A MULTIPLY INTERSECTING ERDŐS–KO–RADO THEOREM — THE PRINCIPAL CASE

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ABSTRACT. Let $m(n,k,r,t)$ be the maximum size of $\mathscr{F} \subset \binom{[n]}{k}$ satisfying $|F_1 \cap \cdots \cap F_r| \geq t$ for all $F_1, \ldots, F_r \in \mathscr{F}$. We prove that for every $p \in (0,1)$ there is some r_0 such that, for all $r > r_0$ and all t with $1 \leq t \leq \lfloor (p^{1-r} - p)/(1-p) \rfloor - r$, there exists n_0 so that if $n > n_0$ and $p = k/n$, then $m(n,k,r,t) = \binom{n-t}{k-t}$. The upper bound for t is tight for fixed p and r.

1. INTRODUCTION

Let n,k,r and t be positive integers, and let $[n] = \{1,2,\ldots,n\}$. A family $\mathscr{G} \subset 2^{[n]}$ is called r-wise t-intersecting if $|G_1 \cap \cdots \cap G_r| \geq t$ holds for all $G_1, \ldots, G_r \in \mathscr{G}$. Let us define a typical r-wise t-intersecting family $\mathscr{G}_i(n,r,t)$ and its k-uniform subfamily $\mathscr{F}_i(n,k,r,t)$, where $0 \leq i \leq \lfloor \frac{n-t}{r} \rfloor$, as follows:

$$\mathscr{G}_i(n,r,t) = \{G \subset [n] : |G \cap [t+ri]| \geq t + (r-1)i\},$$
$$\mathscr{F}_i(n,k,r,t) = \mathscr{G}_i(n,r,t) \cap \binom{[n]}{k}.$$

Two families $\mathscr{G}, \mathscr{G}' \subset 2^{[n]}$ are said to be isomorphic, and denoted by $\mathscr{G} \cong \mathscr{G}'$, if there exists a vertex permutation τ on $[n]$ such that $\mathscr{G}' = \{\{\tau(g) : g \in G\} : G \in \mathscr{G}\}$.

Let $m(n,k,r,t)$ be the maximum size of k-uniform r-wise t-intersecting families on n vertices. To determine $m(n,k,r,t)$ is one of the oldest problems in extremal set theory, which is still widely open. The case $r = 2$ was observed by Erdős, Ko and Rado [6], Frankl [9], Wilson [29], and then $m(n,k,2,t) = \max_i |\mathscr{F}_i(n,k,2,t)|$ was finally proved by Ahlswede and Khachatrian [2]. Frankl [8] showed $m(n,k,r,1) = |\mathscr{F}_0(n,k,r,1)|$ if $(r-1)n \geq rk$. Partial results for the cases $r \geq 3$ and $t \geq 2$ are found in [12, 14, 22, 23, 24, 25, 26, 28]. All known results suggest

$$m(n,k,r,t) = \max_i |\mathscr{F}_i(n,k,r,t)|.$$

In this paper, we will consider the principal case, namely, the case when the maximum is attained by $\mathscr{F}_0(n,k,r,t)$. For fixed $p = k/n \in (0,1)$, r and t, a computation shows that

$$\lim_{n\to\infty} |\mathscr{F}_1(n,k,r,t)|/|\mathscr{F}_0(n,k,r,t)| \leq 1 \text{ iff } 1 \leq t \leq (p^{1-r} - p)/(1-p) - r =: t_{p,r}. \quad (1)$$

To consider the interval for t including $\{1,2,\ldots,\lfloor t_{p,r} \rfloor\}$ let us define $T_{p,r}\ (> t_{p,r})$ by

$$T_{p,r} = p^{1-r}/(1-p) - \log r. \quad (2)$$

Date: March 19, 2009, 11:01am.

2000 *Mathematics Subject Classification.* Primary: 05D05 Secondary: 05C65.

Key words and phrases. Erdős–Ko–Rado Theorem; intersecting family; p-weight.

The author was supported by MEXT Grant-in-Aid for Scientific Research (B) 16340027 and 20340022.

Then we can state a generalized Erdős–Ko–Rado theorem for r-wise t-intersecting families as follows.

Theorem 1. *For all $p \in (0,1)$ there exists r_0 such that the following holds. For all $r > r_0$ and all t with $1 \le t \le T_{p,r}$, there exist positive constants ε, n_0 such that*

$$m(n,k,r,t) = \max\{|\mathscr{F}_0(n,k,r,t)|, |\mathscr{F}_1(n,k,r,t)|\}$$

holds for all $n > n_0$ and k with $|\frac{k}{n} - p| < \varepsilon$. Moreover, $\mathscr{F}_0(n,k,r,t)$ and $\mathscr{F}_1(n,k,r,t)$ are the only optimal families (up to isomorphism).

Now we introduce the p-weight version of the Erdős–Ko–Rado theorem. Throughout this paper, p and $q = 1-p$ denote positive real numbers. For $X \subset [n]$ and a family $\mathscr{G} \subset 2^X$ we define the p-weight of $\mathscr{G}$, denoted by $w_p(\mathscr{G} : X)$, as follows:

$$w_p(\mathscr{G} : X) = \sum_{G \in \mathscr{G}} p^{|G|} q^{|X|-|G|} = \sum_{i=0}^{|X|} \left| \mathscr{G} \cap \binom{X}{i} \right| p^i q^{|X|-i}.$$

We simply write $w_p(\mathscr{G})$ for the case $X = [n]$; for example, we have $w_p(2^{[n]}) = 1$ and $w_p(\mathscr{G}_0(n,r,t)) = p^t$. A direct computation shows that the p-weight of $\mathscr{G}_i(n,r,t)$ is independent of n for $n \ge t + ri$. So let

$$g_i(p,r,t) = w_p(\mathscr{G}_i(n,r,t)).$$

Let $w(n,p,r,t)$ be the maximum p-weight of r-wise t-intersecting families on n vertices. It might be natural to expect

$$w(n,p,r,t) = \max_i w_p(\mathscr{G}_i(n,r,t)) = \max_i g_i(p,r,t).$$

Ahlswede and Khachatrian proved that this is true for $r = 2$ in [3] (cf. [5, 7, 22]). This includes the Katona theorem [18] about $w(n,1/2,2,t)$. It is shown in [13] that

$$w(n,p,r,1) = p \text{ for } p \le (r-1)/r. \tag{3}$$

We can check that $g_0(p,r,t) \ge g_1(p,r,t)$ iff $1 \le t \le t_{p,r}$ cf. (1). In [11], Frankl considered the case $p = 1/2$ and proved $w(n,p,r,t) = p^t$ for $1 \le t \le t_{p,r} = 2^r - r - 1$. This result was extended for the case $|p - 1/2| < \varepsilon$ in [26]. In this paper we will generalize these results from $p \approx 1/2$ to any given $p \in (0,1)$ as follows.

