Tetrahedra passing through a triangular hole, and tetrahedra fixed by a planar frame

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Abstract

We show that a convex body can pass through a triangular hole iff it can do so by a translation along a line perpendicular to the hole. As an application, we determine the minimum size of an equilateral triangular hole through which a regular tetrahedron with unit edge can pass. The minimum edge length of the hole is $(1+\sqrt{2})/\sqrt{6} \approx 0.9856$. One of the key facts for the proof is that no triangular frame can hold a convex body. On the other hand, we also show that every non-triangular frame can fix some tetrahedron.

Keywords: frame, holding a convex body, fixing a convex body, regular tetrahedron, minimal embedding

1. Introduction

Let Ω be a compact convex disk in a plane. By a *frame* we mean the boundary $\partial\Omega$ of Ω. Suppose that the frame $\partial\Omega$ is *attached to* a convex body $K \subset \mathbb{R}^3$, that is, $K \cap \Omega \neq \emptyset$ and $\mathrm{int}(K) \cap \partial\Omega = \emptyset$, where $\mathrm{int}(K)$ denotes the interior of K. If the frame $\partial\Omega$ can be removed away from K by a continuous rigid motion of $\partial\Omega$ (or K) with keeping $\mathrm{int}(K) \cap \partial\Omega = \emptyset$, then we say $\partial\Omega$ can *slip* out of K, otherwise, we say $\partial\Omega$ *holds* K. A *unit regular tetrahedron* is a regular tetrahedron with unit edges. For example, a circular frame of diameter $1/\sqrt{2} + \varepsilon$ can hold a unit regular tetrahedron if ε is sufficiently small, see Figure 1.

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Preprint submitted to Elsevier *August 3, 2011*

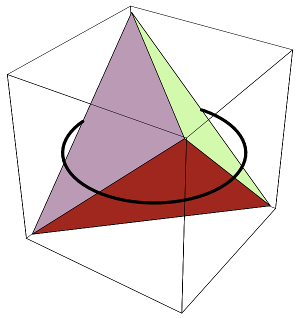

Figure 1: A circular frame fixes a tetrahedron.

Zamfirescu [10] proved that most convex bodies can be held by a circular frame. More precisely, the convex bodies in $\mathbb{R}^3$ that cannot be held by any circular frame form a nowhere dense subset of the space of all convex bodies in $\mathbb{R}^3$ with Hausdorff metric. We first show that a triangular frame is quite different from a circular frame as follows.

Theorem 1. *A triangular frame attached to a convex body can always slip out of the convex body. Thus no triangular frame can hold a convex body.*

Regarding a frame as the boundary of a hole in a plane, we may consider whether a given convex body can pass through the hole. Itoh and Zamfirescu [3] studied the size of a hole (diameter and width) through which a regular simplex of unit edges can pass. Itoh, Tanoue, and Zamfirescu [2] determined the smallest circular hole and the smallest square hole through which a unit regular tetrahedron can pass, see also [6] for the problem in higher dimensions. Concerning a triangular hole, we have the following.

Theorem 2. *A convex body K can pass through a triangular hole Δ iff K can be congruently embedded in a right triangular prism with base Δ.*

Thus, if a convex body can pass through a triangular hole, then it can do so by a continuous translation of the convex body along a line perpendicular to the plane containing the hole. Similar assertion is not true for a circular hole. For example, when a regular tetrahedron passes through a circular hole of the smallest possible size, rotations are necessary, see [2], and [6] for higher dimensional cases.

It is proved in [7] that an equilateral triangular prism can contain a unit regular tetrahedron iff the edge length of the base equilateral triangle of the prism is at least $(1+\sqrt{2})/\sqrt{6}$. Hence we have the following.

Theorem 3. *A unit regular tetrahedron can pass through an equilateral triangular hole iff the edge length of the hole is at least $(1+\sqrt{2})/\sqrt{6}$.*

Finally we consider a fixing problem for non-triangular frames. We say that M_t is a rigid motion if $M_t : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry for each $0 \le t \le 1$ starting with the identity map M_0, and M_t is a continuous function of t for $0 \le t \le 1$. Let P be the xy-plane in $\mathbb{R}^3$, and let $H \subset P$ be a convex disk. We say that H *fixes the convex body* $K \subset \mathbb{R}^3$ if

i. $K \cap P \subset H$, and

ii. if a rigid motion M_t satisfies $(M_t K) \cap P \subset H$ for all $t \in [0,1]$, then $M_t P = P$ for all t.

