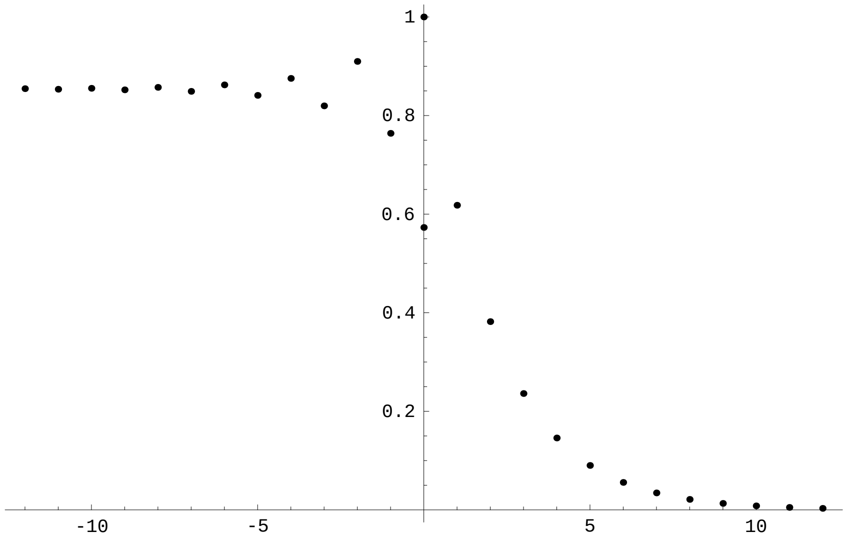
$$P(-n) = \left(p^j q^i \sum_{k\geq 0} \binom{3k+i+j}{2k+j} (p^2 q)^k \right) / \left(\sum_{k\geq 0} \binom{3k}{k} (p^2 q)^k \right). \tag{19}$$

We already computed the above sums in the previous section. Substituting (11) (12) (13) into (19), we have

$$P(-n) = \begin{cases} q^i \left(\dfrac{x_1^{i+1}}{3 - x_1} + \dfrac{x_2^{i+1}}{3 - x_2} \right) / \left(\dfrac{x_3}{x_3 - 3} \right) & \text{if } n = 2i \\[2ex] pq^i \left(\dfrac{x_1^{i+2}}{(x_1 - 1)(3 - x_1)} + \dfrac{x_2^{i+2}}{(x_2 - 1)(3 - x_2)} \right) / \left(\dfrac{x_3}{x_3 - 3} \right) & \text{if } n = 2i - 1 \end{cases} \tag{20}$$

for $n \in \mathbb{N}$.

The following picture shows how $(n, P(n))$ behaves for the case $p = q = \frac{1}{2}$. (We introduce a fake probability $P(0) = 1$, but we can compute the "true" probability by $(P(-1) + P(2))/2 = 3(3 - \sqrt{5})/4$ which is also plotted on the figure.)



The picture suggests the existence of $\lim P(n)$. Let us find the limits. In the language of Problem 1, this is the situation that we release the frog far away from the trap. From (20) we have

$$\lim_{n \to -\infty} P(n) = \begin{cases} \dfrac{\xi(\gamma_+ - 3)}{(3 - \xi)\gamma_+} = \dfrac{-(1+3p) + 3\sqrt{1 + 2p - 3p^2}}{2(2 - 3p)} & \text{if } 0 < p \le \frac{2}{3} \\ 0 & \text{if } \frac{2}{3} < p \le 1, \end{cases}$$

and from (15) we have

$$\lim_{n \to +\infty} P(n) = \begin{cases} 0 & \text{if } 0 \le p < \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \le p \le 1, \end{cases}$$

For the case $p = q = \frac{1}{2}$, it follows that

$$\lim_{n \to -\infty} P(n) = \frac{3\sqrt{5} - 5}{2} \approx 0.854, \quad \lim_{n \to +\infty} P(n) = 0,$$

which of course match the above picture. We considered the case $\ell = 1, r = 2$, and the result can be generalized as follows:

$$\lim_{n \to -\infty} P(n) = \begin{cases} \dfrac{1}{u\{r - (\ell + r)p\}} & \text{if } 0 < p \le \frac{r}{\ell + r} \\ 0 & \text{if } \frac{r}{\ell + r} < p \le 1, \end{cases}$$

$$\lim_{n\to+\infty} P(n) = \begin{cases} 0 & \text{if } 0 \le p < \frac{r}{\ell+r} \\ \dfrac{1}{u\{(\ell+r)p-r\}} & \text{if } \frac{r}{\ell+r} \le p < 1, \end{cases}$$

