INTERSECTING FAMILIES — UNIFORM VERSUS WEIGHTED

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ABSTRACT. What is the maximal size of k-uniform r-wise t-intersecting families? We show that this problem is essentially equivalent to determine the maximal weight of non-uniform r-wise t-intersecting families. Some EKR type examples and their applications are included.

1. INTRODUCTION

Throughout this paper let n,k,r,t denote positive integers with $t \le k \le n$, and let p and q denote positive reals with $p+q=1$. A family $\mathscr{G} \subset 2^{[n]}$ is called r-wise t-intersecting if $|G_1 \cap \cdots \cap G_r| \ge t$ holds for all $G_1,\ldots,G_r \in \mathscr{G}$. Let us define n-vertex k-uniform r-wise t-intersecting family $\mathscr{F}_i(n,k,r,t)$ as follows:

$$\mathscr{F}_i(n,k,r,t) = \{F \in \binom{[n]}{k} : |F \cap [t+ri]| \ge t+(r-1)i\}.$$

Let $m(n,k,r,t)$ be the maximal size of k-uniform r-wise t-intersecting families on n vertices. Can we extend the Erdős–Ko–Rado Theorem [4] in the following way?

Conjecture 1. $m(n,k,r,t) = \max_i |\mathscr{F}_i(n,k,r,t)|$.

The p-weight of a family $\mathscr{G} \subset 2^{[n]}$, denoted by $w_p(\mathscr{G})$, is defined as follows:

$$w_p(\mathscr{G}) = \sum_{G \in \mathscr{G}} p^{|G|} q^{n-|G|} = \sum_{i=0}^{n} \left| \mathscr{G} \cap \binom{[n]}{i} \right| p^i q^{n-i}.$$

Let $w(n,p,r,t)$ be the maximal p-weight of r-wise t-intersecting families on n vertices. Set $\mathscr{G}_i(n,r,t) = \bigcup_{k=0}^{n} \mathscr{F}_i(n,k,r,t)$.

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Conjecture 2. $w(n,p,r,t) = \max_i w_p(\mathcal{G}_i(n,r,t))$.

The aim of this paper is to show that roughly speaking $w(n,p,r,t)$ and $m(n,k,r,t)/\binom{[n]}{k}$ are almost the same if $p \approx \frac{k}{n}$. Therefore the above two problems ask essentially the same thing. We list some known results about the conjectures and related problems in the last two sections.

Our first result says that $w(n,p,r,t)$ can be deduced from $m(n,k,r,t)$ if $\frac{k}{n} \approx p$.

Theorem 1. *Let r,t and p be given. Then (M1) implies (W1).*

(M1) There exist $\varepsilon > 0$ and n_0 such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for all $n > n_0$ and k with $|\frac{k}{n} - p| < \varepsilon$.
(W1) $w(n,p,r,t) = p^t$ holds for all $n \geq t$.

One can slightly generalize the above result as follows.

Theorem 2. *Let r,t,p and c be given. Then (M2) implies (W2).*

(M2) For all $\mu > 0$ there exists $\varepsilon > 0$ such that $m(n,k,r,t) < (c+\mu)\binom{n}{k}$ holds for all $n > n_0(\mu,\varepsilon)$ and k with $|\frac{k}{n} - p| < \varepsilon$.
(W2) $w(n,p,r,t) \leq c$ holds for all $n \geq t$.

Moreover if there is an r-wise t-intersecting family $\mathcal{G} \subset 2^{[n_0]}$ with $w_p(\mathcal{G}) = c$ for some n_0 then $w(n,p,r,t) = c$ holds for all $n \geq n_0$.

Assume (M2). We can choose $\delta, \varepsilon' > 0$ sufficiently small so that

$$\text{if } |p - p'| < \delta \text{ and } |\tfrac{k}{n} - p'| < \varepsilon', \text{ then } |\tfrac{k}{n} - p| < \varepsilon.$$

Then by (M2) we have $m(n,k,r,t) \leq (c+\mu)\binom{n}{k}$ for all n,k with $n > n_0(\mu,\varepsilon')$ and $|\frac{k}{n} - p'| < \varepsilon'$. Thus by (W2) we have $w(n,p',r,t) \leq c$ for all $n \geq t$. This means that we can replace (W2) by

(W2′) There exists $\delta > 0$ such that $w(n,p',r,t) \leq c$ holds for all $n \geq t$ and p' with $|p - p'| < \delta$.

The next results are the reverses of Theorem 1 and Theorem 2, which say that $m(n,k,r,t)$ can be deduced from $w(n,p,r,t)$ if $\frac{k}{n} \approx p$.

Theorem 3. *Let r,t and p be given. Then (W3) implies (M3).*

(W3) $\lim_{n\to\infty} w(n,p,r,t) = p^t$.
(M3) For all $\mu > 0$ and all $0 < \varepsilon < p$ there exists n_0 such that $m(n,k,r,t) < (1+\mu)\binom{n-t}{k-t}$ holds for all $n > n_0$ and k with $\frac{k}{n} < p - \varepsilon$.

Theorem 4. *Let r,t,p and c be given. Then (W4) implies (M4).*

(W4) $\lim_{n\to\infty} w(n,p,r,t) \leq c$.

(M4) For all $\mu > 0$ *and all* $0 < \varepsilon < p$ *there exists* n_0 *such that* $m(n,k,r,t) < (c+\mu)\binom{n}{k}$ *holds for all* $n > n_0$ *and* k *with* $\frac{k}{n} < p - \varepsilon$.

