REGULAR SIMPLICES PASSING THROUGH HOLES

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ABSTRACT. What is the smallest circular or square wall hole that a regular tetrahedron can pass? This problem was solved by Itoh–Tanoue–Zamfirescu [8]. Then, we settled the case of equilateral triangular hole in [1]. Motivated by these results, we consider the corresponding problems in higher dimensions. Among other results, we determine the minimum $(n-1)$-dimensional ball hole that a unit regular n-simplex can pass. The diameter of the minimum hole goes to $3\sqrt{2}/4$ as n tends to infinity.

1. INTRODUCTION

For a given convex body, find a small hole in a wall through which the convex body can pass. This type of problems goes back to Zindler [17] in 1920, who considered a convex polytope which can pass through a fairly small circular holes. A related topic known as Prince Rupert's problem can be found in [4]. Here we concentrate on the case when the convex body is a regular n-simplex.

For a compact convex body $K \subset \mathbb{R}^n$, let $\operatorname{diam}(K)$ and $\operatorname{width}(K)$ denote the diameter and width of K, respectively. For $d > 0$ let dK denote the convex body of diameter $d \times \operatorname{diam}(K)$ and homothetic to K. By S_n, Q_n, and B_n we denote the n-dimensional simplex, hypercube, and ball of unit diameter, respectively. Thus, S_n has side length 1, Q_n has side length $1/\sqrt{n}$, and B_n has radius $1/2$.

Let Θ be an $(n-1)$-dimensional convex body with $\operatorname{diam}(\Theta) = 1$ lying on a hyperplane in $\mathbb{R}^n$, which we will call a hole-shape. For a $d > 0$, let H be the hyperplane containing $d\Theta$. The hyperplane H divides $\mathbb{R}^n$ into two (open) half spaces H^+ and H^-. We want to push S_n from H^+ to H^- through $d\Theta$. In this situation, we are interested in the following two types of minimum diameters γ and Γ of hole-shape Θ, defined by

$$\gamma(n,\Theta) := \min\{d : S_n \text{ can pass through the hole } d\Theta\},$$
$$\Gamma(n,\Theta) := \min\{d : S_n \subset (d\Theta) \times \mathbb{R}\}.$$

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Since S_n can pass through a hole $d\Theta$ by translation perpendicular to the hyperplane containing the hole iff $d \ge \Gamma(n,\Theta)$, we have $\gamma(n,\Theta) \le \Gamma(n,\Theta)$.

As we set the diameters of Q_n, B_n equal to 1, we have $\mathrm{width}(Q_n) = 1/\sqrt{n}$ and $\mathrm{width}(B_n) = 1$. Steinhagen [14] determined the width of S_n as follows.

$$\mathrm{width}(S_n) = \begin{cases} \sqrt{\frac{2}{n+1}} & \text{if } n \text{ is odd,} \\ \sqrt{\frac{2n+2}{n(n+2)}} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

If S_n can pass through a hole $d\Theta$ by translation, then

$$\mathrm{width}(d\Theta) \ge \mathrm{width}(S_n) = (\sqrt{2} - o(1))/\sqrt{n}. \tag{2}$$

Let $n \ge 3$. If S_n can pass through a hole $d\Theta$, then $d \ge \mathrm{width}(S_2) = \sqrt{3}/2$. This gives $\gamma(n,\Theta) \ge \sqrt{3}/2$.

The following result is due to Pukhov [12] and Weißback [15] for n odd, and Brandenberg and Theobald [2, 3] for n even (and odd).

$$\Gamma(n, B_{n-1}) = \begin{cases} \sqrt{\frac{2(n-1)}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{\sqrt{2n(n+1)}} & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

In the next section, we review some results in $\mathbb{R}^3$. Then, in section 3 we consider the problems in higher dimensions. Among other results, we will show that

$$\lim_{n\to\infty} \gamma(n, S_{n-1}) = \lim_{n\to\infty} \Gamma(n, S_{n-1}) = 1,$$
$$\lim_{n\to\infty} \gamma(n, B_{n-1}) = 3\sqrt{2}/4, \quad \lim_{n\to\infty} \Gamma(n, B_{n-1}) = \sqrt{2},$$

and

$$3\sqrt{2}/4 \le \lim_{n\to\infty} \gamma(n, Q_{n-1}) \le 2, \quad \sqrt{2} \le \lim_{n\to\infty} \Gamma(n, Q_{n-1}) \le 2.$$

It would be also interesting to consider problems concerning a general n-simplex of diameter 1 passing through holes.

2. In the 3-space

Itoh, Tanoue, and Zamfirescu [8] proved

$$\gamma(3, Q_2) = \Gamma(3, Q_2) = 1, \quad \gamma(3, B_2) = 2r = 0.8956\ldots, \tag{4}$$

where $r \in (0,1)$ is a unique root of the equation $216x^6 - 9x^4 + 38x^2 - 9 = 0$. We note that $\gamma(3, B_2) < \Gamma(3, B_2) = 1$.

In [1], the following is proved.

$$\gamma(3, S_2) = \Gamma(3, S_2) = \frac{1+\sqrt{2}}{\sqrt{6}} = 0.9855\ldots$$

Zamfirescu [16] proved that most convex bodies can be held by a circular frame. Using (4), one can show that a square frame of diagonal length d

can hold S_3 iff $1/\sqrt{2} < d < 1$, and a circular frame of diameter d can hold S_3 iff $1/\sqrt{2} < d < \gamma(3, B_2)$, see [8].