Theorem 2. *For all $p \in (0,1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \le t \le T_{p,r}$, and all $n \ge t + r$, we have*

$$w(n,p,r,t) = \max\{g_0(p,r,t), g_1(p,r,t)\}.$$

Moreover, $\mathscr{G}_0(n,r,t)$ and $\mathscr{G}_1(n,r,t)$ are the only optimal families (up to isomorphism).

We will deduce Theorems 1 and 2 from slightly stronger, stability type results (cf. [16, 21]). To state our main results let us define some collections of families as follows. For

$0 \le i \le \lfloor (n-t)/r \rfloor$ (but we will actually need the case $i = 0,1$ only), let

$$\begin{aligned}
\mathbf{G}(n,r,t) &= \{\mathscr{G} \subset 2^{[n]} : \mathscr{G} \text{ is } r\text{-wise } t\text{-intersecting}\},\\
\mathbf{G}_i(n,r,t) &= \{\mathscr{G} \subset 2^{[n]} : \mathscr{G} \subset \mathscr{G}' \text{ for some } \mathscr{G}' \cong \mathscr{G}_i(n,r,t)\},\\
\mathbf{X}^i(n,r,t) &= \mathbf{G}(n,r,t) \setminus \textstyle\bigcup_{0 \le j \le i} \mathbf{G}_j(n,r,t),\\
\mathbf{Y}^i(n,k,r,t) &= \{\mathscr{F} \subset \tbinom{[n]}{k} : \mathscr{F} \in \mathbf{X}^i(n,r,t)\},
\end{aligned}$$

and finally let us define

$$\begin{aligned}
m^i(n,k,r,t) &= \max\{|\mathscr{F}| : \mathscr{F} \in \mathbf{Y}^i(n,k,r,t)\},\\
w^i(n,p,r,t) &= \max\{w_p(\mathscr{G}) : \mathscr{G} \in \mathbf{X}^i(n,r,t)\}.
\end{aligned}$$

Ahlswede and Khachatrian [1] determined $m^0(n,k,2,t)$ completely, extending the earlier results by Hilton and Milner [17] and Frankl [10]. Brace and Daykin [4] determined $w^0(n,1/2,r,1)$ and Frankl [11] determined $w^0(n,1/2,r,t)$ for $r \ge 5$ and $1 \le t \le 2^r - r - 1$. More partial results for $m^1(n,k,r,t)$ with $k/n \approx 1/2$ and $w^1(n,p,r,t)$ with $p \approx 1/2$ are found in [15, 26, 27]. Our main results are the following.

Theorem 3. *For all $p \in (0,1)$ there exists r_0 such that the following holds. For all $r > r_0$ and all t with $1 \le t \le T_{p,r}$, there exist positive constants γ, ε, n_0 such that*

$$m^1(n,k,r,t) < (1-\gamma)\max\{|\mathscr{F}_0(n,k,r,t))|, |\mathscr{F}_1(n,k,r,t))|\}$$

holds for all $n > n_0$ and k with $|\frac{k}{n} - p| < \varepsilon$.

Theorem 4. *For all $p \in (0,1)$ there exists r_0 such that the following holds. For all $r > r_0$ and all t with $1 \le t \le T_{p,r}$, there exist there exist positive constants γ, ε such that*

$$w^1(n,\tilde{p},r,t) < (1-\gamma)\max\{g_0(\tilde{p},r,t), g_1(\tilde{p},r,t)\} \tag{4}$$

holds for all n with $n \ge t + r$ and all $\tilde{p}$ with $|\tilde{p} - p| < \varepsilon$.

The condition $r > r_0$ is necessary in the above theorems. To see this, we give an example which violates (4). Let $r < 1/(1-p)$, or equivalently, $p > 1 - \frac{1}{r}$. Consider a family $\mathscr{G} = \{G \subset [n] : |G| \ge (1-\frac{1}{r})n + \frac{t}{r}\}$. Then one can check that $\mathscr{G} \in \mathbf{X}^1(n,r,t)$. As the binomial distribution $B(n,p)$ is concentrated around pn, we see that $\lim_{n\to\infty} w_p(\mathscr{G}) = 1$. Thus, (4) fails even if $\gamma = 0$.

Theorem 3 and Theorem 4 immediately imply Theorem 1 and Theorem 2, respectively. We first prove Theorem 4 in Section 3. Our proof technique is largely based on [11, 26]. Then we deduce Theorem 3 from Theorem 4 in Section 4. We prepare some tools in Section 2.

In our proof of the theorems, we will make no effort to reduce the value of r_0. Instead, we try to give a simpler proof assuming r_0 large enough. Our proof admits to replace $\log r$ in (2) with any function $f(r)$ satisfying $f(r) \to +\infty$ as $r \to +\infty$.

2. Tools

2.1. **Some inequalities.** Let $p, q \in (0,1)$ with $p + q = 1$. We consider the situation that r is large enough for fixed p, and we always assume that $qr > 1$. In this case, the equation $qx^r - x + p = 0$ has unique root $\alpha_{r,p}$ in the interval $(p,1)$. In fact, letting $f(x) = qx^r - x + p$, one can check that $f(0)=p>0$, $f(1)=0$. Also $f'(x)=qrx^{r-1}-1$ has unique real zero $x=(qr)^{-1/(r-1)}\in(0,1)$. We sometimes write α_r for $\alpha_{r,p}$ omitting p if this makes no confusion.

Lemma 1 **([27]).** *Let p,r,t_0,c be fixed constants. Suppose that $w(n,p,r,t_0)\le c$ holds for all $n\ge t_0$. Then we have $w(n,p,r,t)\le c\alpha_{r,p}^{t-t_0}$ for all $t\ge t_0$ and $n\ge t$. In particular, we always have $w(n,p,r,t)\le \alpha_{r,p}^t$.*

Lemma 2. $\mathbf{X}^0(n,r,t)\subset \mathbf{X}^0(n,r-1,t+1)$ *and* $w^0(n,p,r,t)\le w^0(n,p,r-1,t+1)$.