To extend the above result let us introduce non-trivial versions of m and w. An r-wise t-intersecting family $\mathscr{G} \subset 2^{[n]}$ is called non-trivial if $|\bigcap_{G\in\mathscr{G}} G| < t$. Let $m^*(n,k,r,t)$ be the maximal size of k-uniform non-trivial r-wise t-intersecting families on n vertices, and let $w^*(n,p,r,t)$ be the maximal p-weight of non-trivial r-wise t-intersecting families on n vertices.

Theorem 5. *Let* r,t *and* p *be given. Then (W5) implies (M5).*

(W5) There exists $\gamma > 0$ *such that* $\lim_{n\to\infty} w^*(n,p,r,t) < (1-\gamma)p^t$.

(M5) For all $\varepsilon > 0$ *and all* $0 < \eta < \gamma$ *there is* n_0 *such that* $m^*(n,k,r,t) < (1-\eta)\binom{n-t}{k-t}$ *holds for all* $n > n_0$ *and* k *with* $\frac{k}{n} < p - \varepsilon$.

Note that $(M5)$ implies that $m(n,k,r,t) = \binom{n-t}{k-t}$. It would be nice to have the reverse of the above result.

Problem 1. *Let* r,t *and* p *be given. Then does (M6) imply (W6)?*

(M6) There exist $\eta > 0, \varepsilon > 0$ *and* n_0 *such that* $m^*(n,k,r,t) < (1-\eta)\binom{n-t}{k-t}$ *holds for all* $n > n_0$ *and* k *with* $|\frac{k}{n} - p| < \varepsilon$.

(W6) There exists $\gamma > 0$ *such that* $\lim_{n\to\infty} w^*(n,p,r,t) < (1-\gamma)p^t$.

Clearly (M6) implies that $m(n,k,r,t) = \binom{n-t}{k-t}$ for all $n > n_0$ and k with $|\frac{k}{n} - p| < \varepsilon$. On the other hand, by Theorem 5 we know that (W6) implies $m(n,k,r,t) = \binom{n-t}{k-t}$ for all $n > n_1$ and k with $\frac{k}{n} < p - \varepsilon$. Thus if the answer to the problem is affirmative, then (M6) implies $m(n,k,r,t) = \binom{n-t}{k-t}$ for all $n > n_2$ and k with $\frac{k}{n} < p + \varepsilon$ (not only $|\frac{k}{n} - p| < \varepsilon$).

2. Proofs

We will use the following lemma.

Lemma 1. *Let* $t \in \mathbb{N}$ *and* $\varepsilon, p \in \mathbb{R}$ *be fixed positive constants with* $\varepsilon < p$. *For* $n \in \mathbb{N}$ *set*

$$S_n = \sum_{k\in I} \binom{n-t}{k-t} p^k q^{n-k},$$

where $I = ((p-\varepsilon)n, (p+\varepsilon)n) \cap \mathbb{N}$. *Then we have* $\lim_{n\to\infty} S_n = p^t$.

Proof. The upper bound follows from

$$S_n \le \sum_{k\ge t} \binom{n-t}{k-t} p^k q^{n-k} = p^t \sum_{\ell=0}^{n-t} \binom{n-t}{\ell} p^\ell q^{(n-t)-\ell} = p^t.$$

Next let $0 < \mu < p^t$ be given. For the lower bound, we will show that $S_n > p^t - \mu$ for n sufficiently large. Choose a constant c sufficiently large so that

$$\frac{1}{\sqrt{2\pi}}\int_{-c}^{c}\exp(-\frac{z^2}{2})dz > 1-\mu p^{-t},$$

and set $J = (pn - c\sqrt{n}, pn + c\sqrt{n}) \cap \mathbb{N}$. Then we have $J \subset I$ for $n > n_0(\varepsilon, c)$, and

$$S_n \geq \sum_{k\in J}\binom{n-t}{k-t}p^k q^{n-k} \geq \sum_{k\in J}(\frac{k-t+1}{n-t+1})^t\binom{n}{k}p^k q^{n-k}.$$

Now we note that $\lim_{n\to\infty}(k-t+1)/(n-t+1) = p$ for $k \in J$, and

$$\lim_{n\to\infty}\sum_{k\in J}\binom{n}{k}p^k q^{n-k} = \frac{1}{\sqrt{2\pi}}\int_{-c}^{c}\exp(-\frac{z^2}{2})dz,$$

which is the de Moivre–Laplace limit theorem. Thus we have

$$\lim_{n\to\infty} S_n > p^t(1-\mu p^{-t}) = p^t - \mu.$$

□

By setting $t = 0$, it follows from the lemma that

$$\sum_{k\notin I}\binom{n}{k}p^k q^{n-k} = o(1).$$

For a family $\mathscr{G} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓ-th shadow of $\mathscr{G}$, denoted by $\Delta_\ell(\mathscr{G})$, as follows.

$$\Delta_\ell(\mathscr{G}) = \{F \in \binom{[n]}{\ell} : F \subset \exists G \in \mathscr{G}\}.$$

The complement family $\mathscr{G}^c$ is defined by $\mathscr{G}^c = \{[n] - G : G \in \mathscr{G}\}$.

