On a special arrangement of spheres

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Abstract

A sphere-system in $\mathbb{R}^n$ is a family of $n+2$ spheres in $\mathbb{R}^n$ in which each $n+1$ spheres have a unique common point but all $n+2$ have empty intersection. A unit-sphere-system is a sphere-system consisting of all unit spheres. We prove that for every $2 \leq n \neq 3$, there is a unit-sphere-system in $\mathbb{R}^n$. The case $n = 3$ is open. We also prove that if there is a unit-sphere-system in $\mathbb{R}^3$, then there is a tetrahedron in $\mathbb{R}^3$ one of whose "escribed" spheres lies completely inside the circumscribed sphere.

1 Introduction

By a *sphere-system* in the n-dimensional Euclidean space $\mathbb{R}^n$, we mean a family of $n+2$ (hollow) spheres in $\mathbb{R}^n$ such that (i) each $n+1$ spheres have a unique common point, and (ii) the intersection of all $n+2$ spheres is empty. Figure 1 shows a sphere-system (a circle-system) in $\mathbb{R}^2$.

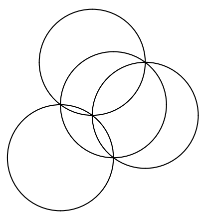

Figure 1: A circle-system

Concerning a circle-system, the following result is known.

Theorem 1. *If some three circles in a circle-system are unit circles, then the remaining circle is also a unit circle.*

This theorem was first discovered by Roger Johnson in 1916, see [6]. A proof is given in Pólya [5] (Chapter 10) to show how a useful idea occurs to us in a process of problem-solving. See also Davis and Hersh [3] Chapter 6.

A sphere-system consisting of all unit spheres is called a *unit-sphere-system*. The circle-system in Figure 1 is a unit-sphere-system (unit-circle-system) in $\mathbb{R}^2$. Theorem 1 implies that if some three circles in a circle-system are unit circles, then it is a unit-circle-system. Thus, there are many unit-circle-systems in $\mathbb{R}^2$. It is obvious that no unit-sphere-system exists in $\mathbb{R}^1$.

In a sphere-system in $\mathbb{R}^n$, $n > 2$, we cannot expect a result similar to Theorem 1. For example, if a sphere-system in $\mathbb{R}^3$ has four unit spheres whose centers span a regular tetrahedron, then the radius of the fifth sphere is 2/3. In $n \geq 3$, even the existence of a unit-sphere-system in $\mathbb{R}^n$ seems doubtful. The first author conjectured [4] that for $n \geq 3$, no unit-sphere-system in $\mathbb{R}^n$ would exist. This was wrong.

A sphere-system in $\mathbb{R}^n$ is called a *right-bipyramid*-type (RB-type) if the spheres have centers $p_0, p_1, \ldots, p_{n+1}$ such that (i) $p_1, p_2, \ldots, p_n$ span an $(n-1)$-dimensional regular simplex Δ^{n-1} in $\mathbb{R}^n$, and (ii) the line segment p_0p_{n+1} intersects Δ^{n-1} orthogonally at their barycenters.

Theorem 2. *There is a unit-sphere-system of RB-type in $\mathbb{R}^n$ if and only if $n \geq 4$.*

Thus, there are unit-sphere-systems in $\mathbb{R}^n$ for all $n \neq 1, 3$. At present, we do not know whether there is a unit-sphere-system in $\mathbb{R}^3$ or not. Several attempts lead us to the following conjecture.

Conjecture 1. *There is no unit-sphere-system in $\mathbb{R}^3$.*

For an n-dimensional simplex σ in $\mathbb{R}^n, n \geq 2$, its *circumscribed sphere* is the sphere passing through the $n+1$ vertices of the simplex. A sphere that is tangent to all $n+1$ facets (or extended hyperplanes) of σ is called a *tangent sphere* of σ. The one that is contained in σ is the *inscribed sphere* of σ, and other tangent spheres are called *escribed spheres* of σ. Every triangle has exactly four tangent circles. For $n \geq 3$, the number of distinct tangent spheres of an n-simplex depends on the shape of the simplex. The number of tangent spheres of a tetrahedron in $\mathbb{R}^3$ can vary from 4 to 8, see, e.g. Berger [2] p.296.

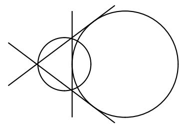

Figure 2: The circumscribed circle and an escribed circle of a triangle

It is obvious that in the plane case, the circumscribed circle of a triangle always cuts all escribed circles of the triangle, see Figure 2. Then, for $n \geq 3$, is there an n-simplex that has an escribed sphere disjoint from its circumscribed sphere?

