Weighted 3-wise 2-intersecting families

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Abstract

Let n and r be positive integers. Suppose that a family $\mathcal{F} \subset 2^{[n]}$ satisfies $|F_1 \cap F_2 \cap F_3| \geq 2$ for all $F_1, F_2, F_3 \in \mathcal{F}$. We prove that if $w < 0.5018$ then $\sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|} \leq w^2$.

1 Introduction

Let n, r and t be positive integers. A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. For a real $w \in (0, 1)$ let us define the weighted size $W_w(\mathcal{F})$ of $\mathcal{F}$ by

$$W_w(\mathcal{F}) := \sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|}.$$

Note that $W_{1/2}(\mathcal{F}) = |\mathcal{F}|/2^n$. Further, define

$$f_{w,r,t}(n) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\}.$$

Let us check

$$f_{w,r,t}(n) \ge w^t. \tag{1}$$

Set $\mathcal{F}_0 := \{F \subset [n] : [t] \subset F\}$. Then $\mathcal{F}_0$ is r-wise t-intersecting for every r, and

$$\begin{aligned} W_w(\mathcal{F}_0) &= w^t \sum_{F \subset [t+1,n]} w^{|F|}(1-w)^{n-t-|F|} \\ &= w^t \sum_{i=0}^{n-t} \binom{n-t}{i} w^i (1-w)^{n-t-i} = w^t. \end{aligned}$$

Problem 1 *Does* $f_{w,r,t}(n) = w^t$ *hold if* $w \le w(r,t)$ *and* $t \le 2^r - r - 1$*?*

For 1-intersecting families, the authors proved the following in [7].

Theorem 1 $f_{w,r,1}(n) = w$ *if* $w \le (r-1)/r$*.*

On the other hand, for all $t \ge 1$ one has

$$\lim_{n\to\infty} f_{w,r,t}(n) = 1 \text{ if } w > (r-1)/r.$$

To obtain an exact formula for $f_{w,r,2}(n)$ seems to be much harder. In this paper, we shall prove

Theorem 2 $f_{w,3,2}(n) = w^2$ *if* $w < 0.5018$*.*

This implies $f_{w,r,2}(n) = w^2$ if $r \ge 3$ and $w < 0.5018$, since $w^t \le f_{w,r+1,t}(n) \le f_{w,r,t}(n)$. Using Theorem 2, the following variation of the Erdős–Ko–Rado theorem is deduced.

Theorem 3 *Let* $\mathcal{F} \subset \binom{[n]}{k}$ *be a* 3*-wise* 2*-intersecting family with* $k/n < 0.501$*. Then* $|\mathcal{F}| \le (1+o(1))\binom{n-2}{k-2}$*.*

A family $\mathcal{F} \subset 2^{[n]}$ is called a Sperner family if $F \not\subset G$ holds for all distinct $F, G \in \mathcal{F}$. The maximum size of 2-wise t-intersecting Sperner families was determined by Milner[17], it is given by the simply formula $\binom{n}{\lceil (n+t)/2 \rceil}$. For 3-wise t-intersecting families, the situation becomes more complicated. For 3-wise 1-intersecting families, it was the subject of several papers of Frankl [3] and Gronau [9, 10, 11] and it is known that for $n \ge 54$ the only optimal families are

$$\begin{array}{ll} \mathcal{F} = \{F \cup [n] : F \in \binom{[n-1]}{n/2}\} \cup \{[n-1]\} & n \text{ even}, \\ \mathcal{F} = \{F \cup [n] : F \in \binom{[n-1]}{(n-1)/2}\} & n \text{ odd}. \end{array}$$

This motivates the following conjecture.

Conjecture 1 *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. Then,*

$$|\mathcal{F}| \leq \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even}, \\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd}, \end{cases}$$

holds for $n \geq n_0$. The corresponding families are

$$\begin{array}{ll} \mathcal{F} = \{F \cup \{n-1, n\} : F \in \binom{[n-2]}{(n-2)/2}\} & n \text{ even}, \\ \mathcal{F} = \{F \cup \{n-1, n\} : F \in \binom{[n-2]}{(n-1)/2}\} \cup \{[n-1]\} \cup \{[n] - \{n-1\}\} & n \text{ odd}. \end{array}$$

Since $\mathcal{F} = \binom{[8]}{6}$ is 3-wise 2-intersecting Sperner and $|\mathcal{F}| = \binom{8}{6} > \binom{6}{3}$, we need the condition $n > n_0$ in the above conjecture.