Proof of Theorem 1. Assume (M1). Since $w(n,p,r,t) \geq w_p(\mathscr{G}_0(n,r,t)) = p^t$ it suffices to show $w(n,p,r,t) \leq p^t$. Set an interval $I = ((p-\varepsilon)n, (p+\varepsilon)n) \cap \mathbb{N}$. Let $\mathscr{G} \subset 2^{[n]}$ be an r-wise t-intersecting family with $w(n,p,r,t) = w_p(\mathscr{G})$. Using the lemma and (M1), we have

$$\begin{aligned} w(n,p,r,t) &\leq \sum_{k\in I}\left|\mathscr{G}\cap\binom{[n]}{k}\right|p^k q^{n-k} + \sum_{k\notin I}\binom{n}{k}p^k q^{n-k} \\ &\leq \sum_{k\in I}\binom{n-t}{k-t}p^k q^{n-k} + o(1) \qquad (1) \\ &= p^t + o(1). \end{aligned}$$

This proves $\lim_{n\to\infty} w(n,p,r,t) \leq p^t$.

Next define $\mathscr{G}' \subset 2^{[n+1]}$ by $\mathscr{G}' = \mathscr{G} \cup \{G \cup \{n+1\} : G \in \mathscr{G}\}$, which is r-wise t-intersecting, too. Then $w_p(\mathscr{G}') = w_p(\mathscr{G})(q+p) = w(n,p,r,t)$, which means

$$w(n+1,p,r,t) \geq w(n,p,r,t). \tag{2}$$

Consequently we have $w(n,p,r,t) = p^t$ for all $n \geq t$. □

Proof of Theorem 2. This is similar to the proof of Theorem 1. In this case (1) is replaced by

$$w(n,p,r,t) \leq \sum_{k \in I} (c+\mu) \binom{n}{k} p^k q^{n-k} + o(1),$$

which implies $\lim_{n\to\infty} w(n,p,r,t) \leq c+\mu$. Since $\mu > 0$ is arbitrary we have $w(n,p,r,t) \leq c$ using (2). □

Proof of Theorem 3. Assume (W3). Let $0 < \mu < q$ be given. We want to show that $m(n,k,r,t) < (1+\mu)\binom{n-t}{k-t}$. Suppose, on the contrary, that there exists $0 < \varepsilon < \min\{p,q\}$ such that for each n_0 we can find an r-wise t-intersecting family $\mathscr{F} \subset \binom{[n]}{k}$ which satisfies $|\mathscr{F}| \geq (1+\mu)\binom{n-t}{k-t}$ for some n,k with $n > n_0$ and $\frac{k}{n} = p - \varepsilon$. Let $\mathscr{G} = \{G : G \supset \exists F \in \mathscr{F}\} = \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathscr{F}^c))^c$. This family is also r-wise t-intersecting (but not necessarily uniform). We will show that $\mathscr{G}$ violates (W3). Set an interval $I = ((p-\varepsilon)n, (p+\varepsilon)n) \cap \mathbb{N}$ and set $\alpha = \frac{q-\varepsilon}{q+\varepsilon} > 0$.

Claim 1. *$|\Delta_{n-i}(\mathscr{F}^c)| \geq (1+\alpha\mu)\binom{n-t}{n-i}$ for $i \in I$.*

Proof. Choose a real x so that $\mu\binom{n-t}{k-t} = \binom{x}{n-k-1}$. Since $\mu < q$ we have $x < n-t-1$. In fact if $x \geq n-t-1$ then we have $\mu \geq \binom{n-t-1}{n-k-1} / \binom{n-t}{k-t} = \frac{1-(k/n)}{1-(t/n)} > 1 - \frac{k}{n} = 1-(p-\varepsilon) = q+\varepsilon > q$.

Since $|\mathscr{F}^c| = |\mathscr{F}| \geq (1+\mu)\binom{n-t}{k-t} = \binom{n-t}{n-k} + \binom{x}{n-k-1}$, the Kruskal–Katona Theorem [18, 17] implies that $|\Delta_{n-i}(\mathscr{F}^c)| \geq \binom{n-t}{n-i} + \binom{x}{n-i-1}$. Thus it suffices to show that $\binom{x}{n-i-1} \geq \alpha\mu\binom{n-t}{n-i}$, or equivalently,

$$\frac{\binom{x}{n-i-1}}{\binom{x}{n-k-1}} \geq \frac{\alpha\mu\binom{n-t}{n-i}}{\mu\binom{n-t}{k-t}}.$$

Since $i \geq k$ this is equivalent to $\frac{i-t}{x-n+i+1} \cdots \frac{k-t+1}{x-n+k+2} \geq \alpha\,\frac{n-k}{n-i}$. The LHS is at least 1, in fact, $\frac{i-t}{x-n+i+1} > 1$ follows from $x < n-t-1$. On the other hand, using $i \leq (p+\varepsilon)n$ we have RHS $= \alpha\,\frac{1-(k/n)}{1-(i/n)} \leq \alpha\,\frac{1-(p-\varepsilon)}{1-(p+\varepsilon)} = \alpha\,\frac{q+\varepsilon}{q-\varepsilon} = 1$, which proves the claim. □

Let us finish the proof of Theorem 3. Using the claim and the lemma, we have

$$\begin{aligned} w_p(\mathscr{G}) &> \sum_{i\in I}\left|\mathscr{G}\cap\binom{[n]}{i}\right|p^iq^{n-i} \\ &= \sum_{i\in I}|\Delta_{n-i}(\mathscr{F}^c)|p^iq^{n-i} \\ &\geq \sum_{i\in I}(1+\alpha\mu)\binom{n-t}{n-i}p^iq^{n-i} \\ &= (1+\alpha\mu)(p^t-o(1)) \\ &> p^t, \end{aligned}$$

which contradicts (M3). □

Proof of Theorem 4. This is similar to the proof of Theorem 3. Suppose that there exists $\varepsilon>0$ such that for all n_0 we can find an r-wise t-intersecting family $\mathscr{F}\subset\binom{[n]}{k}$ which satisfies $|\mathscr{F}|\geq(c+\mu)\binom{n}{k}$ for $n>n_0$ and $\frac{k}{n}=p-\varepsilon$. Let $\mathscr{G}=\{G:G\supset\exists F\in\mathscr{F}\}=\bigcup_{\ell=0}^{n-k}(\Delta_\ell(\mathscr{F}^c))^c$ and $I=((p-\varepsilon)n,(p+\varepsilon)n)$.