On the other hand, it is shown in [1] that

$$\text{no triangular frame can hold a convex body.} \tag{5}$$

This is a special property for triangular frames, and in fact, every non-triangular frame holds some tetrahedron in $\mathbb{R}^3$, see [1].

According to Debrunner and Mani-Levitska [5], Janos Pach asked the following: "If a convec body (say, a stone) can be thrust through a convex hole in a wall by a linear motion (without twisting) then can this be done by a movement perpendicular to the wall?" The answer is positive, and it is proved in [5] that any section of a right cylinder by a plane contains a congruent copy of the base, see also [10], [9]. This together with (5) implies the following: if a convex body, not necessarily smooth, can pass through a triangular hole, then the convex body can pass through the hole by translation perpendicular to the wall, see [1].

Itoh and Zamfirescu [7] found a hole Φ on the plane in $\mathbb{R}^3$ with $\mathrm{diam}(\Phi) = \mathrm{width}(S_2) = \sqrt{3}/2$ and $\mathrm{width}(\Phi) = \mathrm{width}(S_3) = \sqrt{2}/2$, such that S_3 can pass through Φ.

3. Higher dimensions

3.1. **Holes of shape S_{n-1}.** It is known from [5, 9] that any plane section of a right triangular prism contains a congruent copy of a base of the prism. The situation in higher dimension is different. In [5], it is proved that if $n > 3$, then for any right cylinder with convex polytope base, one can find a hyperplane section which does not contain a congruent copy of the base. Nevertheless, we have the following.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a compact convex body, and let Δ_{n-1} be a general $(n-1)$-simplex. If K can pass through the hole Δ_{n-1}, then this can be done by translation only.*

Proof. First consider the case that K is a smooth convex body, that is, the case that, for every point x on the boundary ∂K of K, there is a unique supporting hyperplane of K tangent to K at x. Let $\Delta_{n-1} = p_1 \dots p_n$ be the hole on the hyperplane H, and H_+, H_- be the half spaces such that $H_+ \cap H_- = H$. Let $f : K \times [0,1] \to \mathbb{R}^n$ be a continuous motion of K such that $f(K,0) \subset H_-$, $f(K,1) \subset H_+$, and $H \cap f(K,t) \subset \Delta_{n-1}$ for all $0 < t < 1$. We may further suppose that, for $0 < t < 1$, $f(K,t)$ touchs all facets of Δ_{n-1} except the facet opposite to the vertex p_n. Let $f(K,t)_- = f(K,t) \cap H_-$ and $f(K,t)_+ = f(K,t) \cap H_+$.

Now, for $0 < t < 1$ and $1 \le i < n$, let $H(i,t)$ denote the supporting hyperplane of $f(K,t)$ containing the facet of Δ_{n-1} opposite to p_i. Then the intersection $L(t) = H(1,t) \cap H(2,t) \cap \cdots \cap H(n-1,1)$ is a line. Let $H(n,t)$ be the hyperplane parallel to the line $L(t)$ and containig the facet of Δ_{n-1} opposite to p_n. Then the n hyperplanes $H(1,t),\ldots,H(n,t)$ determine a prism $P(t)$ that intersects H at Δ_{n-1}. Note that, for each $1 \le i < n$ and $0 < t < 1$, $f(K,t)$ and $P(t)$ lie in the same side of $H(i,t)$. Furthermore, since $f(K,t)$ is convex, one of $f(K,t)_-$, $f(K,t)_+$ lies in the same side of $H(n,t)$ as P. Therefore, one of $f(K,t)_-$, $f(K,t)_+$ is contained in $P(t)$. We may suppose that $f(K,t)_+ \subset P(t)$ for $t < \delta$, and $f(K,t)_- \subset P(t)$ for $t > 1-\delta$, where δ is some fixed small positive constant.

Let $t_0 = \max\{t : f(K,t)_+ \subset P(t), \delta \le t \le 1-\delta\}$. Then, for every $\varepsilon > 0$, $f(K,t_0+\varepsilon)_- \subset P(t_0+\varepsilon)$. Since $f(K,t)$ and each hyperplane $H(i,t)$ $(1 \le i \le n)$ move continuously on t, this implies that $f(K,t_0)_- \subset P(t_0)$, and hence $f(K,t_0) \subset P(t_0)$. Thus the prism $P(t_0)$ contains a congruent copy of K, and hence K can pass through Δ_{n-1} by a translation.

Next, let us consider the case that the convex body K is not smooth. We may suppose that K contains the origin 0 of $\mathbb{R}^n$. For $r > 0$, let $B(r)$ denote the ball with center 0 and radius r. Then, for each integer $k > 0$, $K + B(1/k)$ is a smooth convex body, and if K can pass through Δ_{n-1}, then $K + B(1/k)$ can pass through the hole of $(1+2/k)\Delta_{n-1}$. Hence there is a prism P_k containing $K+B(1/k)$ and $P_k \cap H = (1+2/k)\Delta_{n+1}$, where H is the hyperplane containing Δ. Let $r_0 \gg \operatorname{diam}(K)$, and put $X_k = P_k \cap B(r_0)$. Then, in $\{X_k : k = 1,2,\ldots\}$, Blaschke selection theorem guarantees the existence of a subsequence $\{X_{k_m}\}$ and a convex set X_∞ such that X_{k_m} converges to X_∞ in the Hausdorff metric. This X_∞ determines a prism P_∞ with $P_\infty \cap H = \Delta_{n-1}$ such that $K \subset P_\infty$. Hence K can pass through Δ_{n-1} by a translation. □