As an application of Theorem 3, we prove the following weaker result, conjectured in [3].

Theorem 4 *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. Then $|\mathcal{F}| \leq (1 + o(1))\binom{n-2}{\lceil (n-2)/2 \rceil}$.*

Using the same technique, we can remove the above $o(1)$ term for 4-wise case as follows:

Theorem 5 *Let $\mathcal{F} \subset 2^{[n]}$ be a 4-wise 2-intersecting Sperner family. Then $|\mathcal{F}| \leq \binom{n-2}{\lceil (n-2)/2 \rceil}$ holds for $n > n_0$.*

Note that the same upper bound is valid for r-wise 2-intersecting Sperner families if $r \geq 4$.

2 Tools

2.1 Shifting

For integers $1 \leq i < j \leq n$ and a family $\mathcal{F} \subset 2^{[n]}$, define the (i,j)-shift S_{ij} as follows.

$$S_{ij}(\mathcal{F}) := \{S_{ij}(F) : F \in \mathcal{F}\},$$

where

$$S_{ij}(F) := \begin{cases} (F-\{j\})\cup\{i\} & \text{if } i \notin F,\ j \in F,\ (F-\{j\})\cup\{i\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

A family $\mathcal{F} \subset 2^{[n]}$ is called shifted if $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$. We call $\mathcal{F}$ co-complex if $G \supset F \in \mathcal{F}$ implies $G \in \mathcal{F}$. It is not difficult to check that $f_{w,r,t}(n)$ (the maximal weighted size of r-wise t-intersecting families) is attained by a shifted co-complex. See [5] for details.

Let us introduce a partial order in $2^{[n]}$ by using shifting. Let $A, B \subset [n]$. Define $A \succ B$ if there exists $A' \subset [n]$ such that $A \subset A'$ and B is obtained by repeating a shifting to A'. The following fact is trivial but useful.

Fact 1 *Let $\mathcal{F} \subset 2^{[n]}$ be a shifted co-complex. If $A \in \mathcal{F}$ and $A \succ B$, then $B \in \mathcal{F}$.*

Let us see how to apply the above fact.

Fact 2 *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting shifted co-complex. Set $G_0 := \{1, 3, 4, 6, 7, \ldots, 3i, 3i+1, \ldots\} \cap [n]$. Then $G_0 \notin \mathcal{F}$.*

Proof. Set $G_1 := \{1, 2, 4, 5, 7, \ldots, 3i-1, 3i+1, \ldots\} \cap [n]$, and $G_2 := \{1, 2, 3, 5, 6, \ldots, 3i-1, 3i, \ldots\} \cap [n]$. Then, $G_0 \cap G_1 \cap G_2 = \{1\}$ and $G_0 \succ G_1 \succ G_2$. If $G_0 \in \mathcal{F}$ then $G_1, G_2 \in \mathcal{F}$ must hold by Fact 1. But this is impossible because $\mathcal{F}$ is 3-wise 2-intersecting. □

In the same reason, an r-wise t-intersecting family can not contain the set $[n] - \{t, t+r, t+2r, \ldots\}$.

2.2 Random walk

Let $w \in (0, 2/3)$ be a fixed real number, and let $\alpha \in (0,1)$ be the root of the equation $(1-w)x^3 - x + w = 0$, more explicitly, $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$. Consider the infinite random walk, starting from the origin, in which at each step we move one unit up with probability w or move one unit right with probability $1-w$. Then the probability that we ever hit the line $y = 2x + s$ is given by α^s. (See [4] or [5] for details.)

Let $F \in \mathcal{F} \subset 2^{[n]}$. We define the corresponding (finite) walk to F, denoted by walk(F), in the following way. If $i \in F$ (resp. $i \notin F$) then we move one unit up (resp. one unit right) at the i-th step.