Claim 2. $|\Delta_{n-i}(\mathscr{F}^c)|\geq(c+\mu)\binom{n}{i}$ *for* $i\in I$.

Proof. Choose a real $x\leq n$ so that $(c+\mu)\binom{n}{k}=\binom{x}{n-k}$. Since $|\mathscr{F}^c|=|\mathscr{F}|\geq\binom{x}{n-k}$ the Kruskal–Katona Theorem implies that $|\Delta_{n-i}(\mathscr{F}^c)|\geq\binom{x}{n-i}$. Thus it suffices to show that $\binom{x}{n-i}\geq(c+\mu)\binom{n}{i}$, or equivalently,

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}}\geq\frac{(c+\mu)\binom{n}{i}}{(c+\mu)\binom{n}{k}}.$$

Using $i\geq k$ this is equivalent to $i\cdots(i-k+1)\geq(x-n+i)\cdots(x-n+k+1)$, which follows from $x\leq n$. □

Using the claim we have

$$\begin{aligned} w_p(\mathscr{G}) &\geq \sum_{i\in I}|\Delta_{n-i}(\mathscr{F}^c)|p^iq^{n-i}\geq\sum_{i\in I}(c+\mu)\binom{n}{i}p^iq^{n-i} \\ &= (c+\mu)(1-o(1))>c, \end{aligned}$$

which contradicts (M4). □

Proof of Theorem 5. The proof is almost identical to the proof of Theorem 3. The only difference is that instead of Claim 1 we use the following fact here:

If $|\mathscr{F}|\geq(1-\eta)\binom{n-t}{k-t}$ *then* $|\Delta_{n-i}(\mathscr{F}^c)|\geq(1-\eta)\binom{n-t}{n-i}$ *holds for* $i\in I$. □

3. EKR TYPE EXAMPLES

We list some known results about $m(n,k,r,t)$ and $w(n,p,r,t)$ in this section.

3.1. **The case $r = 2$.** Ahlswede and Khachatrian settled Conjecture 1 for this case. (The earlier results for the case $n \geq (t+1)(k-t+1)$ can be found in [6, 26].)

Example 1 ([1])**.** *$m(n,k,r=2,t) = \max_i |\mathscr{F}_i(n,k,r=2,t)|$ for $n > 2k-t$.*

Using Example 1 and Theorem 2 we will prove the following result, which confirms Conjecture 2 for the case $r = 2$.

Example 2. *$w(n,p,r=2,t) = \max_i w_p(\mathscr{G}_i(n,r=2,t))$.*

We note that

$$|\mathscr{F}_i(n,k,r=2,t)| \geq |\mathscr{F}_{i-1}(n,k,r=2,t)|$$

iff $\frac{k-t+1}{n} \geq \frac{i}{2i+t-1}$, and

$$m(n,k,r=2,t) = |\mathscr{F}_i(n,k,r=2,t)| = \sum_{j=t+i}^{t+2i} \binom{t+2i}{j}\binom{n-t-2i}{k-j} \tag{3}$$

for $\frac{i}{2i+t-1} \leq \frac{k-t+1}{n} \leq \frac{i+1}{2i+t+1}$ where $i = 0,1,\ldots,k-t$. Similarly we have

$$w_p(\mathscr{G}_i(n,r=2,t)) \geq w_p(\mathscr{G}_{i-1}(n,r=2,t))$$

if $p \geq \frac{i}{2i+t-1}$. One can show this fact by calculating $w_p(\mathscr{G}_i - \mathscr{G}_{i-1}) \geq w_p(\mathscr{G}_{i-1} - \mathscr{G}_i)$. Let $i_{\max} = \lfloor \frac{n-t}{2} \rfloor$. Then we have have

$$w(n,p,r=2,t) = w_p(\mathscr{G}_i(n,r=2,t)) = \sum_{j=t+i}^{t+2i} \binom{t+2i}{j} p^j q^{t+2i-j} \tag{4}$$

for $\frac{i}{2i+t-1} \leq p \leq \frac{i+1}{2i+t+1}$ where $i = 0,1,\ldots,i_{\max}-1$, and

$$w(n,p,r=2,t) = w_p(\mathscr{G}_{i_{\max}}(n,r=2,t))$$

for $p \geq \frac{i_{\max}}{2i_{\max}+t-1} = \frac{n-t-\varepsilon}{2n-2}$ where $\varepsilon = n-t-2i_{\max} \in \{0,1\}$. In particular, for the case $p = 1/2$ we get the Katona Theorem [16], i.e.,

$$w(n,p=1/2,r=2,t) = w_p(\mathscr{G}_{i_{\max}}(n,r=2,t)) \to 1/2 \quad (n \to \infty).$$

On the other hand, for the case $p > 1/2$ we have

$$w(n,p>1/2,r=2,t) = w_p(\mathscr{G}_{i_{\max}}(n,r=2,t)) \to 1 \quad (n \to \infty).$$

The corresponding non-trivial t-intersecting version is as follows.