Proof. Let $\mathscr{G}\in\mathbf{X}^0(n,r,t)$. If $\mathscr{G}$ is not $(r-1)$-wise $(t+1)$-intersecting, then we can find $G_1,\dots,G_{r-1}\in\mathscr{G}$ such that $|G_1\cap\cdots\cap G_{r-1}|=t$. But $\mathscr{G}$ is r-wise t-intersecting and so every $G\in\mathscr{G}$ must contain $G_1\cap\cdots\cap G_{r-1}$. This means $\mathscr{G}\not\in\mathbf{X}^0(n,r,t)$, a contradiction. Thus, $\mathscr{G}\in\mathbf{G}(n,r-1,t+1)$. If $\mathscr{G}$ fixes $t+1$ vertices, then $\mathscr{G}\not\in\mathbf{X}^0(n,r,t)$. Therefore we have $\mathscr{G}\in\mathbf{X}^0(n,r-1,t+1)$. □

Lemma 3. *For any i with $0\le i\le\lfloor(n-t)/r\rfloor$, we have $w^i(n+1,p,r,t)\ge w^i(n,p,r,t)$.*

Proof. Choose $\mathscr{G}\in\mathbf{X}^i(n,r,t)$ with $w_p(\mathscr{G})=w^i(n,p,r,t)$. Then $\mathscr{G}':=\mathscr{G}\cup\{G\cup\{n+1\}:G\in\mathscr{G}\}\in\mathbf{X}^i(n+1,r,t)$ and $w_p(\mathscr{G}':[n+1])=w_p(\mathscr{G}:[n])(q+p)=w^i(n,p,r,t)$, which means $w^i(n+1,p,r,t)\ge w^i(n,p,r,t)$. □

For a positive integer i and a real $p\in(0,1)$, let

$$c_i:=c_i(p)=-i(p/q)\log p. \tag{5}$$

Lemma 4. *For any positive integer i and any real $p\in(0,1)$ there exists $r_1\in\mathbb{N}$ such that $\alpha_r^{y+i}<p^y$ holds for all $r\ge r_1$ and all $y=dp^{-r}$ with $0<d\le c_i$.*

Proof. Set $\alpha=\alpha_r$ and $\beta=1/(y+i)$. We want to show that $\alpha^{y+i}<p^y$, that is, $\alpha<p^{1-i\beta}$. Let $f(x)=qx^r-x+p$. Since $f(x)\ge 0$ for $0<x\le\alpha$ and $f(x)<0$ for $\alpha<x<1$, it suffices to show that $f(p^{1-i\beta})<0$, that is,

$$(q/p)p^{-i\beta r}<p^{-r}(p^{-i\beta}-1). \tag{6}$$

Noting that $p^{-i\beta}=\exp(\log p^{-i\beta})>1+\log(p^{-i\beta})=1-i\beta\log p$, the RHS of (6) is more than

$$p^{-r}(-i\beta\log p)=\frac{-ip^{-r}\log p}{y+i}=\frac{-ip^{-r}\log p}{dp^{-r}+i}\to\frac{-i\log p}{d}\text{ as } r\to\infty.$$

On the other hand, the LHS of (6) is

$$(q/p)(p^{-i})^{\frac{r}{y+i}}=(q/p)(p^{-i})^{\frac{r}{dp^{-r}+i}}\to q/p\text{ as } r\to\infty.$$

Thus (6) holds for sufficiently large r if $q/p\le -i(\log p)/d$, that is, $d\le c_i$. □

Lemma 5. *For all $p\in(0,1)$ there exist $r_1\in\mathbb{N}$ and $\mu\in(0,1)$ such that the following holds. For all $r\ge r_1+1$ and all t with $1\le t\le\lfloor c_1p(p^{-r}-p^{-r_1})/q\rfloor$, where c_1 is defined by* (5)*, and all $n\ge t+r$, it follows $w^0(n,p,r,t)\le\mu p^t$.*

Proof. Choose r_1 from Lemma 4 for $i=1$. For $r \geq r_1+1$ define a_r by

$$a_r = c_1 \sum_{j=r_1}^{r-1} p^{-j} = c_1 p(p^{-r} - p^{-r_1})/q. \tag{7}$$

Then we have $a_{r_1+1} = c_1 p^{-r_1}$ and $a_{r+1} - a_r = c_1 p^{-r}$ for $r \geq r_1+1$.

Let $r \geq r_1+1$. We will show $w^0(n,p,r,t) \leq \mu p^t$ for all t with $1 \leq t \leq \lfloor a_r \rfloor$, and $n \geq r+t$, by induction on r. For the base case $r = r_1+1$, by Lemmas 2 and 1, we have

$$w^0(n,p,r_1+1,t) \leq w^0(n,p,r_1,t+1) \leq \alpha_{r_1}^{t+1}.$$

Then using Lemma 4 for $y=t$ and $i=1$, we have $\alpha_{r_1}^{t+1} < p^t$ for $t \leq c_1 p^{-r_1} = a_{r_1+1}$. Let $\mu = \max\{\alpha_{r_1}(\alpha_{r_1}/p)^t : 1 \leq t \leq \lfloor a_{r_1+1} \rfloor\}$. The maximum is attained when $t = \lfloor a_{r_1+1} \rfloor$. This $\mu = \mu(p) \in (0,1)$ satisfies $w^0(n,p,r_1+1,t) \leq \mu p^t$ for all $1 \leq t \leq \lfloor a_{r_1+1} \rfloor$.