This, of course, means that the frame ∂H holds K because then no rigid motion can move K away from P. In this definition one cannot require that M_t equals the identity. This is shown by the example in Figure 1: if $\varepsilon = 0$, then the regular tetrahedron is fixed by the circle but it can clearly be rotated.

Theorem 4. *Every non-triangular frame fixes some tetrahedron.*

2. A convex body through a triangular hole

Proof of Theorem 1. Suppose that the boundary $\partial\Delta$ is a triangular frame attached to a convex body K. Let $\partial\Delta = a \cup b \cup c$ with three edges a, b, c. The triangle Δ divides K into two parts K^+ and K^-. Let H^a be a supporting plane of K containing the edge a. Then, $a \subset H^a$ and $\text{int}(K) \cap H^a = \emptyset$. Define H^b similarly. Let H be the plane containing c and parallel to the line $\ell := H^a \cap H^b$. Then H^a, H^b, H determine a prism $\mathcal{P}$. One of K^+, K^- is contained in $\mathcal{P}$. (For otherwise, we can find a point $p \in K^+$ and a point $q \in K^-$ both lying in the same side of H opposite to the prism $\mathcal{P}$. Then the line segment pq does not intersects Δ, contradicting that Δ cuts the convex body K.) If $K^+ \subset \mathcal{P}$ (resp. $K^- \subset \mathcal{P}$), then K can slip out of the frame $\partial\Delta$ by moving parallel to the line ℓ towards K^- (resp. K^+) side. □

Let P be the xy-plane in $\mathbb{R}^3$. For a convex disk $\Omega \subset P$, the right Ω-prism (denoted by $\Omega \times \mathbb{R}$) is the set obtained as the union of those lines that intersect Ω perpendicularly. The set Ω is called the *base* of $\Omega \times \mathbb{R}$. If Ω is an equilateral triangle of edge length t, then the prism is called an *equilateral triangular prism of size* t.

Lemma 1. *Let $\Omega \subset P$ be a convex disk, and let $\mathcal{P} = \Omega \times \mathbb{R}$. Then, for any convex disk $\tilde{\Omega}$ obtained as a section of $\mathcal{P}$ by a plane, Ω can be congruently embedded in $\tilde{\Omega}$.*

Lemma 1 is a result due to Kovalyov [5] (answering a question of Zalgaller [9]), and independently, Debrunner and Mani-Levitska [1] (answering a question of Pach [8]), see also Kós and Törőcsik [4].

Now, let us regard a triangle $\Delta \subset P$ as a *hole*.

Proof of Theorem 2. If K is congruently embedded in $\Delta \times \mathbb{R}$, then K can pass through Δ by a translation parallel to the z-axis.

Suppose that K can pass through the hole Δ. Let $\partial\Delta = a \cup b \cup c$. Suppose that K can go through the hole Δ from the upper half space $[z \geq 0]$ into the lower half space $[z \leq 0]$. Let K_t, $0 \leq t \leq 1$, denote the continuously moving body congruent with K, passing through the hole Δ from $[z \geq 0]$ to $[z \leq 0]$; $K_0 \subset [z \geq 0]$, $K_1 \subset [z \leq 0]$. For each $t \in [0,1]$, the plane P divides K_t into two parts, $K_t^+ = K \cap [z \geq 0]$ and $K_t^- = K \cap [z \leq 0]$. Let H_t^a be a supporting plane of B_t containing the edge a. Then this is a continuously moving plane such that $a \subset H_t^a$ and $H_t^a \cap \text{int}(K_t) = \emptyset$. Define H_t^b similarly. Let H_t be the plane containing c and parallel to the line $L_t := H_t^a \cap H_t^b$. Then H_t^a, H_t^b, H_t determine a continuously moving triangular prism $\mathcal{P}_t$. Note that $\emptyset = K_0^- \subset \mathcal{P}_0$, and $\emptyset = K_1^+ \subset \mathcal{P}_1$. Furthermore, for each $t \in [0,1]$, one of K_t^+, K_t^- is contained in $\mathcal{P}_t$ as in the proof of Theorem 1. Let $\alpha = \sup\{t \in [0,1] : K_t^- \subset \mathcal{P}_t\}$. Then, there is a monotone increasing sequence $0, t_1, t_2, t_3, \dots$ such that $K^-_{t_n} \subset \mathcal{P}_{t_n}$ and $\lim_{n\to\infty} t_n = \alpha$. Hence, by the continuity, we have $K^-_\alpha \subset \mathcal{P}_\alpha$. Similarly, since $t > \alpha$ implies $K^+_t \subset \mathcal{P}_t$, we have $K^+_\alpha \subset \mathcal{P}_\alpha$. Therefore, $K_\alpha \subset \mathcal{P}_\alpha$.