Problem 1. *Is it possible to take the translation in Theorem 1 perpendicular to the wall? Or equivalently, do $\gamma(n, S_{n-1})$ and $\Gamma(n, S_{n-1})$ coincide?*

Theorem 2.

$$\gamma(n, S_{n-1}) \ge \begin{cases} \sqrt{1-\frac{1}{n}} & \text{if } n \text{ is odd},\\ \sqrt{1-\frac{1}{n+2}} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Suppose that S_n can pass through the hole of dS_{n-1}. By Theorem 1, this can be done by translation only. Thus we can apply (2) with (1), which implies the desired inequality. □

The above result together with $\gamma(n, S_{n-1}) \le \Gamma(n, S_{n-1}) \le 1$ gives

$$\lim_{n\to\infty} \gamma(n, S_{n-1}) = \lim_{n\to\infty} \Gamma(n, S_{n-1}) = 1.$$

If the simplex does pass through a hole, then in particular the volume of some central hyperplane section of that simplex is no bigger than the volume of the hole. Jiří Matoušek suggested showing $\gamma(n, S_{n-1}) \to 1$ by using this simple observation. Then, Matthieu Fradelizi told us that a result in [6] implies that the volume of the smallest central hyperplane section of S_n is more than $\mathrm{vol}(S_{n-1})/(2\sqrt{3})$, and this is enough for proving $\gamma(n,S_{n-1}) \to 1$. It is conjectured that the smallest central hyperplane section of S_n is obtained by a hyperplane parallel to a facet of the simplex.

Since the diameter of circumsphere of S_n is $\sqrt{2(n-1)/n}$, we have

$$\Gamma(n,S_{n-1})\sqrt{\tfrac{2(n-1)}{n}} \geq \Gamma(n,B_{n-1}).$$

This together with (3) implies

$$\Gamma(n,S_{n-1}) \geq \sqrt{1-\tfrac{1}{n+1}}$$

for n odd. (For n even, Theorem 2 gives a better lower bound for $\Gamma(n,S_{n-1})$.)

Actually S_n can pass through a hole smaller than its facet.

Theorem 3. $\Gamma(n,S_{n-1}) < 1$ *for all* $n \geq 2$.

Before the formal proof, let us try the case $n=3$ to get a feel. Let $S_2 = A_0A_1A_2$, $A_0 = (0,1/2), A_1 = (0,-1/2), A_2 = (\sqrt{3}/2,0)$, and let $\mathscr{P}$ be the right triangular prism with base $A_0A_1A_2$. We put the unit regular tetrahedron $S_3 = B_0B_1B_2B_3$ in the prism, namely, we set

$$B_0 = (0,1/2,0), B_1 = (0,-1/2,0), B_2 = (1/\sqrt{2},0,1/2), B_3 = (1/\sqrt{2},0,-1/2).$$

Now we move the tetrahedron very slightly keeping it inside $\mathscr{P}$ so that all vertices are off the faces of $\mathscr{P}$. This can be done by rotating the tetrahedron along the x-axis, and push it in the direction of x-axis. This gives $\Gamma(3,S_2) < 1$.

Proof. For $n \geq 2$, let

$$S_n = A_0A_1\cdots A_n \subset \mathbb{R}^n$$

be a unit regular n-simplex with vertices $A_0,\ldots,A_n$. We may assume that

$$A_i = (0,*,\ldots,*) \in \mathbb{R}^n \text{ for } 0 \leq i < n,$$
$$A_n = (h,0,\ldots,0) \in \mathbb{R}^n,$$

where $h = \sqrt{(n+1)/(2n)}$ is the height of S_n. Let S_{n-1} be the facet spanned by $A_0,\ldots,A_{n-1}$. We may assume that the center of S_{n-1} coincides with the origin.

Then we can construct a unit regular $(n+1)$-simplex

$$S_{n+1} = B_0B_1\cdots B_nB_{n+1} \subset \mathbb{R}^{n+1}$$

by setting

$$B_i = (A_i, 0) \in \mathbb{R}^{n+1} \text{ for } 0 \le i < n,$$
$$B_n = (k, 0, \ldots, 0, 1/2) \in \mathbb{R}^{n+1},$$
$$B_{n+1} = (k, 0, \ldots, 0, -1/2) \in \mathbb{R}^{n+1},$$

where $k = \sqrt{h^2 - (1/4)}$. Since $0 < k < h$ we notice that this S_{n+1} is contained in the prism $\mathscr{P} = S_n \times \mathbb{R}$. More precisely, two vertices B_n and B_{n+1} are interior points of $\mathscr{P}$, while a face $F = B_0 B_1 \cdots B_{n-1}$ is contained in a facet $S_{n-1} \times \mathbb{R}$ of $\mathscr{P}$.

We will show by induction on n that we can continuously move our S_{n+1} inside the prism $\mathscr{P}$ so that S_{n+1} will be contained in $\mathscr{P}^\circ$, where $\mathscr{P}^\circ$ denotes the interior of $\mathscr{P}$. Namely, we will find a continuous function $f_{n+1} : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $t_0 \in (0,1)$ with $f_{n+1}(0, S_{n+1}) = S_{n+1}$ (the starting position) and $f_{n+1}(t, S_{n+1}) \subset \mathscr{P}^\circ$ for all $t \in (0, t_0]$.