For the induction step, Lemmas 2 and 1 imply that

$$w^0(n,p,r+1,t) \leq w^0(n,p,r,t+1) \leq w^0(n,p,r,\lfloor a_r \rfloor)\alpha_r^{t+1-\lfloor a_r \rfloor}.$$

Using the induction hypothesis $w^0(n,p,r,\lfloor a_r \rfloor) \leq \mu p^{\lfloor a_r \rfloor}$, we have

$$w^0(n,p,r+1,t) \leq \mu p^{\lfloor a_r \rfloor} \alpha_r^{t+1-\lfloor a_r \rfloor} \leq \mu p^{a_r} \alpha_r^{t+1-a_r}.$$

The RHS is at most μp^t iff $\alpha_r^{(t-a_r)+1} \leq p^{t-a_r}$. Applying Lemma 4 for $y = t - a_r$ and $i=1$, this is true if $t - a_r \leq c_1 p^{-r}$, that is, $t \leq a_r + c_1 p^{-r} = a_{r+1}$. □

2.2. **Shifting.** For integers $1 \leq i < j \leq n$ and a family $\mathscr{G} \subset 2^{[n]}$, we define the (i,j)-shift σ_{ij} as follows:

$$\sigma_{ij}(\mathscr{G}) = \{\sigma_{ij}(G) : G \in \mathscr{G}\},$$

where

$$\sigma_{ij}(G) = \begin{cases} (G - \{j\}) \cup \{i\} & \text{if } i \notin G,\ j \in G,\ (G-\{j\}) \cup \{i\} \notin \mathscr{G}, \\ G & \text{otherwise.} \end{cases}$$

A family $\mathscr{G} \subset 2^{[n]}$ is called *shifted* if $\sigma_{ij}(\mathscr{G}) = \mathscr{G}$ for all $1 \leq i < j \leq n$, and $\mathscr{G}$ is called *tame* if it is shifted and $\bigcap \mathscr{G} = \emptyset$. If $\mathscr{G}$ is r-wise t-intersecting, then so is $\sigma_{ij}(\mathscr{G})$. We notice that $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ does not necessarily imply $\sigma_{ij}(\mathscr{G}) \in \mathbf{X}^0(n,r,t)$, because $\sigma_{ij}(\mathscr{G})$ may fix t vertices.

Lemma 6. *If $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ is p-weight maximum, then we can find a tame $\mathscr{G}' \in \mathbf{X}^0(n,r,t)$ with $w_p(\mathscr{G}') = w_p(\mathscr{G})$.*

Proof. If $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ then $\mathscr{G} \in \mathbf{X}^0(n,r-1,t+1)$ by Lemma 2. We apply all possible shifting operations to $\mathscr{G}$ to get a shifted family $\mathscr{G}' \in \mathbf{G}(n,r,t) \subset \mathbf{G}(n,r-1,t+1)$. Since each shifting operation preserves the p-weight, we have $w_p(\mathscr{G}) = w_p(\mathscr{G}')$.

We have to show that $\bigcap \mathscr{G}' = \emptyset$. Otherwise we may assume that $1 \in \bigcap \mathscr{G}'$ and $H = [2,n] \notin \mathscr{G}'$. Since $\mathscr{G}'$ is p-weight maximum we can find $G_1, \ldots, G_{r-1} \in \mathscr{G}'$ such that $|G_1 \cap \cdots \cap G_{r-1} \cap H| < t$. Then we have $|G_1 \cap \cdots \cap G_{r-1}| < t+1$, which is a contradiction. □

A family $\mathscr{G} \subset 2^{[n]}$ is called a *filter* if it is closed upwards: if $G \in \mathscr{G}$ and $G \subset G'$ then $G' \in \mathscr{G}$. If $\mathscr{G}$ is a filter, then so is $\sigma_{ij}(\mathscr{G})$. We also notice that if $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ is p-weight maximum then $\mathscr{G}$ is necessarily a filter.

3. Proof of Theorem 4

We start with the following simple observation.

Claim 1. *Let $\mathscr{G} \in \mathbf{X}^1(n,r,t)$ be fixed, and let $f(p) := \max\{g_0(p,r,t), g_1(p,r,t)\}$. If $w_p(\mathscr{G}) < f(p)$ for some p, then there exist $\gamma, \varepsilon > 0$ such that $w_{\tilde{p}}(\mathscr{G}) < (1-\gamma) f(\tilde{p})$ for all $|\tilde{p} - p| < \varepsilon$.*

This is because both $w_p(\mathscr{G})$ and $f(p)$ are continuous functions of variable p. So, to prove Theorem 4, it is enough to show that $w_p(\mathscr{G}) < f(p)$ for given p and $\mathscr{G} \in \mathbf{X}^1(n,r,t)$ provided $r \geq r_0$, $1 \leq t \leq T_{p,r}$.

The actual proof goes as follows. Let $\mathscr{G} \in \mathbf{X}^1(n,r,t)$ be p-weight maximum. Choose a tame $\mathscr{G}^* \in \mathbf{X}^0(n,r,t)$ with $w_p(\mathscr{G}^*) = w_p(\mathscr{G})$ by Lemma 6. Then we will show the following.

Case 1. If $\mathscr{G}^* \subset \mathscr{G}_1(n,r,t)$ then $w_p(\mathscr{G}^*) < (1-\gamma) g_1(p,r,t)$.
Case 2. If $\mathscr{G}^* \not\subset \mathscr{G}_1(n,r,t)$ then $w_p(\mathscr{G}^*) < (1-\gamma) g_0(p,r,t)$.

In the proof, after having p, r and t, we may assume that n is large enough by Lemma 3.

For Case 1, we show the following.

Lemma 7. *For all $p \in (0,1)$, $r \geq 2 + 1/q$, t with $1 \leq t \leq T_{p,r+1}$, and all $n \geq t + r$, the following holds. Let $\mathscr{G} \in \mathbf{X}^1(n,r,t)$ be p-weight maximum and let $\mathscr{G}^* \in \mathbf{X}^0(n,r,t)$ be a tame family obtained by shifting from $\mathscr{G}$. If $\mathscr{G}^* \subset \mathscr{G}_1(n,r,t)$ then $w_p(\mathscr{G}^*) \leq (1-\gamma) g_1(p,r,t)$, where $\gamma = \frac{q}{(r-2)}\left(\frac{t+r}{p} + \frac{1}{q}\right)^{-1}$.*

Proof of Lemma 7. Let p, r, t, n be given. Set $\mathscr{G}_1 = \mathscr{G}_1(n,r,t)$. Let $\mathscr{G}' \in \mathbf{X}^1(n,r,t)$ be p-weight maximum. Note that $\mathscr{G}'$ is not necessarily shifted. By Lemma 6 we can find a tame $\mathscr{G}^* \in \mathbf{X}^0(n,r,t)$ in a sequence of shifting $\mathscr{G}' \to \cdots \to \mathscr{G}^*$ with $w_p(\mathscr{G}') = \cdots = w_p(\mathscr{G}^*)$. Suppose that $\mathscr{G}^* \subset \mathscr{G}_1$. Then we find some $\mathscr{G} \in \mathbf{G}(n,r,t)$ in the sequence such that $\mathscr{G} \not\subset \mathscr{G}_1$ and $\sigma_{xy}(\mathscr{G}) \subset \mathscr{G}_1$, where we may assume that $x = t + r$, $y = x + 1$. We note that $|[x] \cap G| \geq x - 2$ for all $G \in \mathscr{G}$. Moreover, if $|[x] \cap G| = x - 2$ then $G \cap \{x, y\} = \{y\}$ and $(G - \{y\}) \cup \{x\} \notin \mathscr{G}$.