Thus K can be congruently embedded in a triangular prism $\mathcal{P}_\alpha$ with $\mathcal{P}_\alpha \cap P = \Delta$. By Lemma 1, $\mathcal{P}_\alpha$ is congruently embedded in $\Delta \times \mathbb{R}$. Hence K can be congruently embedded in $\Delta \times \mathbb{R}$. □

Corollary 1. *If a convex body can pass through a triangular hole, then a whole process of passing through the hole can be realized by a translation along a line perpendicular to the plane having the hole.*

Proof of Theorem 3. Let $\Delta(d)$ denote an equilateral triangle with edge length d. Two congruent regular tetrahedra $T_1, T_2 \subset \Delta(d) \times \mathbb{R}$ are said to be equivalent if it is possible to superpose T_1 on T_2 by a continuous rigid motion of T_1 within the prism. Let $\nu(d)$ denote the maximum number of mutually non-equivalent embeddings of a unit regular tetrahedron into $\Delta(d) \times \mathbb{R}$. The following result is proved in [7]:

$$\nu(d) = \begin{cases} 0 & \text{for } d < d_0 := 1 + \sqrt{2}/\sqrt{6} \approx 0.9856, \\ 6 & \text{for } d_0 \le d < d_1 := \sqrt{3} + 3\sqrt{2}/6 \approx 0.9958, \\ 18 & \text{for } d_1 \le d < 1, \\ 1 & \text{for } 1 \le d. \end{cases} \tag{1}$$

By (1) we have $\nu(d) \neq 0$ iff $d \geq (1+\sqrt{2})/\sqrt{6}$. In other words, a unit regular tetrahedron can be congruently embedded in $\Delta(d) \times \mathbb{R}$ iff $d \geq (1+\sqrt{2})/\sqrt{6}$. Combining this result with Theorem 2, we get Theorem 3. □

Here we recall two important embeddings which are essentially used to show (1) in [7]. We are going to embed a unit tetrahedron $T = ABCD$ into $\Delta(d)$-prisms. First, let us consider the case $d = d_0$. Let $h = d_0/2 = (1+\sqrt{2})/\sqrt{24}$, and let $\Delta_0 \subset P$ be the triangle with vertices $(\pm h, 0, 0)$, $(0, \sqrt{3}h, 0)$. Then Δ_0 is an equilateral triangle of edge length d_0. Let $\mathcal{P}$ be the $\Delta(d_0)$-prism. Let $k = (\sqrt{2}-1)/\sqrt{24}$, $\ell = 1/\sqrt{2}$, and define four points A, B, C, D by

$$A = (k, \ell, -h),\ B = (-h, 0, -k),\ C = (h, 0, k),\ D = (-k, \ell, h).$$

Then one can check that these four points span a regular tetrahedron of edge length 1, which is contained in the $\Delta(d_0)$-prism $\mathcal{P}$, see Figure 2 left.

Next we consider the case $d = d_1$. Let $\Delta_1 \subset P$ be the triangle with vertices

$$A' = (\tfrac{\sqrt{2}}{3}, 0, 0),\ B' = (-\tfrac{\sqrt{3}+\sqrt{2}}{6}, 0, 0),\ E = (-\tfrac{\sqrt{3}-\sqrt{2}}{12}, \tfrac{\sqrt{6}+1}{4}, 0).$$

A straightforward calculation shows that Δ_1 is an equilateral triangle with edge length d_1. Let $T = ABCD$ be the tetrahedron with vertices

$$A = (\tfrac{\sqrt{2}}{3}, 0, \tfrac{1}{3}),\ B = (-\tfrac{\sqrt{3}+\sqrt{2}}{6}, 0, \tfrac{\sqrt{6}-1}{6}),\ C = (\tfrac{\sqrt{3}-\sqrt{2}}{6}, 0, -\tfrac{\sqrt{6}+1}{6}),$$
$$D = (0, \tfrac{\sqrt{6}}{3}, 0).$$

Then T is a unit regular tetrahedron contained in the Δ_1-prism, see Figure 2 right.