For the base case, we take

$$f_3(t, x, y, z) = (x + t, y\cos t - z\sin t, y\sin t + z\cos t),$$

namely, we rotate the regular tetrahedron along the x-axis and push it in the direction of x-axis. Now let $n \ge 3$ and apply the induction hypothesis to the $(n-1)$-simplex F with the prism $\mathscr{P}' = F \times \mathbb{R}$. Then we can find a continuous function f_n and t_0' with the required properties, in particular, $f_n(t, F) \subset (\mathscr{P}')^\circ$ for $t \in (0, t_0']$. We extend f_n to define

$$f_{n+1}(t, x_0, x_1, \ldots, x_n) = (x_0 + t, f_n(t, x_1, \ldots, x_n)).$$

Then we have $f_{n+1}(0, S_{n+1}) = S_{n+1}$, $f_{n+1}(0, \{B_n, B_{n+1}\}) \subset \mathscr{P}^\circ$ and

$$f_{n+1}(t, F) \subset (t, 0, \ldots, 0) + (\mathscr{P}')^\circ \text{ for } t \in (0, t_0'].$$

Thus, we can choose $0 < t_0 \ll t_0'$ such that $f_{n+1}(t, S_{n+1}) = f_{n+1}(t, F \cup \{B_n, B_{n+1}\}) \subset \mathscr{P}^\circ$ for $t \in (0, t_0]$. □

3.2. **Holes of shape** B_{n-1}**.** We have $\Gamma(n, B_{n-1}) \to \sqrt{2}$ as $n \to \infty$ by (3). On the other hand, we will show $\gamma(n, B_{n-1}) \to 3\sqrt{2}/4$. Namely, rotation does help for escaping through a round hole.

By a hyperdisk we mean an $(n-1)$-ball sitting on a hyperplane in $\mathbb{R}^n$. We say that a hyperplane H cuts a line segments xy if H separates x and y. A hyperdisk $D \subset H$ is said to cut a line segment xy if H cuts xy and D intersects xy. In this case, the intersection of the hyperdisk D and the line segment xy is a singleton. The radius of a hyperdisk D is denoted by $r(D)$.

Let $S_n \subset \mathbb{R}^n$ be a unit regualr n-simplex with vertex set $V(S_n)$. Choose $p \in V(S_n)$. By a proper partition, we mean to partition $V(S_n)$ into three nonempty sets $V(S_n) = \{p\} \cup X \cup Y$. Then, X and Y are divided by a hyperplane. Let b_X, b_Y be the barycenters of X, Y, respectively. For $0 < s < 1$, let

$$z_s = (1-s)b_X + sb_Y.$$

This point divides the line segment $b_X b_Y$ into $s : 1-s$. Let H_p denote the hyperplane containing $V(S_n) \setminus \{p\}$, and let $I(X,Y,s) \subset H_p$ be the $(n-2)$-dimensional flat passing through z_s and perpendicular to $b_X b_Y$. For each $-1 \le t < 1$, let

$$w_t = \begin{cases} (1-t)p + tb_Y & \text{if } 0 \le t < 1, \\ (1-|t|)p + |t|b_X & \text{if } -1 \le t < 0. \end{cases}$$

Let $H(X,Y,s,t)$ denote the hyperplane obtained by rotating H_p around $I(X,Y,s)$ so that it comes to the position where it intersects pb_X or pb_Y at the point w_t. Since $\dim I(X,Y,s) = n-2$, it is possible to rotate H_p around $I(X,Y,s)$. We sometimes use a redundant notation $I_p(X,Y,s)$ or $H_p(X,Y,s,t)$ to emphasize that p is the chosen vertex.

For $x \in X$ and $y \in Y$, we notice that $H(X,Y,s,t)$ divides the edge xy with ratio $s : 1-s$, and it divides the edge px or py with ratio $t : 1-t$ (resp. $|t| : 1-|t|$) if $t \ge 0$ (resp. if $t < 0$). The radius of the smallest hyperdisk on $H(X,Y,s,t)$ that contains $S_n \cap H(X,Y,s,t)$ will be denoted by

$$r(|X|,s,t),$$

which only depends on s,t and $|X|$, the cardinality of X.

Lemma 1. *Let $V(S_n) = \{p\} \cup X \cup Y$ be a proper partition. Suppose that D is a (p,X,Y)-hyperdisk, that is, D contains p, and D cuts all edges xy, where $x \in X$, $y \in Y$. Then $r(|X|,s,0) \le r(D)$ holds for some $0 < s < 1$.*

Proof. For a (p,X,Y)-hyperdisk D, let D_{xy} denote the point where D cuts the edge connecting $x \in X$ and $y \in Y$. For $X' \subset X, Y' \subset Y$, let

$$\begin{aligned} d(D,X',Y') &= \{|x - D_{xy}| : x \in X', y \in Y'\}, \\ \delta(D,X',Y') &= \max d(D,X',Y') - \min d(D,X',Y'). \end{aligned}$$

We define the irregularity of D by $\delta(D) := \delta(D,X,Y)$. If $\delta(D) = 0$, namely, if $d(D,X,Y) = \{s\}$ for some $0 < s < 1$, then each D_{xy} lies on $H(X,Y,s,0)$ and $r(|X|,s,0) \le r(D)$. So we may assume that $\delta(D) > 0$. Let δ_0 be the infimum of the irregularities of (p,X,Y)-hyperdisks with radius at most $r(D)$. By applying Blaschke selection theorem, we can find a (p,X,Y)-hyperdisk D_∞ with $r(D_\infty) \le r(D)$ and $\delta(D_\infty) = \delta_0$. Then, the following claim implies that $\delta_0 = 0$, which completes the proof of the lemma.