$$w^*(n,p,r=2,t) = w(n,p,r=2,t) \text{ for } p \geq \tfrac{1}{t+1},$$

$$\lim_{n\to\infty} w^*(n,p,r=2,t) = p^t \text{ for } p \leq \tfrac{1}{t+1}.$$

Proof of Example 2. Write $\mathscr{F}_i = \mathscr{F}_i(n,k,r=2,t)$ and $\mathscr{G}_i = \mathscr{G}_i(n,r=2,t)$. We distinguish two cases. The first case is that p satisfies $\frac{i}{2i+t-1} < p < \frac{i+1}{2i+t+1}$. Let $\mu > 0$ be given and let p be fixed. Then we can choose $\varepsilon = \varepsilon(\mu,p) > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{i}{2i+t-1} < \frac{k-t+1}{n} < \frac{i+1}{2i+t+1}$ and $\binom{n-t-2i}{k-j} < (p^j q^{t+2i-j} + \mu)\binom{n}{k}$ hold for all $n > n_0$ and k with $|\frac{k}{n} - p| < \varepsilon$. Then from (3) we have

$$m(n,k,r=2,t) = |\mathscr{F}_i| < (w_p(\mathscr{G}_i) + \mu)\binom{n}{k}$$

and the desired result (4) follows from Theorem 2.

The next case is that $p = \frac{i}{2i+t-1}$. Set intervals $I_- = ((p-\varepsilon)n, pn+t-1] \cap \mathbb{N}$ and $I_+ = (pn+t-1, (p+\varepsilon)n) \cap \mathbb{N}$. Then we have

$$\begin{aligned} w(n,p,r=2,t) &\leq \sum_{k\in I_-\cup I_+} \left|\mathscr{G} \cap \binom{[n]}{k}\right| p^k q^{n-k} + \sum_{k\notin I_-\cup I_+} \binom{n}{k} p^k q^{n-k}. \\ &\leq \sum_{k\in I_-} |\mathscr{F}_{i-1}| p^k q^{n-k} + \sum_{k\in I_+} |\mathscr{F}_i| p^k q^{n-k} + o(1). \end{aligned}$$

Now set $I'_- = ((p-\varepsilon)n, pn] \cap \mathbb{N}$ and $I'_+ = (pn, (p+\varepsilon)n) \cap \mathbb{N}$. Changing from I_-, I_+ to I'_-, I'_+ only causes $o(1)$ effect and we still have

$$w(n,p,r=2,t) \leq \sum_{k\in I'_-} |\mathscr{F}_{i-1}| p^k q^{n-k} + \sum_{k\in I'_+} |\mathscr{F}_i| p^k q^{n-k} + o(1).$$

Noting that $|\mathscr{F}_i| = w_p(\mathscr{G}_i)\binom{n}{k} + o(1)$, we have

$$w(n,p,r=2,t) \leq \frac{1}{2} w_p(\mathscr{G}_{i-1}) + \frac{1}{2} w_p(\mathscr{G}_i) + o(1).$$

Since $w_p(\mathscr{G}_{i-1}) = w_p(\mathscr{G}_i)$ for $p = \frac{i}{2i+t-1}$ we have

$$w(n,p,r=2,t) \leq w_p(\mathscr{G}_i) + o(1),$$

which actually implies $w(n,p,r=2,t) = w_p(\mathscr{G}_i)$ for all $n \geq t$ by (2). $\square$

3.2. **The case $t = 1$.** In this case both Conjecture 1 and Conjecture 2 are known to be true.

Example 3 ([5, 9]). *We have*

$$m(n,k,r,t=1)/\binom{n}{k} = \tfrac{k}{n} \quad \text{for } p \le \tfrac{r-1}{r},$$
$$\lim_{n\to\infty} m(n,k,r,t=1)/\binom{n}{k} = 1 \quad \text{for } p > \tfrac{r-1}{r}.$$

Example 4 ([9]). *We have*

$$w(n,p,r,t=1) = p \quad \text{for } p \le \tfrac{r-1}{r},$$
$$\lim_{n\to\infty} w(n,p,r,t=1) = 1 \quad \text{for } p > \tfrac{r-1}{r}.$$

Let $\mathscr{G}_1 = \mathscr{G}_1(n,r,t=1) = \{G \subset [n] : |G \cap [r+1]| \ge r\}$. Then this is a non-trivial r-wise 1-intersecting family with $w_p(\mathscr{G}_1) = p^r(r+1-pr)$. Brace and Daykin proved that $\mathscr{G}_1$ is the optimal family if $p = 1/2$.

Example 5 ([2]). $w^*(n,p=1/2,r,t=1) = (\frac{1}{2})^r(\frac{r}{2}+1)$.

We can slightly extend the above result as follows.

Example 6 ([25]). *There exists $\varepsilon > 0$ such that*

$$w^*(n,p,r,t=1) = |w_p(\mathscr{G}_1))| = p^r(r+1-pr)$$

holds for all $n \ge t$, $r \ge 8$ and p with $|p-\frac{1}{2}| < \varepsilon$. Moreover $\mathscr{G}_1$ is the only optimal configuration (up to isomorphism).

The above result fails if $r \le 5$ as follows.

Example 7 ([11]). $\lim_{n\to\infty} w^*(n,p,r=5,t=1) \ge p^3 > p^5(6-5p)$ *holds for* $1/2 < p < \frac{1+\sqrt{21}}{10}$.

Conjecture 3. *There exists $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} w^*(n,p,r,t=1) = p^r(r+1-pr)$$

holds for all $n \ge t$, $r \ge 6$ and $|p-\frac{1}{2}| < \varepsilon$.