The following example shows how to use the random walk to bound the weighted size of families.

Fact 3 *Let $\mathcal{F} \subset 2^{[n]}$ be 3-wise 2-intersecting shifted co-complex. Then $W_w(\mathcal{F}) \leq \alpha^2$.*

Proof. Set $G_0 := \{1, 3, 4, 6, 7, \ldots, 3i, 3i+1, \ldots\} \cap [n]$. Note that walk$(G_0)$ is the maximal walk which does not touch the line $\ell : y = 2x + 2$. We know that $G_0 \notin \mathcal{F}$ by Fact 2. Thus, if $G \succ G_0$ then $G \notin \mathcal{F}$ by Fact 1. In other words, for every $F \in \mathcal{F}$, walk(F) must touch the line ℓ. Therefore,

$$W_w(\mathcal{F}) \leq \text{Prob(a random walk of } n\text{-steps touches the line } \ell) \leq \alpha^2. \qquad \square$$

For an r-wise t-intersecting family, we consider the equation $(1-w)x^r - x + w = 0$, its root $\alpha_r \in (0,1)$, and the line $y = (r-1)x + t$. Then the weight of the family is at most α_r^t.

2.3 Shadow

For a family $\mathcal{F} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓ-th shadow of $\mathcal{F}$, denoted by $\Delta_\ell(\mathcal{F})$, as follows.

$$\Delta_\ell(\mathcal{F}) := \{G \in \binom{[n]}{\ell} : G \subset \exists F \in \mathcal{F}\}.$$

Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $|\mathcal{F}| = \binom{m}{k} + \binom{x}{k-1}$ where $m \in \mathbf{N}$, $x \in \mathbf{R}$, $x \leq m-1$. Then, by the Kruskal–Katona theorem [15, 14] and its Lovász version [16], it follows that

$$|\Delta_\ell(\mathcal{F})| \geq \binom{m}{\ell} + \binom{x}{\ell-1}.$$

We shall use the above inequality to prove Theorem 3.

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 2-wise t-intersecting family. Katona[13] found the following bound for the ℓ-th shadow ($t \leq \ell < k$):

$$\frac{|\Delta_\ell(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{\binom{2k-t}{\ell}}{\binom{2k-t}{k}}.$$

We need the above inequality to prove Theorem 4.

See [5] or [6] for the detail of inequalities concerning the size of shadows.

3 Proof of Theorem 2

Let $\mathcal{F} \subset 2^{[n]}$ be 3-wise 2-intersecting. Further, we assume that $\mathcal{F}$ is shifted co-complex. Fix a constant w, $0 < w < 0.5018$. In this section, we write $W(\mathcal{F})$ instead of $W_w(\mathcal{F})$. Set $\alpha := \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$, $v := 1 - w$.

Let us define the following.

$$\begin{aligned}
*(i) &:= \{i, i+1, i+3, i+4, i+6, i+7, \ldots\} \cap [n] \\
&= [n] - ([i-1] \cup \{i+3j+2 : 0 \le j \le \lfloor \frac{n-i-2}{3} \rfloor\}) \\
P_i &:= \{1,2\} \cup *(i+4), \\
Q_i &:= \{1,2,i+4\} \cup *(i+6), \\
\mathcal{F}_{12} &:= \{F \in \mathcal{F} : \{1,2\} \subset F\}, \\
\mathcal{F}_{1\bar{2}} &:= \{F \in \mathcal{F} : 1 \in F, 2 \notin F\}, \\
\mathcal{F}_{\bar{1}2} &:= \{F \in \mathcal{F} : 1 \notin F, 2 \in F\}, \\
\mathcal{F}_{\bar{1}\bar{2}} &:= \{F \in \mathcal{F} : 1 \notin F, 2 \notin F\}.
\end{aligned}$$

By definition, it follows that $P_{i+1} \succ Q_i \succ P_i$, $|\mathcal{F}| = |\mathcal{F}_{12}| + |\mathcal{F}_{1\bar{2}}| + |\mathcal{F}_{\bar{1}2}| + |\mathcal{F}_{\bar{1}\bar{2}}|$. If $\{1,2\} = P_{n-3} \in \mathcal{F}$ then $\mathcal{F} = \{F \subset [n] : \{1,2\} \subset F\}$ and $W(\mathcal{F}) = w^2$. From now on, we assume $P_{n-3} \notin \mathcal{F}$ and we shall prove $W(\mathcal{F}) < w^2$.