Claim 1. *If $\delta(D) > 0$, then there is a (p,X,Y)-hyperdisk D^* with $r(D^*) \le r(D)$ and $\delta(D^*) < \delta(D)$.*

Now we prove the claim by introducing an operation which rotates D to get D^*. Suppose that

$$\delta(D) = |a - D_{ac}| - |b - D_{bd}| > 0, \tag{6}$$

where $a, b \in X$ and $c, d \in Y$. Let K be the hyperplane contianing D. Then the line ab or cd intersects K. By symmetry, we may assume that the line ab intersects K. Let L be the hyperplane that perpendicularly bisects the edge ab, and let $J = K \cap L$. Since $\dim J = d - 2$, we can rotate K around J until it comes to the position where the hyperplane becomes parallel to the line ab. Let K^* be the resulting hyperplane. Let D' be the mirror image of D with respect to L, and let B be the minimum ball containing $D \cup D'$. Finally let $D^* = B \cap K^*$. Since the distance from the center of B to $J (\subset K)$ is greater than or equal to the distance between the center of B and K, it follows that $r(D^*) \leq r(D)$.

We will show that D^* is a (p, X, Y)-hyperdisk with $\delta(D^*) < \delta(D)$. Let $x \in X \setminus \{a, b\}$ and $y \in Y$. Since $V(S_n) \setminus \{a, b\}$ lies on L, we have $D_{xy} \in D \cap L = D \cap K \cap L = D \cap J \subset D^*$, and $D_{xy} = D^*_{xy}$. This gives

$$\delta(D, X \setminus \{a, b\}, Y) = \delta(D^*, X \setminus \{a, b\}, Y).$$

If $x \in \{a, b\}$ and $y \in Y$, then K^* cuts the line segment $D_{xy}D'_{xy}$, because both D and D' are (p, X, Y)-hyperdisks. Thus the intersection is inside B and we can write it D^*_{xy}. We may assume that the line ab intersects K at the extention beyond b. Since $D^*_{ay}D^*_{by} \parallel ab$ and two edges $D^*_{ay}D^*_{by}$ and $D_{ay}D_{by}$ meet inside the equilateral $\triangle aby$, we have $|a - D_{ay}| > |a - D^*_{ay}|$, and $|b - D_{by}| < |b - D^*_{by}|$. Thus we have

$$\delta(D, \{a, b\}, Y) > \delta(D^*, \{a, b\}, Y).$$

If $\delta(D) = \delta(D, X \setminus \{a, b\}, Y)$, then we have succeeded to decrease the number of pairs $\{ac, bd\}$ which attain (6) by changing from D to D^*. So, by repeating this operation, we may assume that $\{ac, bd\}$ is the only pair which attains (6), or equivalently, $\delta(D) > \delta(D, X \setminus \{a, b\}, Y)$. In this case, we have $\delta(D) > \delta(D^*)$ as desired. $\square$

For computation, it is convenient to put our S_n in $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$. So let $S_n = p_1 p_2 \cdots p_{n+1} \subset \mathbb{R}^{n+1}$ with $p_j = e_j / \sqrt{2}$, where e_j is the j-th standard base of $\mathbb{R}^{n+1}$. Fix $s, t \in (0, 1)$ and $1 \leq i < n$. Let $V(S_n) = \{p\} \cup X \cup Y$ be a proper partition with $|X| = i$. To compute $r(i, s, t)$, we look at the hyperplane $H := H_p(X, Y, s, t)$. Recall that b_X, b_Y are the barycenters of X, Y, respectively. Fix two vertices $x \in X$, $y \in Y$. Let u be the intersection of xy and H, and let $\alpha = |u - z_s|$, where $z_s = (1 - s)b_X + sb_Y$. Then we have

$$2\alpha^2 = (1 - s)^2 (1 - \frac{1}{i}) + s^2 (1 - \frac{1}{n - i}). \tag{7}$$

Let v be the intersection of py and H, and let $\beta = |v - w_t|$, where $w_t = (1-t)p + tb_Y$. Then we have

$$2\beta^2 = t^2(1 - \frac{1}{n-i}). \tag{8}$$

Notice that the distances α, β are independent of the choices of x, y. Also one can see that the two lines uz_s and vw_t are both perpendicular to $z_s w_t$. The distance $\lambda = |z_s - w_t|$ satisfies

$$2\lambda^2 = \frac{(1-s)^2}{i} + (1-t)^2 + \frac{(s-t)^2}{n-i}. \tag{9}$$

Let D be the smallest hyperdisk on H that contains $S_n \cap H$, and let o be its center. Then o is on the line segment $z_s w_t$, and $|u - o| = |v - o| = r(i,s,t)$. Let $\zeta = |z_s - o|$. Then, looking at two right triangles $\triangle ouz_s$ and $\triangle ovw_t$, we have $r(i,s,t)^2 = \alpha^2 + \zeta^2 = (\lambda - \zeta)^2 + \beta^2$. This gives $\zeta = (\alpha^2 - \beta^2 - \lambda^2)/(2\lambda)$, and

$$r(i,s,t)^2 = \alpha^2 + \frac{1}{4\lambda^2}(\lambda^2 - \alpha^2 + \beta^2)^2. \tag{10}$$

By substituting (7), (8), (9) into (10), we can rewrite $r(i,s,t)^2$ in terms of s, t, i (and n) only. We record a special case for later use:

$$2r(i,s,0)^2 = \frac{i(n-i)(s^2 - s + 1)^2}{(1-s)^2 n + (n + 2s - 1)i - i^2}. \tag{11}$$

Lemma 2. *Let $V(S_n) = \{p\} \cup X \cup Y$ be a proper partition with $|X| = i$, and let $0 < s < 1$. Then we have $r(i,s,t) \leq r(i,s,0)$ for all $t \in [s-1, s]$.*

Proof. First assume that $0 < t \leq s$. Let D' be the hyperdisk with center o' on $H(X,Y,s,0)$ that is obtained by rotating D around $I(X,Y,s)$ through the angle $\angle w_t z_s p$. Notice that o' is on the line segment $z_s p$, and $\{u, z_s\} \subset D \cap D'$. Since $r(i,s,t) = r(D) = |u - o|$, to show $r(i,s,t) < r(i,s,0)$ it suffices to show $p \notin D'$, or equivalently, $|p - o'| > r(D') = r(D)$.