Conjecture 2. *In any tetrahedron in $\mathbb{R}^3$, each escribed sphere intersects the circumscribed sphere.*

Theorem 3. *Let $n \geq 3$. If there is a unit-sphere-system in $\mathbb{R}^n$, then there is an n-dimensional simplex one of whose escribed spheres lies completely inside the circumscribed sphere.*

The case $n = 3$ of this theorem implies that Conjecture 1 follows from Conjecture 2. It also follows from Theorems 2 and 3 that for $n \geq 4$, there is an n-simplex in $\mathbb{R}^n$ one of whose escribed spheres lies completely inside the circumscribed sphere.

In the final section (Section 5), we consider unit-sphere-systems from the view point of unit-distance representations of the graph $K_{d+2,d+2}-$ "1-factor."

2 Proof of Theorem 2

In a sphere-system in $\mathbb{R}^n$, the unique common point of each $n+1$ spheres is called the *junction* of the $n+1$ spheres.

First recall here that the radius of the circumscribed sphere of a k-dimensional regular simplex of unit side-length is equal to

$$\sqrt{\frac{k}{2(k+1)}}. \tag{1}$$

We prove that there is an RB-type sphere-system consisting of the spheres of the *same size* in $\mathbb{R}^n$ if and only if $n \geq 4$. Suppose that there is an RB-type sphere-system $\{S_0, S_1, \ldots, S_{n+1}\}$ in $\mathbb{R}^n$ consisting of $n+2$ spheres of the same radius r. Let p_i denote the center of S_i. Since the sphere-system is RB-type, we may suppose that $p_1, \dots, p_n$ are the vertices of a regular $(n-1)$-simplex of unit side-length centered at the origin $o = (0, \dots, 0)$, lying on the hyperplane H defined by setting the last coordinate to be zero. Moreover, we may put

$$p_0 = (0, \dots, 0, x),\ p_{n+1} = (0, \dots, 0, -x),\ x > 0. \tag{2}$$

For each $i = 0, 1, \dots, n+1$, let q_i denote the junction determined by the $n+1$ spheres other than S_i. Since $S_0 \cap S_{n+1} \subset H$, the junctions $q_1, \dots, q_n$ lie on H. Since $\overline{p_i q_0} = \overline{p_i q_{n+1}} = r$ for $i = 1, \dots, n$, we may put

$$q_0 = (0, \dots, 0, y),\ q_{n+1} = (0, \dots, 0, -y). \tag{3}$$

Since $r = \overline{p_0 q_{n+1}} = \overline{p_0 q_1} > \overline{p_0 o}$, it follows that $y > 0$ and

$$r = \overline{p_0 q_{n+1}} = x + y. \tag{4}$$

Since $= \overline{op_i} = \sqrt{(n-1)/(2n)}$ by (1), we have

$$r = \overline{p_1 q_0} = \sqrt{y^2 + (n-1)/(2n)}. \tag{5}$$

Since, for each $1 \le i \le n$, the line op_i is the locus of those points on H that are equidistant from the $n-1$ points p_k $(k = 1, \dots, n;\ k \neq i)$, the junction q_i lies on the line op_i. Hence we may put

$$q_i = tp_i, i = 1, 2, \dots, n. \tag{6}$$

For each $1 \le i \le n$, let z_i denote the center of the $(n-2)$-dimensional simplex spanned by $\{p_1, \dots, p_n\} - \{p_i\}$, that is,

$$z_i = \frac{1}{n-1}\left(\sum_{k=1}^{n} p_k - p_i\right) = \frac{-1}{n-1} p_i,\ i = 1, 2, \dots, n.$$

Then

$$\overline{p_1 z_n} = \sqrt{\frac{n-2}{2(n-1)}},$$

by (1). Since $\overline{p_1 q_n}^2 = \overline{p_1 z_n}^2 + \overline{z_n q_n}^2$, we have

$$r = \overline{p_1 q_n} = \sqrt{\frac{n-2}{2(n-1)} + \left(t + \frac{1}{n-1}\right)^2 \frac{n-1}{2n}}. \tag{7}$$