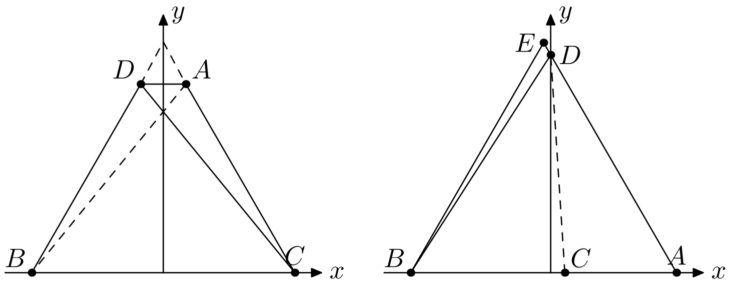

Figure 2: Top views

What is the minimal area of a hole such that a unit regular tetrahedron $ABCD$ can pass through it? This problem is raised in [3]. Let $ABCD$ be a unit regular tetrahedron in $\mathbb{R}^3$ such that the edge AB lies on the z-axis. Then, by projecting $ABCD$ to P, we get an isosceles triangle with sides $1, \sqrt{3}/2, \sqrt{3}/2$, whose area is $1/\sqrt{8}$. Hence $ABCD$ can pass through a triangular hole of area $1/\sqrt{8}$. In fact, this is the minimum area hole that a unit regular tetrahedron can pass through by translation only. So, if we could find a smaller hole by allowing rotation for escape, then the hole would be of non-triangular shape.

Problem 1. *Is $1/\sqrt{8}$ the minimal area of a hole through which a unit regular tetrahedron can pass?*

In this paper, we have considered problems in $\mathbb{R}^3$. In higher dimensions, the following is proved in [6]. If a regular n-simplex Δ^n in $\mathbb{R}^n$ can pass through a hole of a regular $(n-1)$-simplex with side length ℓ_n, then $\sqrt{1-(1/n)} < \ell_n < 1$.

3. Tetrahedra fixed by a non-triangular frame

Let P be the xy-plane in $\mathbb{R}^3$, and let $H \subset P$ be a convex disk. An alternative description of fixing is the following: H fixes the convex body $K \subset \mathbb{R}^3$ if $K \cap P \subset H$ and if a rigid motion $M_t : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $K \cap (M_t^{-1}P) \subset M_t^{-1}H$ for all $t \in [0,1]$, then $M_tP = P$ for all t. We need one more definition. A convex disk $C \subset \mathbb{R}^3$ *fits into* H if H contains a congruent copy of C. It is clear that if C fits into H, then the diameter, width, area of C is at most as large as that of H.

We will use two easy facts (Lemma 2 and Lemma 3 below) from elementary plane geometry. Let R be the first quadrant of P. For positive reals p, q and ε, let $D_\varepsilon(p,q)$ be the ε-disk centered at (p,q), that is, $D_\varepsilon(p,q) = \{(x,y) : (x-p)^2 + (y-q)^2 < \varepsilon^2\}$.

Lemma 2. *Let $\varepsilon > 0$ and $p_1, q_1 > 2\varepsilon$. Then, for all $(x_1, y_1) \in D_\varepsilon(p_1, q_1) \cap R$, the maximum*

$$\max\{(x_1 - x)^2 + (y_1 - y)^2 : (x,y) \in D_\varepsilon(0,0) \cap R\}$$

is attained only at $(x,y) = (0,0)$.

In other words, the origin is the unique farthest point in $D_\varepsilon(0,0) \cap R$ from any point in $D_\varepsilon(p_1, q_1) \cap R$, which easily follows from the positions of $(x,y), (x_1,y_1)$ and $(0,0)$.

For $a, b, c \in \mathbb{R}^3$, we write $[a, b]$ for the line segment from a to b, and $\mathrm{dist}(c, [a, b])$ for the distance from c to $[a, b]$.

Lemma 3. *Let $a = (\alpha, 0, 0), b = (\beta, 0, 0)$ and $c = (\gamma, h, 0)$, where $h > 0$. Suppose that the triangle abc has a unique longest side $[a, b]$. Then,*

$$L(c) := \{(x, y, 0) : 0 \leq y < h, x \in \mathbb{R}\} \subset P$$

cannot contain a congruent copy of $\triangle abc$.

Proof. The width of $\triangle abc$, that is, the shortest height of the triangle, is $\mathrm{dist}(c, [a, b]) = h$. So, the result follows. □

We also need a stronger version of Lemma 1, namely, the embedding obtained in Lemma 1 is continuous in the sense described below. For an isometry f and a compact set C, let $\|f\|_C := \max_{z \in C} |f(z) - z|$.