For $i \in [x]$ set $\mathscr{G}(i) = \{G \in \mathscr{G} : [y] \setminus G = \{i\}\}$, and for $j \in [x-1]$ and $z \in \{x, y\}$ let $\mathscr{G}_z(j) = \{G \in \mathscr{G} : [y] \setminus G = \{j, z\}\}$, $\mathscr{H}_z(j) = \{G \setminus [y] : G \in \mathscr{G}_z(j)\}$. Since $\sigma_{xy}(\mathscr{G}) \subset \mathscr{G}_1$ we have $\mathscr{H}_x(j) \cap \mathscr{H}_y(j) = \emptyset$ and so $w_p(\mathscr{G}_x(j)) + w_p(\mathscr{G}_y(j)) \leq p^{x-1} q^2$. Set $\mathscr{G}_\emptyset = \{G \in \mathscr{G} : [x] \subset G\}$, $\mathscr{G}_{xy} = \{G \in \mathscr{G} : G \cap [y] = [x-1]\}$, and let $e = \min_{i \in [x]} w_p(\mathscr{G}(i))$. Then we have

$$w_p(\mathscr{G}) = \sum_{i \in [x]} w_p(\mathscr{G}(i)) + \sum_{j \in [x-1]} \big(w_p(\mathscr{G}_x(j)) + w_p(\mathscr{G}_y(j))\big) + w_p(\mathscr{G}_\emptyset) + w_p(\mathscr{G}_{xy}) \tag{8}$$

$$\leq e + (x-1) p^x q + (x-1) p^{x-1} q^2 + p^x + p^{x-1} q^2 = e + (\eta - 1) p^x q, \tag{9}$$

where $\eta = \frac{x}{p} + \frac{1}{q}$. Note that $e \leq p^x q$, and (9) coincides with $w_p(\mathscr{G}_1) = x p^{x-1} q + p^x = \eta p^x q$ iff $e = p^x q$. If there is some $j \in [x-1]$ such that $\mathscr{G}_x(j) \cup \mathscr{G}_y(j) = \emptyset$, then by (8) we get $w_p(\mathscr{G}) \leq w_p(\mathscr{G}_1) - p^{x-1} q^2 = \big(1 - q/(\eta p)\big) w_p(\mathscr{G}_1) = (1 - (r-2)\gamma) w_p(\mathscr{G}_1)$, and we are done. Thus we may assume that

$$\mathscr{G}_x(j) \cup \mathscr{G}_y(j) \neq \emptyset \text{ for all } j \in [x-1]. \tag{10}$$

To prove $w_p(\mathscr{G}) \leq (1-\gamma) w_p(\mathscr{G}_1)$ by contradiction, let us assume that

$$w_p(\mathscr{G}) > (1-\gamma) w_p(\mathscr{G}_1) = (1-\gamma) \eta p^x q. \tag{11}$$

By (9) and (11) we have $e > (1-\gamma\eta)p^x q$. This means, letting $\mathscr{H}(i) = \{G \setminus [y] : G \in \mathscr{G}(i)\}$ and $Y = [y+1, n]$, we have

$$w_p(\mathscr{H}(i) : Y) > 1 - \gamma\eta \text{ for all } i \in [x]. \tag{12}$$

Since $\mathscr{G} \not\subset \mathscr{G}_1$, both $\bigcup_{j\in[x-1]} \mathscr{G}_x(j)$ and $\bigcup_{j\in[x-1]} \mathscr{G}_y(j)$ are non-empty. Using this with (10), we can choose $G \in \mathscr{G}_x(j)$ and $G' \in \mathscr{G}_y(j')$ with $j \neq j'$, say, $j = x-1, j' = x-2$. Let $L = [r-2]$ and $\mathscr{H}^* = \bigcap_{\ell\in L} \mathscr{H}(\ell)$. Then by (12) we have

$$w_p(\mathscr{H}^* : Y) > 1 - (r-2)\gamma\eta. \tag{13}$$

If $\mathscr{H}^* \subset 2^Y$ is not $(r-2)$-wise 1-intersecting, then we can find $H_1, \dots, H_{r-2} \in \mathscr{H}^*$ such that $H_1 \cap \cdots \cap H_{r-2} = \emptyset$. Setting $G_\ell = ([y] \setminus \{\ell\}) \cup H_\ell \in \mathscr{G}$ we have $|G_1 \cap \cdots \cap G_{r-2} \cap G \cap G'| = t-1$, which contradicts the r-wise t-intersecting property of $\mathscr{G}$. Thus $\mathscr{H}^*$ is $(r-2)$-wise 1-intersecting and $w_p(\mathscr{H}^* : Y) \leq p$ by (3), where we need $(r-2)q \geq 1$. But this contradicts (13) because we chose γ so that $p = 1-(r-2)\gamma\eta$. This completes the proof of Lemma 7. □

Next we consider Case 2. Rename $\mathscr{G}^*$ by $\mathscr{G}$. Here, to make the proof notationally simpler, we consider the case $r+1$ instead of the case r. Then, it suffices to show the following lemma for Case 2.

Lemma 8. *For all $p \in (0,1)$ there exists r_0 such that the following holds. For all $r > r_0$, all t with $1 \leq t \leq T_{p,r+1}$, there exists $\gamma \in (0,1)$ such that for all $n \geq t + (r+1)$ and all tame $\mathscr{G} \in \mathbf{X}^0(n, r+1, t)$ with $\mathscr{G} \not\subset \mathscr{G}_1(n, r+1, t)$, it follows that $w_p(\mathscr{G}) < (1-\gamma)p^t$.*

Proof of Lemma 8. Let $p \in (0,1)$ be given. We choose $r_0 = r_0(p)$ sufficiently large, which will be specified in the proof. Then, let $r > r_0$ and $1 \leq t \leq T_{p,r+1}$ be given. We choose $\gamma = \gamma(p,r,t) \in (0,1)$ close enough to 1, and the closeness will be specified in the proof. Finally let $\mathscr{G} \in \mathbf{X}^0(n, r+1, t)$ be given with $\mathscr{G} \not\subset \mathscr{G}_1(n, r+1, t)$, where $n \geq t + (r+1)$.