Recall that $\alpha = |u - z_s|$, $\beta = |v - w_t|$, $\lambda = |z_s - w_t|$, $\zeta = |z_s - o| = |z_s - o'|$, and $|w_t - o| = \lambda - \zeta$. Let $\eta = |p - w_t|$.

In $\triangle pb_X b_Y$, the edge $b_X b_Y$ is the shortest one. In fact, we have $|p - b_X|^2 = \frac{1}{2}(1 + \frac{1}{i})$, $|p - b_Y|^2 = \frac{1}{2}(1 + \frac{1}{n-i})$, and $|b_X - b_Y|^2 = \frac{1}{2}(\frac{1}{i} + \frac{1}{n-i})$. Hence $\angle b_X p b_Y < \pi/3$. Since $z_s w_s \parallel b_X p$, we have $\angle z_s w_t p \geq \angle z_s w_s p > 2\pi/3$. Looking at $\triangle z_s w_t p$, we get $|p - z_s|^2 > \lambda^2 + \eta^2 + \lambda\eta$. Thus we have

$$|p - o'| = |p - z_s| - |z_s - o'| > \sqrt{\lambda^2 + \eta^2 + \lambda\eta} - \zeta. \tag{12}$$

Using $\triangle pvw_t \sim \triangle pyb_Y$, we have

$$|v - w_t| = t|y - b_Y| = \tfrac{t}{2}(1 - \tfrac{1}{n-i}) < \tfrac{t}{2}(1 + \tfrac{1}{n-i}) = t|p - b_Y| = |p - w_t| = \eta,$$

which implies

$$r(D)^2 = |v-o|^2 = |v-w_t|^2 + |w_t - o|^2 < \eta^2 + (\lambda-\zeta)^2. \qquad (13)$$

Since $0 < t \le s < 1$, it follows from (7) and (8) that $|u - z_s| = \alpha > \beta = |v - w_t|$. Then looking at two triangles $\triangle ouz_s$ and $\triangle ovw_t$ having hypotenuses of the same length ($= r(D)$), we get $|z_s - o| < |w_t - o|$, that is,

$$2\zeta < \lambda. \qquad (14)$$

By (12) (13) and (14), we have

$$\begin{aligned}|p-o'|^2 - r(D)^2 &> (\sqrt{\lambda^2+\eta^2+\lambda\eta} - \zeta)^2 - (\eta^2 + (\lambda-\zeta)^2)\\ &= \lambda\eta - 2\zeta(\sqrt{\lambda^2+\eta^2+\lambda\eta} - \lambda)\\ &> \lambda(\lambda+\eta-\sqrt{\lambda^2+\eta^2+\lambda\eta}) > 0,\end{aligned}$$

as desired.

Next assume that $s-1 \le t < 0$. By comparing two proper partitions $\{p\} \cup X \cup Y$ and $\{p\} \cup X' \cup Y'$, where $X' = Y$ and $Y' = X$, we have $r(|X|, s, t) = r(|X'|, 1-s, -t,)$. Then, using the inequality for $0 < -t \le 1-s < 1$, we have $r(i,s,t) = r(n-i, 1-s, -t) \le r(n-i, 1-s, 0) = r(i,s,0)$, as desired. □

Lemma 3. *For $0 < s \le 1/2$ and $m = \lfloor n/2 \rfloor$, we have*

$$r(m,s,0) = \max\{r(i,s,0) : 1 \le i \le m\}.$$

Proof. By (11), we have

$$\frac{\partial}{\partial i}(2r(i,s,0)^2) = \frac{N_1 N_2}{D},$$

where

$$\begin{aligned}N_1 &= (s^2 - s + 1)^2,\\ N_2 &= (1-2s)i^2 - 2n(1-s)^2 i + n^2(1-s)^2,\\ D &= (1-s)^2 n + \big((n-i) - (1-2s)\big)i.\end{aligned}$$

We will show that $r(i,s,0)$ is an increasing function of i. It is easy to see that $N_1/D > 0$. So, it suffices to show $N_2 \ge 0$. Fix s, n, and let $f(i) := N_2$. If $s = 1/2$, then $f(i) = n(n-2i)/4 \ge 0$, as desired. If $s < 1/2$, then $f(i)$ is a quadratic function of i with positive coefficient for i^2. Since $f(i)$ takes minimum at $i = n(1-s)^2/(1-2s) > n/2$ and $f(n/2) = n^2(1-2s)/4 > 0$, we can conclude that $f(i) > 0$ for $1 \le i \le n/2$. □