Case 1 $P_0 \notin \mathcal{F}$

Since $P_0 = *(1) \notin \mathcal{F}$, we have $W(\mathcal{F}_{12}) \le w^2\alpha$. Since $\{1,3\} \cup *(4) \succ *(1)$, we have $\{1,3\} \cup *(4) \notin \mathcal{F}$, and $W(\mathcal{F}_{1\bar{2}}) \le wv\alpha^4$. Since $\{2,3\} \cup *(4) \succ *(1)$, we have $\{2,3\} \cup *(4) \notin \mathcal{F}$, and $W(\mathcal{F}_{\bar{1}2}) \le vw\alpha^4$. In the same way, we have $\{3,4,5,6\} \cup *(7) \notin \mathcal{F}$, and $W(\mathcal{F}_{\bar{1}\bar{2}}) \le v^2\alpha^7$. Therefore,

$$\begin{aligned}
W(\mathcal{F}) &= W(\mathcal{F}_{12}) + W(\mathcal{F}_{1\bar{2}}) + W(\mathcal{F}_{\bar{1}2}) + W(\mathcal{F}_{\bar{1}\bar{2}}) \\
&\le w^2\alpha + 2wv\alpha^4 + v^2\alpha^7 < w^2. \qquad \square
\end{aligned}$$

Case 2 $P_i \in \mathcal{F}$, $P_{i+1} \notin \mathcal{F}$, $i \ge 1$.

Case 2.1 $Q_i \notin \mathcal{F}$.

Since $Q_i \notin \mathcal{F}$, we have

$$\begin{aligned}
W(\mathcal{F}_{12}) &\le w^2(v^{i+1}\alpha^2 + (1 - v^{i+1})) \\
&= w^2(1 - v^{i+1}(1 - \alpha^2)).
\end{aligned}$$

Set

$$\begin{aligned} F &:= [1, i+3] \cup \{i+6, i+9, i+12, \ldots, 4i, 4i+3\} \cup *(4i+5), \\ G &:= \{1\} \cup [3, 4i+4] \cup *(4i+6). \end{aligned}$$

Since $P_i \in \mathcal{F}$ and $P_i = \{1,2\} \cup *(i+4) \succ F$, we have $F \in \mathcal{F}$. Note that $P_i \cap F \cap G = \{1\}$. Thus $G \notin \mathcal{F}$ follows from the assumption that $\mathcal{F}$ is 3-wise 2-intersecting. Therefore,

$$W(\mathcal{F}_{1\bar{2}}) \leq wv\alpha^{4i+3}.$$

In the same way, we have $W(\mathcal{F}_{\bar{1}2}) \leq wv\alpha^{4i+3}$. Next, set $H := [3, 4i+7] \cup *(4i+9)$. Since $P_i \cap F \cap H = \{4i+5\}$, we have $H \notin \mathcal{F}$, which implies

$$W(\mathcal{F}_{\bar{1}\bar{2}}) \leq v^2\alpha^{4i+6}.$$

Therefore,

$$W(\mathcal{F}) \leq w^2(1 - v^{i+1}(1-\alpha^2)) + 2wv\alpha^{4i+3} + v^2\alpha^{4i+6} < w^2. \tag{2}$$

(This is equivalent to $(\frac{\alpha^4}{v})^i < \frac{v(1-\alpha^2)}{\alpha^3(2(v/w)+(v/w)^2\alpha^3)}$.) □

Case 2.2 $Q_i \in \mathcal{F}$.
Since $P_{i+1} \notin \mathcal{F}$, we have

$$W(\mathcal{F}_{12}) \leq w^2(v^{i+1}\alpha + (1 - v^{i+1})) = w^2(1 - v^{i+1}(1-\alpha)).$$