Let $t^{(i)} = \max\{j : \mathscr{G} \text{ is } i\text{-wise } j\text{-intersecting}\}$. We may assume that $t^{(r+1)} = t$ and $\mathscr{G}$ is p-weight maximum among all tame $\mathscr{G} \in \mathbf{X}^0(n, r+1, t)$ with $\mathscr{G} \not\subset \mathscr{G}_1(n, r+1, t)$. Let $t^{(r)} = t + s$. We have $s \geq 1$ by Lemma 2. Choose r_1 from Lemma 5. Using Lemma 1 with Lemma 5, we have

$$w_p(\mathscr{G}) \leq w^0(n, p, r, t+s) \leq w^0(n, p, r, \lfloor a_r \rfloor)\alpha_r^{(t+s)-\lfloor a_r \rfloor} \leq \mu p^{a_r} \alpha_r^{(t+s)-a_r},$$

for some $\mu = \mu(p) \in (0,1)$, where a_r is defined in (7). We want to show the RHS is at most μp^t, or equivalently, $\alpha_r^{t-a_r+s} \leq p^{t-a_r}$. Choosing r sufficiently large, that is, $r > r_1$, this is true if $t - a_r \leq c_s p^{-r}$ by Lemma 4. Thus we get the desired inequality $w_p(\mathscr{G}) \leq \mu p^t$ if

$$(t \leq)\, T_{p,r+1} \leq c_s p^{-r} + a_r. \tag{14}$$

The LHS is $T_{p,r+1} = p^{-r}/q - \log r$, while the RHS is

$$c_s p^{-r} + a_r = s c_1 p^{-r} + a_r = c_1 p^{-r}(s + p/q) - c_1 p^{1-r_1}/q.$$

We choose $r > r_0 \gg r_1$ so that $-\log r < -c_1 p^{1-r_1}/q = (p^{2-r_1}/q)\log p$. Then we have (14) if $p^{-r}/q \leq c_1 p^{-r}(s + p/q)$, that is, $-p(\log p)(s + p/q) \geq 1$. This is true if

$$s \geq s_0 := (-p \log p)^{-1} - p/q. \tag{15}$$

So we may assume that $1 \le s < s_0$. After [11] let $h = \min\{i : |G \cap [t+i]| \ge t \text{ for all } G \in \mathscr{G}\}$. This is the minimum size of "holes" in $[t+h]$.

Claim 2. $1 \le h \le s\,(< s_0)$.

Proof. Since $\mathscr{G} \in \mathbf{X}^0(n, r+1, t)$, we have $h \ge 1$. By the definition of s and the shiftedness of $\mathscr{G}$, we have $G_1, \dots, G_r \in \mathscr{G}$ such that $G_1 \cap \cdots \cap G_r = [t+s]$. Then it follows from $t^{(r+1)} = t$ that $|[t+s] \cap G| \ge t$ for all $G \in \mathscr{G}$, which implies, $t+h \le t+s$. ☐

Let $b = t+h-1$ and let $T_i = [b+1-i, b]$ be the right-most i-set in $[b]$. For $A \subset [b]$ let

$$\mathscr{G}(A) = \{G \cap [b+1, n] : G \in \mathscr{G},\ [b] \setminus G = A\}.$$

Since $\mathscr{G}$ is shifted, we have $\mathscr{G}(A) \subset \mathscr{G}(T_i)$ for all $A \in \binom{[b]}{i}$. Thus, for each $G \in \mathscr{G}$ with $|[b] \setminus G| = i$, we can find $G' \in \mathscr{G}(T_i)$ such that $G = ([b] \setminus G) \cup G'$. By considering the weight of $\mathscr{G}$ on $[b]$ and $[b+1, n]$ separately, we have

$$w_p(\mathscr{G}) \le \sum_{i=0}^{h} \binom{b}{i} p^{b-i} q^i\, w_p(\mathscr{G}(T_i) : [b+1, n]). \tag{16}$$

Claim 3. *For* $0 \le i < h$ *and* $2 \le j \le r$, $\mathscr{G}(T_i)$ *is* j*-wise* $(ij + (r-j)h + 1)$*-intersecting.*

Proof. Suppose that $\mathscr{G}(T_i)$ is not j-wise v-intersecting, where $v = ij + (r-j)h + 1$. Then we can find $G_1, \dots, G_j \in \mathscr{G}(T_i)$ such that $|G_1 \cap \cdots \cap G_j| < v$. Since $\mathscr{G}$ is a shifted filter, we may assume that $G_1 \cap \cdots \cap G_j = [b+1, b+v-1]$. By shifting $(G_\ell \cup [b]) \setminus T_i \in \mathscr{G}$, we get $G'_\ell := (G_\ell \cup [b]) \setminus [b+1+(\ell-1)i, b+\ell i] \in \mathscr{G}$ for $1 \le \ell \le j$. Then, $G'_1 \cap \cdots \cap G'_j = [b] \cup [b+ij+1, b+v-1]$.

By the definition of h we have some $H \in \mathscr{G}$ such that $|H \cap [h+t-1]| = |H \cap [b]| = t-1$ and due to the shiftedness of $\mathscr{G}$ we may assume that $H = [n] \setminus [t, b]$. By shifting H, we get $G'_\ell := [n] \setminus [b+ij+1+(\ell-1-j)h, b+ij+(\ell-j)h] \in \mathscr{G}$ for $j < \ell \le r$. Then, $G'_{j+1} \cap \cdots \cap G'_r = [n] \setminus [b+ij+1, b+v-1]$. Thus we have $G'_1 \cap \cdots \cap G'_r \cap H = [t-1]$, which contradicts the $(r+1)$-wise t-intersecting property of $\mathscr{G}$. ☐

Claim 4. *If* $\mathscr{G} \subset \mathscr{G}_h(n, r+1, t)$ *then* $w_p(\mathscr{G}) < (1-\gamma)p^t$.