Example 8 ([25]). *Let $r \ge 8$. Then there exists $\varepsilon_r > 0$ and n_r such that*

$$m^*(n,k,r,t=1) = |\mathscr{F}_1(n,k,r,t=1)| = (r+1)\binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1}$$

holds for all $n > n_r$ and k with $|\frac{k}{n}-\frac{1}{2}| < \varepsilon_r$. Moreover $\mathscr{F}_1(n,k,r,1)$ is the only optimal configuration (up to isomorphism).

3.3. **The case** $r = 3$. Let $p_t = \frac{2}{\sqrt{4t+9}-1}$. Then we have $w_p(\mathscr{G}_0(n, r = 3, t)) \geq w_p(\mathscr{G}_1(n, r = 3, t))$ iff $p \leq p_t$. If Conjecture 2 is true then we have $w(n, p, r = 3, t) = p^t$ for $p \leq p_t$.

Example 9 ([8]). *$w(n, p, r = 3, t = 2) = p^2$ for $p \leq 0.5018$. Moreover $\mathscr{G}_0(n, r = 3, t = 2)$ is the only optimal configuration (up to isomorphism).*

Comparing $p_2 = (\sqrt{17}+1)/8 \approx 0.64$, the bound for p in the above example seems to be far from best possible. Theorem 3 and Example 9 with some additional argument give the following.

Example 10 ([10]). *$m(n, k, r = 3, t = 2) = \binom{n-2}{k-2}$ for $\frac{k}{n} \leq 0.501$ and $n > n_0$. Moreover $\mathscr{F}_0(n, k, r = 3, t = 2)$ is the only optimal configuration (up to isomorphism).*

For larger t, we can get the sharp bound for k/n and p.

Example 11 ([22]). *$m(n, k, r = 3, t) = \binom{n-t}{k-t}$ for $t \geq 26$, $\frac{k}{n} \leq p_t$ and $n > n_0(t)$. Moreover $\mathscr{F}_0(n, k, r = 3, t)$ is the only optimal configuration (up to isomorphism).*

This together with Theorem 1 implies $w(n, p, r = 3, t) = p^t$ for $t \geq 26$ and $p \leq p_t$.

3.4. **The case** $p \approx 1/2$. Let $T_r = 2^r - r - 1$. Then we have

$$w_{1/2}(\mathscr{G}_0(n, r, t)) \geq w_{1/2}(\mathscr{G}_1(n, r, t))$$

iff $t \leq T_r$. Frankl proved Conjecture 2 for the case $p = 1/2$.

Example 12 ([7]). *$w(n, p = 1/2, r, t) = w_{1/2}(\mathscr{G}_0(n, r, t)) = (1/2)^t$ for $t \leq T_r$.*

Using Theorem 3 we have

$$m(n, k, r, t) = (1 + o(1))\binom{n-t}{k-t}$$

for $t \leq T_r$, $\frac{k}{n} < \frac{1}{2}$ and n sufficiently large. Conjecture 1 suggests that the $o(1)$ term could be removed. In fact this was confirmed for $4 \leq r \leq 10$ and smaller t in [24]. Let us define t_r for $4 \leq r \leq 10$ as in the following table.

r	4	5	6	7	8	9	10
t_r	7	18	41	89	184	377	762
T_r	11	26	57	120	247	502	1013

Example 13 ([24]). *For $4 \le r \le 10$ there exists $\varepsilon > 0$ and $n_0 = n_0(\varepsilon)$ such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for $t \le t_r$, $n > n_0$ and $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$. Moreover there exists $\gamma = \gamma(\varepsilon) > 0$ such that $m^*(n,k,r,t) < (1-\gamma)\binom{n-t}{k-t}$ holds for $n > n_1(\gamma)$.*

Thus it follows from Theorem 1 that for $4 \le r \le 10$ there exists $\varepsilon > 0$ such that $w(n,p,r,t) = p^t$ holds for all $n \ge t$, $t \le t_r$, $|p - \frac{1}{2}| < \varepsilon$.

3.5. General case.

Example 14 ([21]). *We have $m(n,k,r,t) = \binom{n-t}{k-t}$ if $p = \frac{k}{n}$ satisfies $p < \frac{r-2}{r}$,*

$$q\, p^{\frac{t}{t+1}(r-1)} - p^{\frac{t}{t+1}} + p < 0 \tag{5}$$

and $n > n_0(r,t,p)$.

For $d \in \mathbb{N}$ let $f_d(x) = qx^d - x + p$ and let $\alpha_d \in (p,1)$ be the root of the equation $f_d(x) = 0$. Then α_d satisfies the following identity for $s \in \mathbb{C}$:

$$\alpha_d^s = \sum_{j \ge 0} \frac{s}{dj+s} \binom{dj+s}{j} p^{(d-1)j+s} q^j,$$

and if $0 < p \le \frac{1}{2}$ then $p^{d+1} < \alpha_d - p < p^d$ (see [20]). We note that $f_d(x) > 0$ for $0 < x < \alpha_d$ and $f_d(x) < 0$ for $\alpha_d < x < 1$. Thus we have the following equivalent conditions.

$$(5) \iff f_{r-1}(p^{\frac{t}{t+1}}) < 0 \iff \alpha_{r-1} < p^{\frac{t}{t+1}} \iff t \le \lfloor \tfrac{-\log \alpha_{r-1}}{\log \alpha_{r-1} - \log p} \rfloor. \tag{6}$$

Example 14 and Theorem 1 give $w(n,p,r,t) = p^t$ if (5) holds. On the other hand we have $w_p(\mathscr{G}_0(n,r,t)) \ge w_p(\mathscr{G}_1(n,r,t))$ iff

$$(t+r)p^{r-1} - (t+r-1)p^r - 1 \le 0, \tag{7}$$

or equivalently, $t \le \sum_{i=0}^{r-1}(p^{-i} - 1)$. Thus if Conjecture 2 is true then we can replace (5) by (7).