Set

$$\begin{aligned} F &:= [1, i+3] \cup \{i+5, i+8, i+11, \ldots, 4i+5\} \cup *(4i+7), \\ G &:= \{1\} \cup [3, 4i+6] \cup *(4i+8). \end{aligned}$$

Since $Q_i \in \mathcal{F}$ and $Q_i \succ F$, we have $F \in \mathcal{F}$. Note that $Q_i \cap F \cap G = \{1\}$. Thus $G \notin \mathcal{F}$ follows from the assumption that $\mathcal{F}$ is 3-wise 2-intersecting. Therefore,

$$W(\mathcal{F}_{1\bar{2}}) \leq wv\alpha^{4i+5}.$$

In the same way, we have $W(\mathcal{F}_{\bar{1}2}) \leq wv\alpha^{4i+5}$. Set $H := [3, 4i+9] \cup *(4i+11)$. Since $Q_i \cap F \cap H = \{4i+7\}$, we have $H \notin \mathcal{F}$, which implies

$$W(\mathcal{F}_{\bar{1}\bar{2}}) \leq v^2\alpha^{4i+8}.$$

Therefore,

$$W(\mathcal{F}) \leq w^2(1 - v^{i+1}(1-\alpha)) + 2wv\alpha^{4i+5} + v^2\alpha^{4i+8} < w^2. \tag{3}$$

(This is equivalent to $(\frac{\alpha^4}{v})^i < \frac{v(1-\alpha)}{\alpha^5(2(v/w)+(v/w)^2\alpha^3)}$.) $\square$

Now we may assume that $P_0 \in \mathcal{F}$ and $P_1 \notin \mathcal{F}$.

Case 3 $P_1 \notin \mathcal{F}$, $Q_0 \in \mathcal{F}$.
Set $F := \{1,2,3,5\} \cup *(7)$, $G := \{1,3,4,5,6\} \cup *(8)$. Since $Q_0 \in \mathcal{F}$ and $Q_0 \succ F$, we have $F \in \mathcal{F}$. Note that $Q_0 \cap F \cap G = \{1\}$. Thus, $G \notin \mathcal{F}$, and

$$W(\mathcal{F}_{1\bar{2}}) \leq wv\alpha^5.$$

In the same way, we have $W(\mathcal{F}_{\bar{1}2}) \leq wv\alpha^5$. Next set $H := \{3,4,5,6,7\} \cup *(8)$. Since $Q_0 \cap F \cap H = \{7\}$, we have $H \notin \mathcal{F}$, which implies

$$W(\mathcal{F}_{\bar{1}\bar{2}}) \leq v^2\alpha^8.$$

Case 3.1 $S_1 := \{1,2,5\} \cup *(6) \notin \mathcal{F}$.
Since $S_1 \notin \mathcal{F}$, we have

$$W(\mathcal{F}_{12}) \leq w^2(w^2 + 2wv + v^2\alpha^4) =: W_{31}.$$

Therefore, we have

$$W(\mathcal{F}) \leq W_{31} + 2wv\alpha^5 + v^2\alpha^8 < w^2. \qquad \square$$

Case 3.2 $S_2 := \{1,2,5,6,8,9\} \cup *(10) \notin \mathcal{F}$.
Since $S_2 \notin \mathcal{F}$, we have

$$\begin{aligned} W(\mathcal{F}_{12}) \leq\ & w^2(w^5 + 5w^4v + 10w^3v^2 + w^2v^3(7+3\alpha^5) \\ & + wv^4(2+3\alpha^8) + v^5\alpha^{11}) := W_{32}. \end{aligned}$$

Therefore, we have

$$W(\mathcal{F}) \leq W_{32} + 2wv\alpha^5 + v^2\alpha^8 < w^2.$$

This is the hardest case and the above inequality fails if $w \geq 0.5019$. $\square$