Lemma 4. *Let n be even and let $m = n/2$. Then we have*

$$\min\{r(m,s,0) : s \in [0,1]\} = r(m, 1/2, 0), \qquad (15)$$

and

$$r(m,1/2,0)=\frac{3}{8}\sqrt{\frac{2n}{n+1}} \tag{16}$$

Proof. By (11) we have

$$\frac{\partial}{\partial s}(r(n/2,s,0)^2)=\frac{-n(1-2s)(s^2-s+1)(2s^2-2s+n)}{(4s^2-4s+n+2)^2},$$

which implies (15). By (11) we have (16). □

Lemma 5. *Let n be odd and let $m=\lfloor n/2\rfloor$. Let $s_0\in(0,1)$ be a unique real root of a the cubic equation*

$$4nX^3-6(n+1)X^2+2(n^2+n+2)X-n^2+1=0. \tag{17}$$

Then we have

$$\min\{r(m,s,0):s\in[0,1]\}=r(m,s_0,0), \tag{18}$$

and

$$\max\{r(m,s,s):s\in[s_0,1/2]\}=r(m,s_0,s_0). \tag{19}$$

Moreover, $r(m,s_0,0)^2$ is a unique real root of the following cubic equation with integral coefficients:

$$2048(n+1)n^3X^3+a_2X^2+a_1X+a_0=0, \tag{20}$$

where

$$\begin{aligned} a_0&=-9(n^2-1)^2(n^4-4n^3+2n^2+4n+13),\\ a_1&=16(n^2-1)(2n^6-6n^5-15n^4+38n^3+42n^2+48n-29),\\ a_2&=64(8n^6-8n^5-41n^4-28n^3-10n^2+36n+27). \end{aligned}$$

Proof. We start with (11). Then we get (18) by computing $\frac{\partial}{\partial s}(r(m,s,0)^2)$. Moreover, it follows from (17) that $1/2-1/n^2<s_0<1/2$ for $n\geq 3$. In fact, we have

$$s_0=\frac{1}{2}-\frac{3}{4n^2}-\frac{3}{8n^3}+O(n^{-4}). \tag{21}$$

By (10) we have $2r(m,s,s)^2=(2s^2-2s+1)m/(1+m)$, which implies (19).

Finally we outline how to show that $r(m,s_0,0)^2$ satisfies (20). This can be done by direct computation (with aid of computer). By (11) we have

$$2r(m,s,0)^2=\frac{m(1+m)(s^2-s+1)^2}{m^2+(1-s)^2+2m(s^2-s+1)}. \tag{22}$$

The numerator of $r(m,s_0,0)^2$ is a biquadratic polynomial of s_0, and one can reduce it to a quadratic polynomial using (17). Substitute the reduced $r(m,s_0,0)^2$ into (20), and reduce these fractions to a common denominator. Then one can check that the numerator vanishes by reducing it using (17). □

Define

$$s^* = \begin{cases} 1/2 & \text{if } n \text{ is even,} \\ s_0 & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

Lemma 6. *Let $m = \lfloor n/2 \rfloor$, and let $V(S_n) = \{p\} \cup X \cup Y$ be a proper partition with $i = |X| \leq m$. Then $S_n \cap H_p(X,Y,s^*,t)$ is contained in a hyperdisk of radius $r(m,s^*,0)$ for all $t \in [s^* - 1, s^*]$.*

Proof. We have $r(i,s^*,t) \leq r(i,s^*,0) \leq r(m,s^*,0)$, where the first inequality follows from Lemma 2, and the second one from Lemma 3. □

Theorem 4. *Let $m = \lfloor n/2 \rfloor$, and define s^* by* (23)*. Then we have $\gamma(n, B_{n-1}) = r(m,s^*,0)$.*

For $n = 2m$, it follows from (16) that

$$r(m,1/2,0) = \frac{3}{4\sqrt{2}}\left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{5}{16n^3} + O(n^{-4})\right). \tag{24}$$

On the other hand, for $n = 2m+1$, it follows from (20) that

$$r(m,s_0,0) = \frac{3}{4\sqrt{2}}\left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{13}{16n^3} + O(n^{-4})\right). \tag{25}$$

One can also get (25) by substituting (21) directly into (22). By Theorem 4 with (16), (24) and (25), we have the following.

Corollary 5.

$$\gamma(n, B_{n-1}) = \begin{cases} \frac{3}{4}\sqrt{\frac{2n}{n+1}} & \textit{if } n \textit{ is even,} \\ \frac{3}{4}\sqrt{\frac{2n}{n+1}} - \frac{3\sqrt{2}}{8n^3} + O(n^{-4}) & \textit{if } n \textit{ is odd,} \end{cases}$$

and

$$\lim_{n\to\infty} \gamma(n, B_{n-1}) = \frac{3\sqrt{2}}{4}.$$

Proof of Theorem 4. Let D be a hyperdisk on a hyperplane in $\mathbb{R}^n$ through which S_n can pass. We may suppose that the vertices $p_1, p_2, \ldots, p_{n+1}$ can pass through D one by one in this order. (If this seems to be impossible, then by replacing D with a hyperdisk of radius $r(D) + \varepsilon$, this would become possible, where $\varepsilon > 0$ can be chosen arbitrarily small.) Consider the moment when p_{m+1} passes through D. By Lemma 1, $r(D) + \varepsilon$ is at least $r(m,s,0)$ for some $0 < s < 1$, which is at least $r(m,s^*,0)$ by (15) and (18). Thus $r(D) + \varepsilon \geq r(m,s^*,0)$. Since ε can be chosen arbitrarily small, we have $r(D) \geq r(m,s^*,0)$, and hence $\gamma(n,B_{n-1}) \geq 2r(m,s^*,0)$.