4. Applications

4.1. Intersecting Sperner families.

A family $\mathscr{G} \subset 2^{[n]}$ is called a Sperner family if $G \not\subset G'$ holds for all distinct $G, G' \in \mathscr{G}$. Let $s(n,r,t)$ be the maximal size of r-wise t-intersecting Sperner families on n vertices.

Problem 2. *Determine $s(n,r,t)$.*

Milner settled the case $r = 2$.

Example 15 ([19]). $s(n, r=2, t) = \binom{n}{\lceil \frac{n+t}{2} \rceil}$.

Frankl and Gronau settled the case $r=3$ and $t=1$.

Example 16 ([5, 13, 14]). *$s(n=2\ell,r=3,t=1)=\binom{n-1}{\ell}+1$ for $\ell>\ell_0$ and $s(n=2\ell+1,r=3,t=1)=\binom{n-1}{\ell}$ for $\ell>\ell_1$.*

Gronau also settled the case $r\geq 4$ and $t=1$ completely.

Example 17 ([13]). *$s(n,r\geq 4,t=1)=\binom{n-1}{\lceil\frac{n-1}{2}\rceil}$.*

Based on Example 11, the case $r=3$ and $t=2$ was settled for large n as follows.

Example 18 ([10]). *$s(n=2\ell,r=3,t=2)=\binom{n-2}{\ell-1}$ for $\ell>\ell_0$ and $s(n=2\ell+1,r=3,t=2)=\binom{n-2}{\ell}+2$ for $\ell>\ell_1$. Moreover $\mathscr{F}_0(n=2\ell,k=\ell+1,r=3,t=2)$ and $\mathscr{F}_0(n=2\ell+1,k=\ell,r=3,t=2)\cup\{[n]-\{1\}\}\cup\{[n]-\{2\}\}$ are the only optimal configurations (up to isomorphism).*

Problem 3. *Does $s(n,r,t)=\binom{n-t}{\lceil\frac{n-t}{2}\rceil}$ hold for $r\geq 4$, $t\leq 2^r-r-1$ and $n>n_0(r,t)$?*

Example 19 ([24]). *Let r and t be fixed positive integers. Suppose that there exists $\gamma=\gamma(r,t)>0$ and $\varepsilon=\varepsilon(\gamma)>0$ such that $m^*(n,k,r,t)=(1-\gamma)\binom{n-t}{k-t}$ holds for $n>n_0(\varepsilon)$ and $|\frac{k}{n}-\frac{1}{2}|<\varepsilon$. Then we have $s(n,r,t)=\binom{n-t}{\lceil\frac{n-t}{2}\rceil}$ for $n>n_0(\varepsilon)$.*

This together with Example 13 gives the following.

Example 20 ([24]). *For $4\leq r\leq 10$ we have $s(n,r,t)=\binom{n-t}{\lceil\frac{n-t}{2}\rceil}$ for $t\leq t_r$ and $n>n_0$. Moreover $\mathscr{F}_0(n,k,r,t)$ is the only optimal configuration (up to isomorphism), where $k=t+\lceil\frac{n-t}{2}\rceil$ or $k=t+\lfloor\frac{n-t}{2}\rfloor$.*

The proof of Example 14 given in [21] can be extended without much changes to prove the following.

Example 21. *Let r,t,p be given with $p<\frac{r-2}{r}$ and (5). Then there exist $\gamma=\gamma(r,t,p)>0$ and $\varepsilon=\varepsilon(\gamma)>0$ such that $m^*(n,k,r,t)<(1-\gamma)\binom{n-t}{k-t}$ holds for all $n>n_0(\varepsilon)$ and k with $|\frac{k}{n}-p|<\varepsilon$.*

Example 21 for $p=1/2$ and Example 19 give the following.

Example 22. *Let $r\geq 5$ and let $\alpha_{r-1}\in(1/2,1)$ be the root of the equation $2x=1+x^{r-1}$. Suppose that $t\leq\frac{-\log\alpha_{r-1}}{\log\alpha_{r-1}+\log 2}$ (or one of (6)). Then we have $s(n,r,t)=\binom{n-t}{\lceil\frac{n-t}{2}\rceil}$ for $n>n_0$.*

In this case $p = 1/2$ we note that $t \le 2^{r-2}\log 2 - 1$, i.e., $\exp(\frac{t+1}{2^{r-2}}) \le 2$ implies (6). To see this fact we use $\alpha_{r-1} < p + p^{r-1}$. Then we have

$$\alpha_{r-1}^{t+1} < p^{t+1}(1+p^{r-2})^{t+1} < p^{t+1}\exp(p^{r-2}(t+1)) \le p^t.$$

4.2. **Intersecting and union families.** A family $\mathscr{G} \subset 2^{[n]}$ is called q-wise t-union if $|G_1 \cup \cdots \cup G_r| \le n-t$ holds for all $G_1,\ldots,G_r \in \mathscr{G}$. This is equivalent to the property that $\mathscr{G}^c = \{[n]-G : G \in \mathscr{G}\}$ is q-wise t-intersecting. Let $f(n,k,(r,s),(q,t))$ be the maximal size of n-vertex k-uniform r-wise s-intersecting and q-wise t-union families.