Case 3.3 $S_1, S_2 \in \mathcal{F}$.
Set $F := \{1,2,3,4,8,9\} \cup *(10)$, $G := \{1,3,4,5,6,7,8\} \cup *(11)$. Since $S_2 \in \mathcal{F}$ and $S_2 \succ F$, we have $F \in \mathcal{F}$. Note that $S_1 \cap F \cap G = \{1\}$ and $G \notin \mathcal{F}$. Thus, we have

$$\begin{aligned} W(\mathcal{F}_{1\bar{2}}) &\leq wv(w^7 + v\sum_{i=1}^{7} w^{7-i}\alpha^i) \\ &= wv(w^7 + v\alpha\frac{\alpha^7 - w^7}{\alpha - w}) =: W_{33}. \end{aligned}$$

In the same way, we have $W(\mathcal{F}_{\bar{1}2}) \leq W_{33}$. Next set $H := \{3,4,5,6,7,8,9\} \cup *(11)$. Since $S_1 \cap F \cap H = \{9\}$, we have $H \notin \mathcal{F}$, which implies

$$W(\mathcal{F}_{\bar{1}\bar{2}}) \leq v^2\alpha^8.$$

Finally $P_1 \notin \mathcal{F}$ implies

$$W(\mathcal{F}_{12}) \leq w^2(w^2 + 2wv + v^2\alpha^3).$$

Therefore, we have

$$W(\mathcal{F}) \leq w^2(w^2 + 2wv + v^2\alpha^3) + 2W_{33} + v^2\alpha^8 < w^2. \qquad \square$$

Now we may assume $P_0 \in \mathcal{F}$ and $Q_0 \notin \mathcal{F}$.

Case 4 $P_0 \in \mathcal{F}$, $Q_0 \notin \mathcal{F}$, $R := \{1,2,3\} \cup *(6) \in \mathcal{F}$.
First, $Q_0 \notin \mathcal{F}$ implies

$$W(\mathcal{F}_{12}) \leq w^2(w + v\alpha^2).$$

Set $G := \{1,3,4\} \cup *(5)$, $H := \{3,4,5,6,7\} \cup *(8)$. Since $P_0 \cap R \cap G = \{1\}$, we have $G \notin \mathcal{F}$, which implies

$$W(\mathcal{F}_{1\bar{2}}) \leq wv\alpha^5.$$

In the same way, we have $W(\mathcal{F}_{\bar{1}2}) \leq wv\alpha^5$. Since $P_0 \cap R \cap H = \{7\}$, we have $H \notin \mathcal{F}$ and

$$W(\mathcal{F}_{\bar{1}\bar{2}}) \leq v^2\alpha^8.$$

Therefore, we have

$$W(\mathcal{F}) \leq w^2(w + v\alpha^2) + 2wv\alpha^5 + v^2\alpha^8 < w^2. \qquad \square$$

At this point, let us summarize what we have proved.

Proposition 1 *Theorem 2 is true if* $P_0 \notin \mathcal{F}$ *or* $Q_0 \in \mathcal{F}$ *or* $\{1,2,3\} \cup *(6) \in \mathcal{F}$.

In order to prove the remaining cases, we need some preparations. For a subset $S \subset [5]$, let us define

$$\begin{aligned} \mathcal{F}(S) &:= \{F - S : F \in \mathcal{F}, F \cap [5] = S\} \subset 2^{[6,n]}, \\ f(S) &:= W(\mathcal{F}(S)). \end{aligned}$$

If $S \succ S'$, the shiftedness of $\mathcal{F}$ implies $\mathcal{F}(S) \subset \mathcal{F}(S')$ (and $f(S) \leq f(S')$). For simplicity, we write $\mathcal{F}(123)$, $f(123)$ instead of $\mathcal{F}(\{1,2,3\})$, $f(\{1,2,3\})$.

Lemma 1 *Let* $S \subset [5]$, $|S| = 3$, *and* $F := \{1,3\} \cup *(4)$. *If* $F \in \mathcal{F}$, *then* $f(S) \leq \alpha^3$.

