WALLACE'S THEOREM AND MIQUEL'S THEOREM IN HIGHER DIMENSIONS

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ABSTRACT. We present a simple proof for the generalizations of a result due to Wallace and a result due to Miquel to higher dimensions.

1. Introduction

Let us recall the following two classic results in plane geometry.

Theorem 1 (Wallace [6] 1804). *Every three extended sides of a general quadrilateral determine a triangle, and therefore determine a circle circumscribed to the triangle. Thus,* $\binom{4}{3} = 4$ *circles are determined by a quadrilateral. These four circles meet at a point.*

Theorem 2 (Miquel [4] 1834). *On each (extended) side of an arbitrary triangle, consider a point different from the two vertices. Then the three circles, each of which is determined by a vertex and the two points on the adjacent side, meet at a point.*

In this note, we present a simple proof for the generalizations of the above results to higher dimensions. In the following, we write $A_1A_2 \dots A_n$ (or $\prod_{i=1}^n A_i$) for $A_1 \cap A_2 \cap \dots \cap A_n$, and if $A_1 \cap A_2 \cap \dots \cap A_n = \{X\}$, then we write $X = A_1A_2 \dots A_n$. Let $[n] = \{1, 2, \dots, n\}$ and $[m, n] = \{i : m \le i \le n\}$. We denote the cardinality of a set A by $|A|$. Then the generalized results are stated below.

Theorem 3. *Let $S_1, \dots, S_{d+2}$ be $d + 2$ hyperplanes in $\mathbb{R}^d$ such that each $d + 1$ of them determine a d-simplex. For each $i \in [d + 2]$, let T_i be the circumsphere of the d-simplex determined by $\{S_j : j \in [d + 2] \setminus \{i\}\}$. Then the intersection $T_1 \dots T_{d+2}$ is a single point if d is even, and empty if d is odd.*

Theorem 4. *Suppose that $d+1$ hyperplanes $S_1, \dots, S_{d+1}$ in $\mathbb{R}^d$ determine a d-simplex with vertices $Q_1, \dots, Q_{d+1}$, where $Q_i = \prod_{j \ne i} S_j$. For distinct $i, j \in [d + 2]$, choose a point Q_{ij} on the line Q_iQ_j, avoiding Q_i, Q_j. For each $i \in [d + 1]$, let T_i be the circumsphere of the d-simplex spanned by the $d+1$ points $\{Q_i\} \cup \{Q_{ij} : j \ne i\}$. Then the $d + 1$ spheres $T_1, \dots, T_{d+1}$ intersect at a single point.*

2. Proofs

We first prove Theorem 4, and then we derive Theorem 3 from Theorem 4. A triple $(d, \{S_i\}, \{T_i\})$, where $i \in [d + 1]$, is called an M-triple if it lists the dimension,

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hyperplanes and spheres under the condition of Theorem 4. We note that this triple is hereditary, namely, if $(d, \{S_i\}, \{T_i\})$ is an M-triple then so is $(d-1, \{S_1S_i\}, \{S_1T_i\})$ $(i \in [2, d+2])$ by identifying S_1 with $\mathbb{R}^{d-1}$. By an "ℓ-point" we mean a point of type Q_{ij} in Theorem 4, namely, a point chosen on a line.

Proof of Theorem 4. We show the following slightly stronger claim.

Claim 1. *For an M-triple $(d, \{S_i\}, \{T_i\})$ and $T := T_1 \dots T_d$, there exist points $X, Y \in \mathbb{R}^d$ such that $TT_{d+1} = X$, $TS_{d+1} = Y$, and $T = \{X, Y\}$.*

The existence of the point X implies the theorem. Note that $X = Y$ may happen. We prove the claim by induction on the dimension d. In the case $d = 1$, we have three distinct reals Q_1, Q_2, Q_{12}, and $S_1 = \{Q_2\}$, $S_2 = \{Q_1\}$, $T_1 = \{Q_1, Q_{12}\}$ $T_2 = \{Q_2, Q_{12}\}$. Thus it follows that $X = Q_{12}$ and $Y = Q_1$. The planar case $d = 2$ follows from Theorem 2. Let $d \geq 3$. We show the case d assuming that the claim is true up to $(d-1)$-dimensions. We divide the remaining part of the proof into two steps. In the first step we apply the induction hypothesis in 2 dimensions to get points of type Q_{ijk}. These points will be used as ℓ-points in the second step, where we apply the induction hypothesis in $(d-1)$-dimensions. Notice that

$$Q_a = T_a \prod_{i \neq a} S_i, \quad Q_{ab} = T_a T_b \prod_{i \neq a,b} S_i.$$