Next we show $\gamma(n,B_{n-1}) \leq 2r(m,s^*,0)$. Instead of pushing S_n through the hole, we fix our S_n and move the hyperplane containing the hole. Let D be the hyperdisk of radius $r(m,s^*,0)$ in the moving wall hyperplane. Let

$V(S_n) = \{p_1, \dots, p_{n+1}\}$, $X_i = \{p_1, \dots, p_i\}$, and $Y_i = \{p_{i+2}, \dots, p_{n+1}\}$. Since $\{p_{i+2}\} \cup X_{i+1} \cup Y_{i+1}$ is also a proper partition, we have $H_{p_{i+1}}(X_i, Y_i, s^*, s^*) = H_{p_{i+2}}(X_{i+1}, Y_{i+1}, s^*, s^* - 1)$.

Let $H_i = H(X_i, Y_i, s^*, s^* - 1)$. Suppose that our hole D sits in H_i for some $1 \le i \le m$. By rotating this wall hyperplane around $I(X_i, Y_i, s^*)$, we get $H(X_i, Y_i, s^*, s^*) = H_{i+1}$. By Lemma 6, we may assume that, in this rotation process, the intersection of the fixed S_n and the wall hyperplane is always contained in D. Namely, we can move the wall hyperplane from H_i to H_{i+1} so that the intersection of S_n and the wall is always contained in the hole.

Therefore, starting from H_1, we can send the wall hyperplane to the position of H_{m+1}. If n is even, then in the process from H_m to H_{m+1}, we get $H(X_m, Y_m, 1/2, 0)$. This hyperplane divides S_n into two parts that are mutually congruent, and we are done (by repeating the same procedure in the reverse order). If n is odd, then by translating $H_{m+1} = H(X_m, Y_m, s^*, s^*)$, we get $H(X_m, Y_m, 1/2, 1/2)$, which divides S_n into mutually congruent parts. Moreover, by (19), the intersection of S_n and the hyperplane is always contained in D in the process of this translation. This completes the proof. □

3.3. **Holes of shape Q_{n-1}.** In [11] the following is proved: for every $\varepsilon > 0$ there is an N such that for every $n > N$ one has

$$S_n \subset (2 + \varepsilon) Q_n.$$

This gives

$$\lim_{n \to \infty} \Gamma(n, Q_{n-1}) \le 2.$$

Since $Q_{n-1} \subset B_{n-1}$, we get lower bounds for $\gamma(n, Q_{n-1})$ and $\Gamma(n, Q_{n-1})$ from $\gamma(n, B_{n-1})$ and $\Gamma(n, B_{n-1})$, respectively. Here we include a simple proof of the following slightly weaker bound for $\Gamma(n, Q_{n-1})$.

Theorem 6. *We have*

$$\Gamma(n, Q_{n-1}) \ge \sqrt{\frac{2(n-1)}{n+1}}, \tag{26}$$

with equality holding iff there exists an Hadamard matrix of order $n+1$.

Proof. Let $d = \Gamma(n, Q_{n-1})$. Then S_n can pass through a hole of dQ_{n-1} by translation. So (2) and (1) imply

$$\mathrm{width}(dQ_{n-1}) = \frac{d}{\sqrt{n-1}} \ge \mathrm{width}(S_n) \ge \sqrt{\frac{2}{n+1}},$$

which gives (26). Moreover, if $S_n \subset \ell Q_n$, then we have

$$\ell \ge \frac{\sqrt{n}}{\sqrt{n-1}} \Gamma(n, Q_{n-1}) \ge \sqrt{\frac{2n}{n+1}}.$$

It is known that $\ell = \sqrt{(2n)/(n+1)}$ iff there exists an Hadamard matrix of order $n+1$, see e.g., [13]. □

Problem 2. *Do we have* $\gamma(n, Q_{n-1}) = \Gamma(n, Q_{n-1}) = \sqrt{2} - o(1)$*?*

3.4. **Holes having minimum volumes.** In [7], the following problem is posed.

Problem 3. *Find the minimum* $(n-1)$*-dimensional volume of a compact hole in a hyperplane of* $\mathbb{R}^n$ *such that* S_n *can pass through it.*

The following variation seems to be easier.

Problem 4. *Find the minimum* $(n-1)$*-dimensional volume of a compact hole in a hyperplane of* $\mathbb{R}^n$ *such that* S_n *can pass through it by translation perpendicular to the hyperplane.*

We list possible candidates. Put $\sqrt{2}S_n$ in $\mathbb{R}^{n+1}$ so that the vertices are $e_1, \ldots, e_{n+1}$, where e_i is the i-th standard base of $\mathbb{R}^{n+1}$.

Project the $\sqrt{2}S_n$ in the direction of

$$(1, -1, \overbrace{0, \ldots, 0}^{n-1}).$$

Then the hole created by the shadow has volume

$$\frac{1}{(n-1)!}\sqrt{\frac{n+1}{2}}. \tag{27}$$

Next suppose that n is odd and write $n = 2k+1$. Project the $\sqrt{2}S_n$ in the direction of

$$(\overbrace{1, \ldots, 1}^{k+1}, \overbrace{-1, \ldots, -1}^{k+1}).$$

Then the corresponding hole has volume

$$\frac{2}{(n-1)!}. \tag{28}$$

Finally suppose that n is even and write $n = 2k$. Project the $\sqrt{2}S_n$ in the direction of

$$(\overbrace{k+1, \ldots, k+1}^{k}, \overbrace{-k, \ldots, -k}^{k+1}).$$

In this case, the volume of the hole is

$$\frac{2}{(n-1)!}\sqrt{\frac{n}{n+2}}. \tag{29}$$

Among the above examples, the smallest one is (27) for $n \le 5$. For $n = 7$, (27) and (28) coincide. For the other cases, (28) and (29) give the smallest one.

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