where $u = \sum_{k\ge0} \binom{(\ell+r)k}{\ell k} (p^r q^\ell)^k$. □

4. A VARIATION OF THE FROG PROBLEM

In Example 2, we considered the probability that the frog "lands on" the line $3x-2y+2=0$. Now we consider a variation of the problem. We say that the frog "attains" the line if he lands on the line or jumps over the line. Then what is the probability P^* that the frog "attains" the line $3x-2y+2=0$ assuming that he starts from the origin and he jumps one unit up with probability p, or he jumps one unit right with probability $q=1-p$. Since the frog lands only on lattice points, P^* is the probability that he lands on a point in $L=\{s_i : i \in \mathbb{N}\}$, where

$$s_i = \begin{cases} (2i, 3i+1) & \text{if } i \text{ is even} \\ (2i+1, 3i+3) & \text{if } i \text{ is odd.} \end{cases} \tag{21}$$

In the language of Problem 1, the situation is the following. The frog starts from $x=2$ and he jumps 2 steps to the left with probability p, or 3 steps to the right with probability $q=1-p$. Then what is the probability P^* that we can catch him by setting the traps at $x=0$ and $x=-1$? Let us solve the problem in this version here.

For $\{i,j\}=\{-1,0\}$ let $P_i(n)$ be the probability that the frog starts from $x=n$ and ends up with landing on $x=i$ without landing on $x=j$ in his journey. Then we can divide $P^*=P_0(2)+P_{-1}(2)$ according to his last landing position. Let $P(n\to m)$ be the probability that the frog ever lands on $x=m$ starting from $x=n$, and set $P(n):=P(n\to 0)$. As we saw in the previous section, we can compute $P(n)$ by using (14) and Theorem 2.

Noting that

$$P(n) = P_0(n) + P_{-1}(n)P(-1\to 0)$$

and

$$P(n+1) = P(n\to -1) = P_{-1}(n) + P_0(n)P(0\to -1) = P_{-1}(n) + P_0(n)P(1\to 0),$$

we have

$$\begin{pmatrix} P(n) \\ P(n+1) \end{pmatrix} = \begin{pmatrix} 1 & P(-1) \\ P(1) & 1 \end{pmatrix} \begin{pmatrix} P_0(n) \\ P_{-1}(n) \end{pmatrix}. \tag{22}$$

Moreover by (21) and (14) we have

$$P(n) = \begin{cases} f(0, \frac{n}{2}) & \text{if } i \text{ is even} \\ f(1, \frac{n+3}{2}) & \text{if } i \text{ is odd,} \end{cases}$$

where

$$f(i,j) = \left(p^j q^i \sum_{k\ge0} \binom{5k+i+j}{2k+i} (p^3q^2)^k \right) \Big/ \left(\sum_{k\ge0} \binom{5k}{2k} (p^3q^2)^k \right).$$

We can compute $f(i,j)$ using Theorem 2. Therefore setting $n=2$ in (22) we get $P_0(2)$ and $P_{-1}(2)$ by solving

$$f(0,1) = P_0(2) + P_{-1}(2) f(1,1),$$
$$f(1,3) = P_{-1}(2) + P_0(2) f(1,2).$$

If $p = q = \frac{1}{2}$ then we have $P_0(2) \approx 0.595882$ and $P_{-1}(2) \approx 0.105078$, which give $P^* = P_0(2) + P_{-1}(2) \approx 0.70096$.

It is also possible to compute P^* without using $f(i,j)$ as follows. Let $P_{i,\bar{j}}(n)$ be the probability that the frog starts from $x=n$ and ends up with landing on $x=i$ without landing on $x=j$. For $\{i,j\} = \{0,1\}$, we have $P_{i,\bar{j}}(n) = P_{i-1,\overline{j-1}}(n-1)$. Since

$$P_{0,\overline{-1}}(n) = P_{0,\bar{1}}(n) + P_{1,\bar{0}}(n) P_{0,\overline{-1}}(1) = P_{-1,\bar{0}}(n-1) + P_{0,\overline{-1}}(n-1) P_{0,\overline{-1}}(1),$$

it follows that

$$P_0(n) = P_{-1}(n-1) + P_0(n-1) P_0(1).$$

Also, since

$$P_{-1,\bar{0}}(n) = P_{1,\bar{0}}(n) P_{-1,\bar{0}}(1) = P_{0,\overline{-1}}(n-1) P_{-1,\bar{0}}(1),$$

it follows that

$$P_{-1}(n) = P_0(n-1) P_{-1}(1).$$

Therefore we have

$$\begin{pmatrix} P_0(n) \\ P_{-1}(n) \end{pmatrix} = \begin{pmatrix} P_0(1) & 1 \\ P_{-1}(1) & 0 \end{pmatrix} \begin{pmatrix} P_0(n-1) \\ P_{-1}(n-1) \end{pmatrix} = \begin{pmatrix} P_0(1) & 1 \\ P_{-1}(1) & 0 \end{pmatrix}^{n-1} \begin{pmatrix} P_0(1) \\ P_{-1}(1) \end{pmatrix}. \quad (23)$$