Lemma 4. *Let $\Omega \subset P$ and $\tilde{\Omega}$ be as in Lemma 1. Then, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any rigid motion M_t with $M_1(\tilde{\Omega}) \subset P$ and $\|M_1\|_\Omega < \delta$, one can find an isometry g on P with $g(\Omega) \subset M_1(\tilde{\Omega})$ and $\|g\|_\Omega < \varepsilon$.*

This is an easy consequence of a result from [4]. For convenience we include a sketch of the proof here.

Proof. By choosing a suitable coordinate system on P, we may assume that there exist a $\lambda \geq 1$ and a map $p_\lambda : (x, y) \mapsto (x, \lambda y)$ with $p_\lambda(\Omega) = \tilde{\Omega}'$, where $\tilde{\Omega}' \subset P$ is a congruent copy of $\tilde{\Omega}$. It is proved in [4] that there are two points $E, F \in \partial\Omega$ with the following property:

> Let $E' = p_\lambda(E)$ and $F' = p_\lambda(F)$ be points on $\partial\tilde{\Omega}'$. Choose F'' on the line segment $[E', F']$ so that $|E' - F''| = |E - F|$. Let h be the rotation preserving isometry on P sending E and F to E' and F'', respectively. Then, $h(\Omega) \subset \tilde{\Omega}'$.

Let N_t be a rigid motion with $N_1(\tilde{\Omega}) = \tilde{\Omega}'$. Then $g := M_1 \circ N_1^{-1} \circ h$ is the desired isometry. Indeed, $g(\Omega) \subset M_1(\tilde{\Omega})$ follows from the construction. If $\|M_1\|_\Omega$ is small, then we see that $\|N_1\|_\Omega$, $\lambda - 1$, and $\|h\|_\Omega$ are small as well. In fact, by choosing δ sufficiently small, we can guarantee that $\|M_1\|_\Omega < \delta$ implies $\max\{\|M\|_\Omega, \|N_1\|_\Omega, \|h\|_\Omega\} < \varepsilon/3$. So it follows that $\|g\|_\Omega \leq \|M_1\|_\Omega + \|N_1\|_\Omega + \|h\|_\Omega < \varepsilon$. □

Proof of Theorem 4. Let $H \subset P$ be a non-triangular convex disk. We construct a tetrahedron T fixed by H. Let $f(x, y) = |x - y|$ be the *distance* function, restricted to $(x, y) \in H \times H$.

Case 1. *There is a local maximum of f at (a, b) such that the open segment $(a, b) \subset \mathrm{int}\, H$.*

We may assume that $|a - b| = 1$. So let $a = (0, 0, 0)$ and $b = (1, 0, 0)$. Choose two points $c = (c_x, c_y, 0)$ and $d = (d_x, d_y, 0)$ on ∂H in the opposite side with respect to the x-axis, that is, $c_y d_y < 0$. Let $Q := \mathrm{conv}\{a, c, b, d\} \subset H$ be the convex hull of $\{a, b, c, d\}$. We construct a tetrahedron T fixed by H so that $Q = T \cap P$.

Choose a point A on the z-axis. If the lines ad and bc intersect, then let ℓ be a line passing through the intersection and A, else if $ad \parallel bc$, then let ℓ be a line passing through

A and parallel to ad. Let B be the intersection of the line ℓ and the plane $x = |a-b| = 1$. Let D be the intersection of the lines Bd and Aa. Since two lines Ac and Bb intersects by Desargues's theorem, let C be the intersection. Then,

$$ab \perp AD,\ ab \perp BC, \text{ and two lines } AD \text{ and } BC \text{ are skew,} \tag{2}$$

see Figure 3.

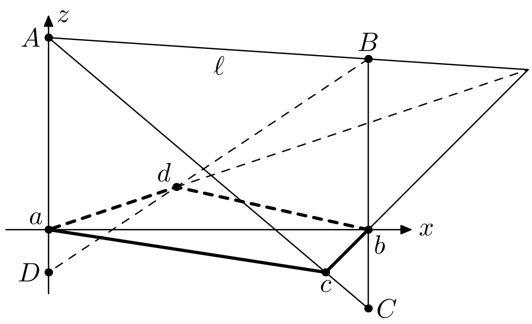

Figure 3: Case 1. $(a, b) \subset \operatorname{int} H$

Let $T = ABCD$ be our tetrahedron. Now let us verify that the four vertices a, c, b, d are all on the edges of T. To see this, it is enough to check that A and B sit in the same half-space according to the plane P, while C and D are in the other half-space. By direct computation, this is equivalent to the condition that x-coordinates of c and d are in $(0, 1)$, and y-coordinates of c and d have opposite signs. In fact, this property is equivalent to our assumption that (a, b) is a local maximum of f. Consequently, we have $Q = T \cap P$.