First we look at the sections by the 2-dimensional plane $H = S_4 \dots S_{d+1}$. For each $i \in [3]$, let $S'_i := S_i H$ and $T'_i := T_i H$. Then for $\{a, b, c\} = [3]$, S'_a is a line on H, and $S'_a S'_b = S_a S_b H = Q_c$, $Q_{bc} = S_a T_b T_c H = S'_a T_b T_c \in S'_a$. Also, T'_a is the circle determined by $Q_a, Q_{ab}, Q_{ac} \in H$. Thus applying the induction hypothesis to the M-triple $(d = 2, \{S'_1, S'_2, S'_3\}, \{T'_1, T'_2, T'_3\})$, we get the intersection $Q_{123} := T'_1 T'_2 T'_3 = T_1 T_2 T_3 H$. In the same way, we get the intersection of three spheres and $d-2$ hyperplanes

$$Q_{ijk} = T_i T_j T_k \prod_{\ell \notin \{i,j,k\}} S_\ell. \tag{1}$$

Next we look at the sections by the $(d-1)$-dimensional hyperplane $\varphi(T_1)$, where φ is an inversion of $\mathbb{R}^d$ with respect to a sphere centered at Q_1. For $i \in [2, d+1]$ let $\sigma_i = \varphi(T_1 S_i)$ and $\tau_i = \varphi(T_1 T_i)$. Since $Q_1 \in T_1 \prod_{i \in [2,d+1]} S_i$, we find that $\sigma_2, \dots, \sigma_{d+1}$ are $(d-2)$-dimensional flats on the hyperplane $\varphi(T_1)$. For $a \in [2, d+1]$, set

$$q_a := \varphi(Q_{1a}) = \varphi(T_1 T_a \prod_{i \notin \{1,a\}} S_i) \in \tau_a \prod_{i \in [2,d+1] \setminus \{a\}} \sigma_i.$$

Then the d flats $\sigma_2, \dots, \sigma_{d+1}$ determine a $(d-1)$-simplex in $\varphi(T_1)$ with vertices $q_2, \dots, q_{d+1}$. Also, for distinct $a, b \in [2, d+1]$, using (1) set

$$q_{ab} := \varphi(Q_{1ab}) = \varphi(T_1 T_a T_b \prod_{i \notin \{1,a,b\}} S_i) \in \tau_a \tau_b \prod_{i \in [2,d+1] \setminus \{a,b\}} \sigma_i.$$

We note that $\varphi(T_{d+1} \prod_{i \notin \{1,a,b\}} S_i) = \prod_{i \in [2,d+1] \setminus \{a,b\}} \sigma_i$ is the line $q_a q_b$ itself, and q_{ab} is an ℓ-point on this line. Moreover, the d points q_2, $q_{23}, q_{24}, \dots, q_{2,d+1}$ determine a $(d-2)$-dimensional sphere, which is τ_2. Similarly, for $a \in [2, d+1]$, the sphere τ_a passes through d points q_a and q_{ab} $(b \in [2, d+1] \setminus \{a\})$. We apply the induction hypothesis to the M-triple $(d-1, \{\sigma_i\}, \{\tau_i\})$ to get $\tau_2 \dots \tau_{d+1} = x$, $\tau_2 \dots \tau_d \sigma_{d+1} = y$, and $\tau_2 \dots \tau_d = \{x, y\}$. In other words, we have $\varphi(TT_{d+1}) = x$, $\varphi(TS_{d+1}) = y$, and $\varphi(T) = \{x, y\}$. Since φ is a bijection on $\mathbb{R}^d \setminus \{Q_1\}$, we have $X = \varphi^{-1}(x)$, $Y = \varphi^{-1}(y)$ as desired. □