Proof. Set $G := \{1,2,4\} \cup *(5)$, $H := \{1,2,3\} \cup *(6)$. Since $F \in \mathcal{F}$ and $F \succ G$, we have $G \in \mathcal{F}$. Note that $F \cap G \cap H = \{1\}$. Thus, we have $H \notin \mathcal{F}$ and $f(123) \leq \alpha^3$. If $S \subset [5]$ and $|S| = 3$ then $S \succ \{1,2,3\}$. Thus, $f(S) \leq f(123) \leq \alpha^3$. □

Lemma 2 *Let* $S \subset [5]$, $|S| \leq 3$, *and* $F := \{1,3\} \cup *(4)$. *If* $F \in \mathcal{F}$, *then* $f(S) \leq \alpha^{3(4-|S|)}$.

Proof. Similar as proof of Lemma 1. Use the fact that $F \in \mathcal{F}$ implies $\{1,2,6,7,8\} \cup *(9) \notin \mathcal{F}$, $\{1\} \cup [6,11] \cup *(12) \notin \mathcal{F}$, $[6,14] \cup *(15) \notin \mathcal{F}$. □

Lemma 3 *Let* $S \subset [5]$ *and* $|S| = 3$. *If* $[2] \not\subset S$, *then* $\mathcal{F}(S)$ *is* 3*-wise* 3*-intersecting (on* $[6,n]$*).*

Proof. By the shiftedness of $\mathcal{F}$, it is sufficient to consider the case $S = \{1,3,4\}$. Suppose, on the contrary, that $\mathcal{F}(S)$ is not 3-wise 3-intersecting. Then, there exist $T_1, T_2, T_3 \in \mathcal{F}(S)$ such that $T_1 \cap T_2 \cap T_3 = \{x, y\}$. Set

$$\begin{aligned} F_1 &:= \{1,3,4\} \cup T_1, \\ F_2 &:= \{1,2,4,5\} \cup (T_2 - \{x\}), \\ F_3 &:= \{1,2,3,5\} \cup (T_3 - \{y\}). \end{aligned}$$

Since $S \cup T_2 \succ F_2$ and $S \cup T_3 \succ F_3$, we have $F_1, F_2, F_3 \in \mathcal{F}$, but $F_1 \cap F_2 \cap F_3 = \{1\}$. This contradicts our assumption that $\mathcal{F}$ is 3-wise 2-intersecting. □

Using the same approach, we can extend the above lemma as follows.

Lemma 4 *If* $[2] \not\subset S \subset [5]$ *and* $|S| \leq 3$*, then* $\mathcal{F}(S)$ *is* 3*-wise* $3(4-|S|)$*-intersecting (on* $[6,n]$*).*

Now, let us leave the proof of Theorem 2 aside for a while, and concentrate on the following stronger proposition.

Proposition 2 *Let* $\mathcal{G} \subset 2^{[n]}$*,* $t \geq 2$*, and* $w < 0.5018$*. If* $\mathcal{G}$ *is* 3*-wise* t*-intersecting, then* $W(\mathcal{G}) \leq w^2\alpha^{t-2}$*.*

Note that the case $t = 2$ in Proposition 2 is exactly Theorem 2. We prove Proposition 2 by double induction on n and t.

First, let us check the cases $t \leq n \leq t+2$. Set $\mathcal{G}_0 := \{G \subset [n] : [t] \subset G\}$. It is easy to verify that if $t \leq n \leq t+2$ then

$$W(\mathcal{G}) \leq W(\mathcal{G}_0) = w^t \leq w^2\alpha^{t-2}.$$

Another initial step of the induction is the case $t = 2$, i.e., Theorem 2. But we postpone this essential case, and check, in advance, that Theorem 2 actually implies the induction step.

Assume that Proposition 2 is true for $t = 2$. Let $\mathcal{G} \subset 2^{[n]}$ be 3-wise t-intersecting and $t \geq 3$. (We also assume that $\mathcal{G}$ is shifted co-complex.) Define $\mathcal{G}_1, \mathcal{G}_{\bar{1}} \subset 2^{[2,n]}$ as follows.

$$\begin{aligned}\mathcal{G}_1 &:= \{G - \{1\} : 1 \in G \in \mathcal{G}\},\\ \mathcal{G}_{\bar{1}} &:= \{G \in \mathcal{G} : 1 \notin \mathcal{G}\}.\end{aligned}$$