We fix the tetrahedron T and we try to move the frame ∂H. If we can move the frame within P only, then, by definition, T is fixed by H. Now suppose that we can move the frame slightly and it is on the plane $\tilde{P} \neq P$. More precisely, we consider a rigid motion M_t such that $T \cap (M_t^{-1}P) \subset M_t^{-1}H$ for all $t \in [0, 1]$ and $M_1^{-1}P = \tilde{P}$. Then, by (2), we have $M_t a = a$ and $M_t b = b$ for all t. So M_t is a rotation around the line ab, and thus $P \cap \tilde{P}$ coincides with the line ab.

Let $\tilde{Q} = \operatorname{conv}\{a, b, \tilde{c}, \tilde{d}\}$ be the section of our tetrahedron by the plane $\tilde{P}$, where $\tilde{c}$ (resp. $\tilde{d}$) is on the edge $[A, C]$, (resp. $[B, D]$), and let $Q' = \operatorname{conv}\{a, b, c', d'\} \subset P$ be the projection of $\tilde{Q}$ to P. Then, c' is on the line ac, because $\tilde{c}$ is on $[A, C]$. On the other hand, $\tilde{c}$ is obtained by rotating c around the line ab, and so c' is an interior point of $\triangle abc$. This contradiction completes the proof of Case 1.

Next we assume that we are not in Case 1, that is, if f has a local maximum at $(a, b) \in H \times H$, then the open segment (a, b) is on the boundary of H. Let $a, b \in H$ and suppose that $[a, b]$ is a diameter of H. Then $[a, b] \subset \partial H$, otherwise we are in Case 1. We may assume that H is contained in the first quadrant of P and $|a - b| = 1$. So put $a = (0, 0, 0)$ and $b = (1, 0, 0)$ on the x-axis. Define a distance function from b by $f_b(x) = |x - b|$ for $x \in H_0 := \partial H \setminus (a, b)$. Then, $f_b(x)$ is monotone increasing as x moves from b to a along H_0. To see this, suppose, to the contrary, that there is $c \in H_0$ such that f_b has a local maximum at $c \in H_0$. Then $[b, c] \subset \partial H$. Since H is not a triangle, we have $(a, c) \subset \operatorname{int} H$. But, by Lemma 2, f has a local maximum at (a, c). This means that we are in Case 1, a contradiction. So f_b is monotone, and similarly $f_a(x) := |x - a|$ for $x \in H_0$ is also monotone.

Case 2. *There is a diameter $[a, b] \subset \partial H$ of H, and f_a is monotone.*

We will choose $c, d \in H_0$, and a_i, b_i, c_i, d_i $(i = 1, 2)$ from P, see Figure 4. We start with the following construction.

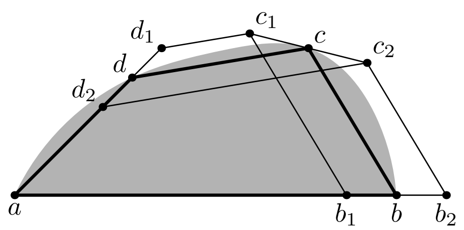

Figure 4: Case 2. $[a, b] = \operatorname{diam} H$

Lemma 5. *There are points $c, d \in H_0$ and $d_1, c_1, c_2 \in P$ such that c is the midpoint of $[c_1, c_2]$ and $[c_1, c_2] \cap H = \{c\}$, $[d_1, c_1]$ is parallel with $[d, c]$, $[a, d_1] \cap H = [a, d]$, $\operatorname{dist}(c, [a, b]) \geq \operatorname{dist}(d, [a, b])$, and the line c_1c_2 intersects the line ab at z with $b \in [a, z]$.*

Proof. Let v be the farthest point of H from $[a, b]$. Suppose $[b, v] \subset \partial H$. Then v would do for c, we just let $z = 2b - a$ and choose a suitable pair of point c_1, c_2 on the line cz. We find d above the chord $[a, v]$ as follows. Let ℓ be the line parallel with $[a, v]$ and supporting H between a and v. As H is not a triangle, $(a, v) \subset \operatorname{int} H$, and so ℓ is disjoint from the chord $[a, v]$. Let d be the point in $\ell \cap H$ closest to $[a, b]$. The position of d_1 on the line ad is determined by the condition that $[c_1, d_1]$ parallel with $[d, c]$.