On the other hand, if the frog starts from $x=1$ then the next jump brings him to $x=-1$ with probability p or to $x=4$ with probability q. So we have

$$P_0(1) = q P_0(4), \quad P_{-1}(1) = p + q P_{-1}(4).$$

Thus we get $x = P_0(1)$ and $y = P_{-1}(1)$ by solving

$$\begin{pmatrix} P_0(4) \\ P_{-1}(4) \end{pmatrix} = \begin{pmatrix} x/q \\ (y-p)/q \end{pmatrix} = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}^3 \begin{pmatrix} x \\ y \end{pmatrix}. \quad (24)$$

(We choose the solutions so that $x \to 0$ and $y \to 0$ as $p \to 0$, and $x \to 0$ and $y \to 1$ as $p \to 1$.) Substituting x and y into (23), we can compute $P^* = P_0(2) + P_{-1}(2)$. If $p = q = \frac{1}{2}$ then we get $x \approx 0.187382$, $y \approx 0.560769$, and we can verify that $P^* = x^2 + xy + y \approx 0.70096$ is the unique root in $(0,1)$ of the following equation:

$$X^6 - 2X^5 + 13X^4 - 44X^3 + 51X^2 - 24X + 4 = 0.$$

Finally let us state the identities corresponding to (22) and (24) for a general case. Namely we consider the following problem.

Problem 3. *Starting from the origin, a frog jumps one unit up with probability p, or he jumps one unit right with probability $q = 1 - p$ at each time. Then what is the probability P^* that he ever "attain" the line $rx - ly + n = 0$?*

This problem is equivalent to the following.

Problem 4. *Starting from the $x = n$, a frog jumps ℓ steps to the left with probability p, or he jumps r steps to the right with probability $q = 1 - p$ at each time. What is the probability P^* that he ever lands on $x \le 0$?*

Let $P(n)$ be the probability that the frog (of Problem 4) ever lands on $x = 0$ starting from $x = n$. Define $P(0) = 1$. Set $I = \{0, -1, \dots, -\ell + 1\}$ and let $P_i(n)$ be the probability that the frog starts from $x = n$ and ends up with landing $x = i$ without landing any point in $I - \{i\}$. Then we have $P^* = P_0(n) + P_{-1}(n) + \cdots + P_{-\ell+1}(n)$. Similar to (22) we can compute $P_i(n)$'s from

$$\begin{pmatrix} P(n) \\ P(n+1) \\ P(n+2) \\ \cdots \\ P(n+\ell-1) \end{pmatrix} = \begin{pmatrix} P(0) & P(-1) & P(-2) & \cdots P(-\ell+1) \\ P(1) & P(0) & P(-1) & \cdots P(-\ell+2) \\ P(2) & P(1) & P(0) & \cdots P(-\ell+3) \\ & & & \cdots \\ P(\ell-1) & P(\ell-2) & P(\ell-3) & \cdots \quad P(0) \end{pmatrix} \begin{pmatrix} P_0(n) \\ P_{-1}(n) \\ P_{-2}(n) \\ \cdots \\ P_{-\ell+1}(n) \end{pmatrix}.$$

On the other hand, the equation corresponding to (24) is the following.

$$\begin{pmatrix} P_0(1)/q \\ P_{-1}(1)/q \\ \cdots \\ P_{-\ell+2}(1)/q \\ (P_{-\ell+1}(1) - p)/q \end{pmatrix} = \begin{pmatrix} P_0(1) & 1 & 0 & \cdots & 0 \\ P_{-1}(1) & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ P_{-\ell+2}(1) & 0 & 0 & \cdots & 1 \\ P_{-\ell+1}(1) & 0 & 0 & \cdots & 0 \end{pmatrix}^r \begin{pmatrix} P_0(1) \\ P_{-1}(1) \\ \cdots \\ P_{-\ell+2}(1) \\ P_{-\ell+1}(1) \end{pmatrix}.$$

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