Proof. Let $1 \le i \le h$ and set $\mathscr{G}_i = \mathscr{G}_i(n, r+1, t)$. We are going to compare

$$w_p(\mathscr{G}_i \setminus \mathscr{G}_{i-1}) = \binom{t+(r+1)(i-1)}{i} p^{t+ri} q^i$$

and

$$w_p(\mathscr{G}_{i-1} \setminus \mathscr{G}_i) = \sum_{j=\max\{0, i-r\}}^{i-1} \binom{t+(r+1)(i-1)}{j} \sum_{\ell=i+1-j}^{r+1} \binom{r+1}{\ell} p^{t+(r+1)i-j-\ell} q^{j+\ell}.$$

For the latter, by choosing $j = i-1$, we have

$$\begin{aligned} w_p(\mathscr{G}_{i-1} \setminus \mathscr{G}_i) &\ge \binom{t+(r+1)(i-1)}{i-1} p^{t+ri-r} q^{i-1} \sum_{\ell=2}^{r+1} \binom{r+1}{\ell} p^{r+1-\ell} q^{+\ell} \\ &= \binom{t+(r+1)(i-1)}{i-1} p^{t+ri-r} q^{i-1} (1 - p^{r+1} - (r+1)p^r q). \end{aligned}$$

Thus,

$$\frac{w_p(\mathscr{G}_{i-1} \setminus \mathscr{G}_i)}{w_p(\mathscr{G}_i \setminus \mathscr{G}_{i-1})} \ge \frac{i}{t+r(i-1)} (p^{-r} q^{-1} - pq^{-1} - (r+1)).$$

The RHS is more than 1 iff

$$t < ip^{-r}/q + r - (2r+1)i - p/q. \tag{17}$$

Using $t \le T_{p,r+1} = p^{-r}/q - \log r$, we can verify (17) for $i \ge 2$ and r large enough, say, $p^{-r} > 2rhq$. Thus we have $\max\{w_p(\mathscr{G}_0), w_p(\mathscr{G}_1)\} > w_p(\mathscr{G}_2) > \cdots > w_p(\mathscr{G}_h)$.

Suppose that $\mathscr{G} \subset \mathscr{G}_h$. Since $\mathscr{G} \not\subset \mathscr{G}_1$ is an assumption of Lemma 8, we may assume that $h \ge 2$. Then we have $w_p(\mathscr{G}) \le w_p(\mathscr{G}_h) \le w_p(\mathscr{G}_2)$. A direct computation using $t \le T_{p,r+1} < p^{-r}/q$ shows that $\lim_{r\to\infty} w_p(\mathscr{G}_2) \le p^t/2$. Thus, for sufficiently large r, we can find some $\gamma \in (0,1)$ satisfying $w_p(\mathscr{G}) < (1-\gamma)p^t$. □

So, we may assume that $\mathscr{G} \not\subset \mathscr{G}_h(n, r+1, t)$.

Claim 5. *If $\mathscr{G} \not\subset \mathscr{G}_h(n,r+1,t)$ then $\mathscr{G}(T_h)$ is r-wise $(rh+2)$-intersecting.*

Proof. Suppose that $\mathscr{G}(T_h)$ is not r-wise $(rh+2)$-intersecting. Then we can find $G_1, \ldots, G_r \in \mathscr{G}(T_h)$ such that $G_1 \cap \cdots \cap G_r = [b+1, b+rh+1] = [t+h, t+(r+1)h]$. By shifting $(G_\ell \cup [b]) \setminus T_h \in \mathscr{G}$ we get $G'_\ell := (G_\ell \cup [b]) \setminus [t+(\ell-1)h, t+\ell h-1] \in \mathscr{G}$ for $1 \le \ell \le r$. Then, $G'_1 \cap \cdots \cap G'_r = [t-1] \cup [t+rh, t+(r+1)h]$. Since $\mathscr{G} \not\subset \mathscr{G}_h(n,r+1,t)$ we have $G'_{r+1} := [n] \setminus [t+rh, t+(r+1)h] \in \mathscr{G}$. Thus, we have $G'_1 \cap \cdots \cap G'_{r+1} = [t-1]$, which contradicts the $(r+1)$-wise t-intersecting property of $\mathscr{G}$. □

Let $0 \le i < h$. By Claim 3, $\mathscr{G}(T_i)$ is $\lfloor \frac{r}{2} \rfloor$-wise u-intersecting, where $u = \lfloor \frac{r}{2} \rfloor i + \lceil \frac{r}{2} \rceil h + 1$. By Lemma 5 we have $w_p(\mathscr{G}(T_i) : [b+1, n]) \le w^0(n-b, p, \lfloor \frac{r}{2} \rfloor, u) \le p^u$ if $u \le a_{\lfloor r/2 \rfloor}$. In fact, we can choose $r \ge r_0(p)$ so that $u \le a_{\lfloor r/2 \rfloor}$, because $u \le rh+1 < rs_0+1$ (by Lemma 2) and $rs_0 + 1 < a_{\lfloor r/2 \rfloor}$ (by (15), (7) and (5)). Using $t \le T_{p,r+1} = p^{-r}/q - \log r$ and $\binom{b}{i} < (t+h)^i < (t+s_0)^i < (p^{-r}/q)^i$ for $r > r_0(p)$, we have

$$\binom{b}{i} p^{b-i} q^i w_p(\mathscr{G}(T_i) : [b+1, n]) < (p^{-r}/q)^i p^{b-i} q^i p^u \le p^{t + (1+\frac{r}{2})(h-i)} < p^{t+\frac{r}{2}}. \tag{18}$$

By Claim 5, $\mathscr{G}(T_h)$ is r-wise $(rh+2)$-intersecting. Thus, by choosing r large enough so that $rh+2 < a_r$, Lemma 5 gives

$$\binom{b}{h} p^{b-h} q^h w_p(\mathscr{G}(T_h) : [b+1, n]) < (p^{-r}/q)^h p^{t-1} q^h p^{rh+2} = p^{t+1}. \tag{19}$$

By (16), (18), (19) we have $w_p(\mathscr{G}) \le hp^{t+\frac{r}{2}} + p^{t+1} = p^t(hp^{r/2} + p) < (1-\gamma)p^t$ by choosing r sufficiently large so that $hp^{r/2} < s_0 p^{r/2} \ll q$. This completes the proof of Lemma 8 and Theorem 4. □