Problem 4. *Determine* $f(n,k,(r,s),(q,t))$.

The case $(r,s) = (q,t) = (2,1)$ is easy. In fact, it follows from the EKR theorem [4] that

$$f(n,k,(2,1),(2,1)) = \begin{cases} \binom{n-1}{k} & \text{if } n < 2k \\ \binom{n-1}{k} = \binom{n-1}{k-1} & \text{if } n = 2k \\ \binom{n-1}{k-1} & \text{if } n > 2k. \end{cases}$$

But the case $r \ge 3$ or $q \ge 3$ is not so easy even if $s = t = 1$. Engel and Gronau settled the case $r \ge 4$, $q \ge 4$ and $s = t = 1$ as follows.

Example 23 ([15, 3])**.** *Let* $r \ge 4$, $q \ge 4$ *and* $\frac{n-1}{q} + 1 \le k \le \frac{r-1}{r}(n-1)$. *Then we have*

$$f(n,k,(r,1),(q,1)) = \binom{n-2}{k-1}.$$

The case $(r,s) = (q,t) = (3,1)$ is more difficult and still open. As a special case the following is known.

Example 24 ([12])**.** *We have* $f(2n,n,(3,1),(3,1)) = \binom{2n-2}{n-1}$. *Moreover* $\{F \in \binom{[2n-1]}{n} : 1 \in F\}$ *is the only optimal configuration (up to isomorphism).*

The following result is based on the result of 4-wise intersecting case of Example 13.

Example 25 ([23, 24])**.** *Let t be an integer with* $1 \le t \le 4$. *Then we have*

$$f(2n,n,(4,t),(4,t)) = \binom{2n-2t}{n-t}$$

for $n > n_0$. *Moreover* $\{F \in \binom{[2n-t]}{n} : [t] \subset F\}$ *is the only optimal configuration (up to isomorphism).*

REFERENCES

[1] R. Ahlswede, L.H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18:125–136, 1997.

[2] A. Brace and D. E. Daykin. A finite set covering theorem. I,II,III,IV. *Bull. Austral. Math. Soc.*, 5:197–202, 1971, 6:19–24, 417–433, 1972, *Infinite and finite sets*, I:199–203, 1975.

[3] K. Engel, H.-D.O.F. Gronau. An Erdős–Ko–Rado type theorem II. *Acta Cybernet.*, 4:405–411, 1986.

[4] P. Erdős, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.

[5] P. Frankl. On Sperner families satisfying an additional condition. *J. Combin. Theory (A)*, 20:1–11, 1976.

[6] P. Frankl. The Erdős–Ko–Rado theorem is true for $n = ckt$. *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthey, 1976), Vol. I*, 365–375, Colloq. math. Soc. János Bolyai, 18, North–Holland, 1978.

[7] P. Frankl. Multiply-intersecting families. *J. Combin. Theory (B)*, 53:195–234, 1991.

[8] P. Frankl, N. Tokushige. Weighted 3-wise 2-intersecting families. *J. Combin. Theory (A)* 100:94-115, 2002.

[9] P. Frankl, N. Tokushige. Weighted multiply intersecting families. *Studia Sci. Math. Hungarica* 40:287–291, 2003.

[10] P. Frankl, N. Tokushige. Random walks and multiply intersecting families. *J. Combin. Theory (A)*, 109:121–134, 2005.

[11] P. Frankl, N. Tokushige. Weighted non-trivial multiply intersecting families. *Combinatorica*, to appear.

[12] P. Frankl, N. Tokushige. The maximum size of 3-wise intersecting and 3-wise union families. *Graphs and Combinatorics*, to appear.

[13] H.-D.O.F. Gronau. On Sperner families in which no k-sets have an empty intersection. *J. Combin. Theory (A)*, 28:54–63, 1980.

[14] H.-D.O.F. Gronau. On Sperner families in which no k-sets have an empty intersection II. *J. Combin. Theory (A)*, 30:298–316, 1981.

[15] H.-D.O.F. Gronau. An Erdős–Ko–Rado type theorem. Finite and infinite sets, Vol. I,II (Eger, 1981) *Colloq. Math. Soc. J. Bolyai*, 37:333–342, 1984.

[16] G.O.H. Katona. Intersection theorems for systems of finite sets. *Acta Math. Acad. Sci. Hung.*, 15:329–337, 1964.

[17] G.O.H. Katona. A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, 1966 (Akademiai Kiadó, 1968) 187–207.

[18] J.B. Kruskal. The number of simplices in a complex, in: Math. Opt. Techniques (Univ. of Calif. Press, 1963) 251–278.

[19] E.C. Milner. A combinatorial theorem on systems of sets. *J. London Math. Soc.*, 43:204–206, 1968.

[20] N. Tokushige. A frog's random jump and the Pólya identity. *Ryukyu Math. Journal*, 17:89–103, 2004.

[21] N. Tokushige. Extending the Erdős–Ko–Rado theorem. *J. Combin. Designs*, 14:52–55, 2006.

[22] N. Tokushige. The maximum size of 3-wise t-intersecting families. *European J. Combin.*, to appear.
[23] N. Tokushige. The maximum size of 4-wise 2-intersecting and 4-wise 2-union families. *European J. Combin.*, to appear.
[24] N. Tokushige. EKR type inequalities for 4-wise intersecting families. *preprint*.
[25] N. Tokushige. Brace–Daykin type inequalities for intersecting families. *preprint*.
[26] R.M. Wilson. The exact bound in the Erdős–Ko–Rado theorem. *Combinatorica*, 4:247–257, 1984.

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