Since $\overline{p_0 q_n}^2 = \overline{p_0 o}^2 + \overline{o q_n}^2$, we have

$$r = \overline{p_0 q_n} = \sqrt{x^2 + t^2 \frac{n-1}{2n}}. \tag{8}$$

From (7)(8), we have

$$t = nx^2 - \frac{n^2 - 2n + 1}{2(n-1)}. \tag{9}$$

From (4)(5) and (4)(7)(9), we have the following simultaneous equation for x, y:

$$2xy = \frac{n-1}{2n} - x^2 \tag{10}$$

$$(x+y)^2 = \frac{n-2}{2(n-1)} + \frac{n-1}{2n}\left(nx^2 - \frac{n^2-2n-1}{2(n-1)}\right)^2. \tag{11}$$

Eliminating y, we get a sectic equation for x with 6 solutions

$$\pm\sqrt{\frac{n-1}{2n}}, \ \pm\sqrt{\frac{n^2-4}{8n(n-1)}} \pm \sqrt{\frac{n-4}{8(n-1)}}.$$

(These solutions can be easily found by using, say, *Mathematica* or *Maple*.) Thus the system of equations (10)(11) has a real solution (x, y) with $x > 0, y > 0$ if and only if the sectic equation has a real solution x such that $0 < x < \sqrt{(n-1)/(2n)}$, that is, if and only if

$$\sqrt{\frac{n^2-4}{8n(n-1)}} \pm \sqrt{\frac{n-4}{8(n-1)}} \tag{12}$$

are real numbers. Hence, we must have $n \geq 4$.

Next, suppose $n \geq 4$. Let x be one of the values in (12) and define p_0, p_{n+1} as in (2). Choose an $(n-1)$-dimensional regular simplex of unit side-length centered at the origin and lying on the hyperplane H, and let $p_1, \ldots, p_n$ be its vertices. Define t, y by (9)(10), and $q_0, q_1, \ldots, q_{n+1}$ by (3)(6). Put $r = x + y$. Then, (4)(5)(7)(8) hold. Hence, the spheres of radius r centered at $p_0, p_1, \ldots, p_{n+1}$ form a sphere-system of RB-type. □

3 Some lemmas

Let φ denote the inversion of $\mathbb{R}^n$ with respect to the unit sphere centered at the origin o.

Lemma 1. *Suppose that in the plane, a circle Γ encloses a triangle ABC, and the triangle ABC encloses a circle γ of radius $1/d$ centered at the origin. Then the radius r of $\varphi(\Gamma)$ satisfies $r \le d/2$, the equality holds only when Γ is the circumscribed circle of ABC and γ is the inscribed circle of ABC.*

Proof. By replacing ABC by a larger triangle if necessary, we may suppose that ABC is inscribed in Γ. Let $A'B'C'$ be the triangle homothetic to ABC and circumscribed to the circle γ. Then, since the center of the homothety is contained in the triangle $A'B'C'$, the circumscribed circle Γ' of $A'B'C'$ lies inside Γ. Hence $\varphi(\Gamma)$ lies inside $\varphi(\Gamma')$, and hence the radius of $\varphi(\Gamma)$ is smaller than the radius of $\varphi(\Gamma')$. Therefore, it is enough to show that the radius of $\varphi(\Gamma')$ is equal to $d/2$.

Now, notice that by the inversion φ, the line $A'B'$ goes to a circle of diameter d passing through the origin. Similarly, the lines $B'C', C'A'$ go to two circles of diameter d passing through the origin. And the points $\varphi(A'), \varphi(B'), \varphi(C')$ are the intersection points of pairs among these three circles. Therefore, these three circles and $\varphi(\Gamma')$ form together a circle-system. Then by Theorem 1, the radius of $\varphi(\Gamma')$ is $d/2$. □

Lemma 2. *Let σ be an n-simplex in $\mathbb{R}^n, n \ge 3$, and let ℓ be a line passing through an interior point of σ. Then there is a plane π containing the line ℓ such that $\sigma \cap \pi$ is a triangle.*

Proof. First consider the case $n = 3$. Let π be any plane containing ℓ. Rotate π around the line ℓ until it meets a vertex of σ. At that time, the section of σ by the plane π is a triangle. Similarly, for $n \ge 3$, we can cut the n-simplex σ by a hyperplane containing ℓ so that the section is an $(n-1)$-simplex. Then the lemma follows by induction on n. □

Lemma 3. *Let $n \ge 3$. Suppose that a sphere Σ in $\mathbb{R}^n$ encloses an n-simplex σ, and σ contains a sphere K of radius $1/d$ centered at the origin. Then the radius of the sphere $\varphi(\Sigma)$ is smaller than $d/2$.*