4. Proof of Theorem 3

Assume the negation of Theorem 3. Then the statement starts with

$$\underline{\exists p}\, \forall r_0\, \underline{\exists r}\, \underline{\exists t}\, \forall \gamma\, \forall \varepsilon\, \forall n_0\, \underline{\exists n}\, \underline{\exists k}\, \cdots, \tag{20}$$

where the underlines will indicate the choice of parameters described below. We will construct a counterexample to Theorem 4 using (20). Recall that Theorem 4 starts with

$$\forall p\, \underline{\exists r_0}\, \forall r\, \forall t\, \underline{\exists \gamma}\, \underline{\exists \varepsilon}\, \cdots. \tag{21}$$

First, assuming the negation of Theorem 3, there exists some $p \in (0,1)$ (corresponding to the first underline in (20)) such that the rest of Theorem 3 does not hold. For this p, Theorem 4 provides some r_0 (corresponding to the first underline in (21)) such that the rest of Theorem 4 holds. With this r_0, the negation of Theorem 3 provides some $r > r_0$ and $1 \le t \le T_{r,p}$ (the second and third underlines in (20)) such that the rest of Theorem 3 does not hold. With this r and t, Theorem 4 provides some $\gamma_0 = \gamma_0(p,r,t)$ and $\varepsilon_0 = \varepsilon_0(p,r,t)$ such that

$$w^1(n,\tilde{p},r,t) < (1-\gamma_0)\,f(\tilde{p}) \tag{22}$$

holds for all $\tilde{p}$ with $|\tilde{p}-p| \le \varepsilon_0$, and all $n \ge t+r$, where $f(\tilde{p}) := \max\{g_0(\tilde{p},r,t), g_1(\tilde{p},r,t)\}$.

For reals $0 < b < a$ we write $a \pm b$ to mean the open interval $(a-b, a+b)$. We note that $f(\tilde{p})$ is a uniformly continuous function of $\tilde{p}$ on $p \pm \varepsilon_0$. Let $\gamma = \frac{\gamma_0}{4}$, $\varepsilon = \frac{\varepsilon_0}{2}$, and $I = p \pm \varepsilon$. Now we are going to define n_0. Choose $\varepsilon_1 \ll \varepsilon$ so that

$$(1-3\gamma)\,f(\tilde{p}) > (1-4\gamma)\,f(\tilde{p}+\delta) \tag{23}$$

holds for all $\tilde{p} \in I$ and all $0 < \delta \le \varepsilon_1$. As the binomial distribution $B(n,p)$ is concentrated around pn, we can choose n_1 so that

$$\sum_{i \in J} \binom{n}{i} p_0^i (1-p_0)^{n-i} > (1-3\gamma)/(1-2\gamma) \tag{24}$$

holds for all $n > n_1$ and all $p_0 \in I_0 := p \pm \frac{3\varepsilon}{2}$, where $J = ((p_0-\varepsilon_1)n, (p_0+\varepsilon_1)n) \cap \mathbb{N}$. A little calculation shows that we can choose n_2 so that

$$(1-\gamma)\max\{|\mathscr{F}_0(n,k,r,t)|, |\mathscr{F}_1(n,k,r,t)|\} > (1-2\gamma)\,f(k/n)\binom{n}{k} \tag{25}$$

holds for all $n > n_2$ and k with $k/n \in I$. Finally set $n_0 = \max\{n_1, n_2\}$.

We plug these γ, ε and n_0 into (20). Then the negation of Theorem 3 gives us some n,k and $\mathscr{F} \in \mathbf{Y}^1(n,k,r,t)$ with $|\mathscr{F}| \ge (1-\gamma)\max\{|\mathscr{F}_0(n,k,r,t)|, |\mathscr{F}_1(n,k,r,t)|\}$, where $n > n_0$ and $\frac{k}{n} \in I$. We fix n,k and $\mathscr{F}$, and let $\tilde{p} = \frac{k}{n}$. By (25) we have $|\mathscr{F}| > c\binom{n}{k}$, where $c = (1-2\gamma)\,f(\tilde{p})$. Let $\mathscr{G} = \bigcup_{k \le i \le n} (\nabla_i(\mathscr{F})) \in \mathbf{X}^1(n,r,t)$ be the collection of all upper shadows of $\mathscr{F}$, where $\nabla_i(\mathscr{F}) = \{H \in \binom{[n]}{i} : H \supset \exists F \in \mathscr{F}\}$. Let $p_0 = \tilde{p} + \varepsilon_1 \in I_0$.

Claim 6. *$|\nabla_i(\mathscr{F})| \ge c\binom{n}{i}$ for $i \in J$.*

Proof. Choose a real $x \le n$ so that $c\binom{n}{k} = \binom{x}{n-k}$. Since $|\mathscr{F}| > c\binom{n}{k} = \binom{x}{n-k}$ the Kruskal–Katona Theorem [20, 19] implies that $|\nabla_i(\mathscr{F})| \ge \binom{x}{n-i}$. Thus it suffices to show that $\binom{x}{n-i} \ge c\binom{n}{i}$, or equivalently,

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}} \ge \frac{c\binom{n}{i}}{c\binom{n}{k}}.$$

Using $i \ge k$ this is equivalent to $i \cdots (k+1) \ge (x-n+i) \cdots (x-n+k+1)$, which follows from $x \le n$. □

By the claim we have

$$w_{p_0}(\mathscr{G}) \ge \sum_{i \in J} |\nabla_i(\mathscr{F})|\, p_0^i (1-p_0)^{n-i} \ge c \sum_{i \in J} \binom{n}{i} p_0^i (1-p_0)^{n-i}. \tag{26}$$

Using (24) and (23), the RHS of (26) is more than

$$c\,(1-3\gamma)/(1-2\gamma) = (1-3\gamma)\,f(\tilde{p}) > (1-4\gamma)\,f(\tilde{p}+\varepsilon_1) = (1-\gamma_0)\,f(p_0).$$

This means $w_{p_0}(\mathscr{G}) > (1-\gamma_0)\,f(p_0)$ which contradicts (22) because $p_0 \in I_0 \subset p \pm \varepsilon_0$. □

ACKNOWLEDGMENT

The author thanks the referees for their careful reading and many helpful suggestions.

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