If both $(a, v), (b, v) \subset \operatorname{int} H$, then let d be the same point as before. We find c above the chord $[b, v]$ just as d was found above $[a, v]$. We assume (by swapping H with its mirror image if necessary), that $\operatorname{dist}(c, [a, b]) \geq \operatorname{dist}(d, [a, b])$. It is clear that there is a supporting line ℓ_c to H with $H \cap \ell_c = \{c\}$, and that ℓ_c intersects the line ab at a point z with $b \in [a, z]$. We can choose the points c_1, c_2 on ℓ_c satisfying all the conditions, and then find d_1 on the line ad such that $[c_1, d_1]$ parallel with $[d, c]$. □

Here the segment $[c_1, c_2]$ can be chosen as small as needed. For $i = 1, 2$, choose b_i on the line ab so that b_ic_i is parallel to bc, and choose d_2 on the line ad so that c_2d_2 is parallel to cd. By choosing $[c_1, c_2]$ sufficiently short we can make sure that d_2 lies in the interior of the segment $[a, d]$. Let $a_1 = a_2 = a$. Set $Q_i = \operatorname{conv}\{a_i, b_i, c_i, d_i\}$ for $i = 1, 2$. Let e be the unit (upward) normal vector of the plane P. Let T be the tetrahedron delimited by the planes $\operatorname{aff}\{a, b, a_1 + e\}$, $\operatorname{aff}\{b, c, b_1 + e\}$, $\operatorname{aff}\{c, d, c_1 + e\}$, and $\operatorname{aff}\{d, a, d_1 + e\}$. By the construction, we have

$$
\begin{aligned}
T \cap P &= Q = \operatorname{conv}\{a, b, c, d\}, \\
T \cap (P + e) &= Q_1 + e = \operatorname{conv}\{a_1 + e, b_1 + e, c_1 + e, d_1 + e\}, \\
T \cap (P - e) &= Q_2 - e = \operatorname{conv}\{a_2 - e, b_2 - e, c_2 - e, d_2 - e\}.
\end{aligned}
$$

We fix the tetrahedron T and we try to move the frame ∂H. Suppose that we can move the frame slightly and it is on the plane $\tilde{P}$. Namely, we consider a rigid motion M_t such that $T \cap (M_t^{-1}P) \subset M_t^{-1}H$ for all $t \in [0,1]$ and $M_1^{-1}P = \tilde{P}$. Our goal is to show that M_t is the identity, which means T is fixed by H. The plane $\tilde{P}$ intersects the edge $[a_1 + e, a_2 - e]$ in the point $\tilde{a}$. Define $\tilde{b}, \tilde{c}$ and $\tilde{d}$ similarly. By the construction, we have $T \cap P = Q \subset H$, and $\tilde{Q} := T \cap \tilde{P} = \mathrm{conv}\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\} \subset M_1^{-1}(H)$ fits into H. Let a' denote the orthogonal projection of $\tilde{a}$ onto the plane P. Define b', c' and d' similarly. Notice that $a' = a$, $b' \in [b_1, b_2]$, $c' \in [c_1, c_2]$, $d' \in [d_1, d_2]$.

Choose $\varepsilon > 0$ so that $6\varepsilon < \min\{c_x, c_y\}$, where $c = (c_x, c_y, 0)$. (We will need this to apply Lemma 2 later.) We plug this ε into Lemma 4 to get δ. Assume that Q and $\tilde{Q}$ differ only slightly. More precisely, we assume that

$$|\tilde{c} - c| < \varepsilon/3, \text{ and } \|M_1\|_H < \delta/3 < \varepsilon/3.$$

By Lemma 1, $a'b'c'd'$ also fits into H, and moreover, by Lemma 4, we can find an embedding close to the original position, that is, there is an isometry $g : P \to P$ satisfying $a''b''c''d'' := g(a'b'c'd') \subset H$ and $\|g\|_H < \varepsilon/3$. Then we have $|c'' - c'| = |g(c') - c'| \leq \|g\|_H < \varepsilon/3$, $|c' - \tilde{c}| \leq \|M_1\|_H < \varepsilon/3$, and $|\tilde{c} - c| < \varepsilon/3$. Thus we get $|c'' - c| \leq |c'' - c'| + |c' - \tilde{c}| + |\tilde{c} - c| < \varepsilon$. Similarly, we get $|M_1\tilde{c} - c| \leq |M_1\tilde{c} - \tilde{c}| + |\tilde{c} - c| \leq \|M_1\|_H + \varepsilon/3 < 2\varepsilon/3$. In summary, we have