Proof. Let ℓ be a line passing through the origin and the center of the sphere Σ. Then there is a plane π containing ℓ such that $\pi \cap \sigma$ is a triangle. Notice that on the plane π, the circle $\Sigma \cap \pi$ is not the circumscribed circle of the triangle $\sigma \cap \pi$. Now, by restricting φ on this plane π, and applying Lemma 1, we can deduce that the radius of $\varphi(\Sigma)$ is *smaller than* $d/2$. $\square$

Lemma 4. *Let* $\{S_0, S_1, \ldots, S_{n+1}\}$ *be a sphere-system in* $\mathbb{R}^n$*, and let* q_i *be the junction of the* $n+1$ *spheres other than* S_i*. Let* φ_0 *denote the inversion of* $\mathbb{R}^n$ *with respect to the unit sphere centered at* q_0*. Then* $n+1$ *points* $\varphi_0(q_i)\,(1 \leq i \leq n+1)$ *span an* n*-simplex* σ*, and* $\varphi_0(S_0)$ *is the circumscribed sphere of* σ*.*

Proof. Since $q_0 \notin S_0$, $\varphi_0(S_0)$ is a sphere, and the $n+1$ points $\varphi_0(q_i)\,(1 \leq i \leq n+1)$ clearly lie on $\varphi_0(S_0)$. For each $i = 1, 2, \ldots, n+1$, $H_i := \varphi_0(S_i)$ is a hyperplane containing the n points $\varphi_0(q_j)\,(1 \leq j \leq n+1, j \neq i)$.

Suppose that $X := \{\varphi_0(q_i) \mid 1 \leq i \leq n+1\}$ does not span an n-simplex. Then X lies on the flat spanned by $X - \{\varphi_0(q_i)\}$, for some $1 \leq i \leq n+1$. Hence X lies on the hyperplane H_i, which contradicts $q_i \notin S_i$. $\square$

The next lemma will be clear.

Lemma 5. *Let* σ *be an* n*-simplex in* $\mathbb{R}^n$*. Then for any point* $q \notin \sigma$*, there is a vertex* p *of* σ *such that the line segment* pq *crosses the hyperplane determined by the facet of* σ *opposite to* p*.*

4 Proof of Theorem 3

Let $\{S_0, S_1, \ldots, S_{n+1}\}$ be a unit-sphere-system in $\mathbb{R}^n$, $n \geq 3$, and let q_i denote the junction of $n+1$ spheres other than S_i. Denote by φ_0 the inversion of $\mathbb{R}^n$ with respect to the unit sphere centered at q_0.

First, we show that there is a j such that q_j lies inside S_j. Suppose that q_0 lies outside S_0. The $n+1$ points $\varphi_0(q_1), \ldots, \varphi_0(q_{n+1})$ span an n-simplex σ and $\varphi_0(S_0)$ is the circumscribed sphere of σ. Since q_0 lies outside $\varphi_0(S_0)$, σ does not contain q_0. Then, by Lemma 5, there is some $j, 1 \leq j \leq n+1$, such that the hyperplane $\varphi_0(S_j)$ separates $\varphi_0(q_j)$ from q_0. This implies that q_j lies inside the sphere S_j.

By changing the indexes if necessary, we may now suppose that q_0 lies inside S_0. Let Λ denote the sphere of radius 2 centered at q_0. Then the $n+1$ spheres $S_i\,(0 \leq i \leq n+1)$ lie inside Λ with each being tangent to Λ, and S_0

lies completely inside Λ. Let

$$\begin{aligned} K &= \varphi(\Lambda), \\ \Sigma &= \varphi(S_0), \\ H_i &= \varphi(S_i), i = 1, 2, \ldots, n+1. \end{aligned}$$

Then, $H_i, i = 1, 2, \ldots, n+1$, are hyperplanes. Let σ be the n-simplex spanned by the $n+1$ points $\varphi_0(q_i)$ $(1 \leq i \leq n+1)$. The sphere Σ is the circumscribed sphere of σ, and K is a tangent sphere of σ. Furthermore, K lies completely inside Σ. So, it is enough to show that K is an escribed sphere of σ. Suppose, on the contrary, that K is the inscribed sphere of σ. Then, by Lemma 3, the radius of $S_0 = \varphi(\Sigma)$ is smaller than 1. This contradicts that $\{S_0, S_1, \ldots, S_{n+1}\}$ is a unit-sphere-system. □