A triple $(d, \{S_i\}, \{T_i\})$ $(i \in [d+2])$ is called a W-triple if it lists the dimension, hyperplanes and spheres under the condition of Theorem 3. This triple is hereditary. Moreover, we observe that if $(d, \{S_i\}, \{T_i\})$ $(i \in [d+2])$ is a W-triple then $(d, \{S_i\}, \{T_i\})$ $(i \in [d+1])$ is an M-triple. To see this, we need to find proper ℓ-points. Let $\mathcal{H}$ be the d-simplex determined by the hyperplanes $S_1, \dots, S_{d+1}$. Then the hyperplane S_{d+2} cuts the extended edges of $\mathcal{H}$, and determines the desired ℓ-points.

Proof of Theorem 3. We prove by induction on the dimension d. The case $d = 1$ is trivial, and the case $d = 2$ is Theorem 1. Let $d \geq 3$ and suppose that the theorem is true up to $(d-1)$-dimensions.

Let $T := T_1 \dots T_d$. By Applying Claim 1 to the M-triple $(d, \{S_i\}, \{T_i\})$ $(i \in [d+1])$, there are $X, Y \in \mathbb{R}^d$ such that $X = TT_{d+1}$, $Y = TS_{d+1}$, and $T = \{X, Y\}$. Similarly, applying Claim 1 to another M-triple $(d, \{S_j\}, \{T_j\})$ $(j \in [d+2] \setminus \{d+1\})$, we have $T = \{X', Y'\}$, where $X' = TT_{d+2}$, $Y' = TS_{d+2}$. Thus $\{TT_{d+1}, TS_{d+1}\} = \{TT_{d+2}, TS_{d+2}\}$, and $TT_{d+1} = TT_{d+2}$ iff $TS_{d+1} = TS_{d+2}$.

Applying the induction hypothesis to the W-triple

$$(d-2, \{S_i S_{d+1} S_{d+2}\}, \{T_i S_{d+1} S_{d+2}\}), \text{ where } i \in [d],$$

we find that $TS_{d+1}S_{d+2}$ is a single point if $d-2$ is even, and empty otherwise. Thus if d is even, then $TS_{d+1} = TS_{d+2}$ and $TT_{d+1} = TT_{d+2}$, namely, $TT_{d+1}T_{d+2}$ is a single point. If d is odd, then $TT_{d+1} \neq TT_{d+2}$ and $TT_{d+1}T_{d+2}$ is empty. □

3. Concluding remarks

Roberts [5] obtained Theorem 3 for $d = 3$. Then Grace [2] proved Theorem 3 and Theorem 4 up to 4 dimensions. He considered intersections of cubic surfaces corresponding to $S_i \cup T_i$. One can extend his algebraic approach to higher dimensions, see "The generalised Miquel theorem" and "Generalisation of Wallace's theorem" in Chapter I of Baker [1].

On the other hand, Longuet-Higgins [3] obtained Theorem 3 as a base case for a version of Clifford's chain. In particular, he found that the $d+2$ hyperplanes and the $d+2$ spheres appearing in Theorem 3 can be viewed as a part of facets of the $(d+2)$-dimensional hemi-cube. Our proof for Theorem 4 is based on his idea. In our case, the corresponding polytope is the $(d+1)$-dimensional hypercube. We just mention that the structure essentially comes from the following extension of Claim 1.

Theorem 5. *Let $(d, \{S_i\}, \{T_i\})$ be an M-triple. Then for each $\emptyset \neq J \subset [d+1]$, there exists a point $Q_J \in \mathbb{R}^d$ such that $Q_J = \prod_{j \in J} T_j \prod_{k \notin J} S_k$.*

Notice that in the cases $|J| = 1, 2$, the point Q_J coincides with a vertex and an ℓ-point of the d-simplex in Theorem 4, respectively. For comparison, we state the corresponding result for a W-triple due to Longuet-Higgins.

Theorem 6. *Let* $(d, \{S_i\}, \{T_i\})$ *be a W-triple. Then for each* $\emptyset \neq J \subset [d+2]$ *there exists a point* $Q_J \in \mathbb{R}^d$ *such that* $Q_J = \prod_{j \in J} T_j \prod_{k \notin J} S_k$ *iff* $d - |J|$ *even.*

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