$$\{c'', M_1\tilde{c}\} \subset D_\varepsilon(c). \tag{3}$$

Since $c'' \in D_\varepsilon(c)$ by (3), we can apply Lemma 2 to get

$$|c'' - a''| \leq |c'' - a'|.$$

By Lemma 3, $\triangle a'b'c'$ does not fit into $L(c')$. The same is true for $\triangle a''b''c''(\equiv \triangle a'b'c')$. So we have $c'' \in H \setminus L(c')$. Let c'_H (resp. c''_H) be the intersection of ∂H and the line ac' (resp. ac''), see Figure 5.

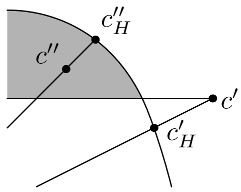

Figure 5: $c'_H, c''_H \in \partial H$

Since $c'' \in H \setminus L(c')$, using the monotonicity of f_a, we have

$$|c''_H - a'| \leq |c'_H - a'|.$$

Therefore we have

$$|c'' - a''| \leq |c'' - a'| \leq |c''_H - a'| \leq |c'_H - a'| \leq |c' - a'| = |c'' - a''|,$$

and thus $|c'' - a''| = |c'' - a'| = |c' - a'|$. Then, by Lemma 2, $|c'' - a''| = |c'' - a'|$ gives $(a =)\ a' = a''$. Also $c'' \in H \setminus L(c')$ and $|c'' - a'| = |c' - a'|$ give $c' = c''$, which is only possible if $c' = c'' = c = \tilde{c}$.

We will show that $a = \tilde{a}$. Observe that $M_1(\tilde{Q}) \subset H$ and

$$\mathrm{dist}(M_1\tilde{c}, M_1[\tilde{a}, \tilde{b}]) = \mathrm{dist}(\tilde{c}, [\tilde{a}, \tilde{b}]) = \mathrm{dist}(c, [\tilde{a}, \tilde{b}]) \geq \mathrm{dist}(c, [a, b]),$$

where the last inequality follows from the fact that $[\tilde{a}, \tilde{b}]$ is contained in the plane $y = 0$, namely, the plane whose distance to c equals $\mathrm{dist}(c, [a, b])$. So, by Lemma 3, the triangle $M_1(\triangle\tilde{a}\tilde{b}\tilde{c})$ does not fit into $L(c)$, and thus $M_1\tilde{c} \in H \setminus L(c)$. Then we have

$$|M_1\tilde{a} - M_1\tilde{c}| \leq |a - M_1\tilde{c}| \leq |a - c|,$$

where we use $M_1\tilde{c} \in D_\varepsilon(c)$ from (3) to apply Lemma 2 for the first inequality, and we use the monotonicity of f_a for the second inequality. On the other hand $|M_1\tilde{a} - M_1\tilde{c}| = |\tilde{a} - \tilde{c}| = |\tilde{a} - c| \geq |a - c|$ where the last inequality follows from the construction. Thus $|M_1\tilde{a} - M_1\tilde{c}| = |\tilde{a} - c| = |a - c|$ and then $\tilde{a} = a$ follows.

Now it follows from $\tilde{a} = a$ and $\tilde{c} = c$ that M_t is a rotation around the line ac. Thus $\tilde{b}$ is obtained by rotating b around ac. In this case, $b \neq \tilde{b}$ is impossible because $bb' \not\perp ac$. Therefore we have $\tilde{a} = a$, $\tilde{b} = b$ and $\tilde{c} = c$. Thus $\tilde{P} = P$ and M_t is the identity. This completes the proof of Case 2 and also of the theorem. □

Similarly to the proof of Theorem 4, one can show the following: for every convex quadrilateral $H \subset P$, there is a tetrahedron T such that T is fixed by H and $H = T \cap P$. Conversely, if we are given a tetrahedron first, then can we find such a quadrilateral frame?

Problem 2. *Let T be a tetrahedron. Is it true that there is a plane P such that $H := T \cap P$ fixes T?*

Acknowledgement

We thank Tudor Zamfirescu and Jin-ichi Itoh for stimulating our interest in the topic discussed in this paper. We also thank the referees for helpful comments and suggestions which have improved the presentation of this paper. The first author was partially supported by Hungarian National Foundation Grant no NK 78439. The second and the last authors were supported by KAKENHI 20340022.

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