5 Unit distance representations

Let G be a finite graph. An injection $f : V(G) \to \mathbb{R}^d$ is called a unit distance representation (u.d.r.) of G if

$$||f(x) - f(y)|| = 1 \quad \text{iff} \quad xy \in E(G).$$

Let us define the dimension of G by

$$\dim(G) = \min\{d : \exists \text{u.d.r. } f : V(G) \to \mathbb{R}^d\}.$$

Let G_d denote the graph $K_{d+2,d+2} - (d+2)K_2$, that is,

$$V(G_d) = A \cup B,\ A = \{a_1, \ldots, a_{d+2}\},\ B = \{b_1, \ldots, b_{d+2}\},$$

$$E(G_d) = \{a_i b_j : 1 \leq i,\ j \leq d+2,\ i \neq j\}.$$

Note that G_d has $2(d+2)$ vertices and $(d+2)^2 - (d+2)$ edges. It is easy to see that $\dim(G_1) = 2$.

Now we consider the unit-sphere-system in $\mathbb{R}^d$. If the centers of the spheres and the junctions are all different (this is the case in RB-type as easily verified), then by taking the centers as $f(A)$ and junctions as $f(B)$, we have a u.d.r. f of G_d in $\mathbb{R}^d$. Hence Theorem 1 (Figure 1) and Theorem 2 give

$$\dim(G_d) \leq d \text{ for } d = 2 \text{ or } d \geq 4. \tag{13}$$

Note that $\dim(G_d) \leq d$ implies the existence of a unit-sphere system in $\mathbb{R}^d$, but there is a unit-sphere-system in which a center and a junction coincide, and which does not give a unit distance representation of G_d.

Lemma 6. $\dim(G_d) \geq d$ *for* $d \geq 1$.

Proof. We prove the above inequality by linear algebra method (cf. [1]). Suppose that G_d has a u.d.r. f in $\mathbb{R}^n$. Then, by definition, we have

$$||f(a_i) - f(b_j)|| = 1 \quad \text{iff} \quad i \neq j.$$

Let us define an n-variable polynomial $g_i : \mathbb{R}^n \to \mathbb{R}$ $(1 \leq i \leq d+2)$ by $g_i(x) = ||x - f(a_i)||^2 - 1$. Namely, setting $x = (x_1, \ldots, x_n)$ and $f(a_i) = (\alpha_1, \ldots, \alpha_n)$,

$$g_i(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2) - 2(\alpha_1 x_1 + \cdots + \alpha_n x_n) + \alpha_1^2 + \cdots + \alpha_n^2 - 1.$$

Since $g_i(f(b_j)) = 0$ iff $i \neq j$, these $d+2$ polynomials $g_1, \ldots, g_{d+2}$ are linearly independent, i.e., $\dim < g_1, \ldots, g_{d+2} > = d+2$.

On the other hand, g_i is a member of the subspace of polynomials spanned by

$$x_1^2 + \cdots + x_n^2,\ x_1,\ x_2, \ldots, x_n,\ 1$$

whose dimension is $n+2$. Therefore, we have $\dim < g_1, \ldots, g_{d+2} > \leq n+2$ and $d \leq n$, which completes the proof. □

The next theorem follows from the Lemma 6 and (13).

Theorem 4. $\dim(G_d) = d$ *for* $d = 2$ *or* $d \geq 4$.

Now we know that $\dim(G_3) = 3$ or 4 and our conjecture is

Conjecture 3. $\dim(G_3) = 4$.

If G_3 has a u.d.r f in $\mathbb{R}^3$, then we may assume that

$$f(a_1) = (0,0,0),\ f(b_2) = (1,0,0),\ f(b_3) = (p, \pm\sqrt{1-p^2}, 0).$$

The authors checked with aid of computer that $f(b_4) \neq (0,0,1)$.

Let $r_1, \ldots, r_{d+2}$ be positive reals. Suppose that an injection $f : V(G_d) \to \mathbb{R}^d$ satisfies

$$||f(a_i) - f(b_j)|| = r_i \quad \text{iff} \quad i \neq j. \tag{14}$$

This representation gives a sphere-system in $\mathbb{R}^d$. For every $d \geq 2$, we can find $r_1, \ldots, r_{d+2}$ with an injection $f : V(G_d) \to \mathbb{R}^n$ satisfying (14) iff $n \geq d$. The proof is the same as the proof of Lemma 6 by setting $g_i(x) = ||x - f(a_i)||^2 - r_i^2$. On the other hand, for every $d \geq 2$ (including $d = 3$) we have a sphere-system in $\mathbb{R}^d$ in which the centers and the junctions are all different. Therefore, the dimension of G_d as a representation satisfying (14) is precisely d for all $d \geq 2$.

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