Note that $\mathcal{G}_1$ is 3-wise $(t-1)$-intersecting, and since $\mathcal{G}$ is shifted, $\mathcal{G}_{\bar{1}}$ is 3-wise $(t+2)$-intersecting. Using the induction hypothesis, we have

$$\begin{aligned}W(\mathcal{G}) &= wW(\mathcal{G}_1) + vW(\mathcal{G}_{\bar{1}}) \leq w^3\alpha^{t-3} + vw^2\alpha^t\\ &= w^2\alpha^{t-3}(v\alpha^3 + w) = w^2\alpha^{t-2}.\end{aligned}$$

(Remember that α is a root of the equation $vx^3 - x + w = 0$.) This completes the induction step for the proof of Proposition 2.

Consequently, all we have to do is to prove the case $t = 2$ (Theorem 2) by induction on n. So let us return to the proof of Theorem 2 again. But this time, we can use the induction hypothesis of Proposition 2 i.e., we assume that Proposition 2 is true for all (n', t') if $t' \leq n' < n$.

Lemma 5 *If* $[2] \not\subset S \subset [5]$ *and* $|S| \le 3$, *then* $f(S) \le w^2\alpha^{10-3|S|}$.

Proof. By Lemma 4, $\mathcal{F}(S) \subset 2^{[6,n]}$ is 3-wise $3(4-|S|)$-intersecting. Using the induction hypothesis, we have $W(\mathcal{F}(S)) \le w^2\alpha^{3(4-|S|)-2} = w^2\alpha^{10-3|S|}$. $\square$

Case 5 $\{1,3\} \cup *(4) \in \mathcal{F}$.
Let $S \subset [5]$. Define

$$\tilde{\mathcal{F}}(S) := \{F \in \mathcal{F} : F \cap [5] = S\} \subset 2^{[n]}.$$

For $S = \{1,2,3\}, \{1,2,4\}, \{1,2,5\}$, we apply Lemma 1 and obtain $f(S) \le \alpha^3$. For the remaining seven 3-sets S, we use Lemma 5 and obtain $f(S) \le w^2\alpha$. Thus, we have

$$\sum_{|S|=3} W(\tilde{\mathcal{F}}(S)) \le (3\alpha^3 + 7w^2\alpha) \cdot w^3v^2.$$

Similarly,

$$\sum_{|S|=2} W(\tilde{\mathcal{F}}(S)) \le (\alpha^6 + 9w^2\alpha^4) \cdot w^2v^3.$$

In this way, we have

$$\begin{aligned}\sum_{|S|\le 3} W(\tilde{\mathcal{F}}(S)) &\le (3\alpha^3 + 7w^2\alpha) \cdot w^3v^2 + (\alpha^6 + 9w^2\alpha^4) \cdot w^2v^3 \\ &\quad +5w^2\alpha^7 \cdot wv^4 + w^2\alpha^{10} \cdot v^5 =: W_5.\end{aligned}$$

Case 5.1 $\{2,3\} \cup *(4) \notin \mathcal{F}$.
Since $f(2345) \le \alpha$, we have $\sum_{|S|=4} W(\tilde{\mathcal{F}}(S)) \le (4+\alpha)w^4v$. We also use $f(12345) \le 1$, i.e., $W(\tilde{\mathcal{F}}(S)) \le w^5$. Therefore, we have

$$W(\mathcal{F}) = \sum_{S \subset [5]} W(\tilde{\mathcal{F}}(S)) \le W_5 + (4+\alpha)w^4v + w^5 < w^2. \qquad \square$$

Case 5.2 $\{2,3\} \cup *(4) \in \mathcal{F}$ and $\{1,3,4,5\} \cup *(8) \notin \mathcal{F}$.
Set $\mathcal{F}_6 := \{H \in \mathcal{F}(1345) : 6 \in F\}$, $\mathcal{F}_{\bar{6}} := \{H \in \mathcal{F}(1345) : 6 \notin F\}$. Then we have $W(\mathcal{F}_6) \le w^5v$ and $W(\mathcal{F}_{\bar{6}}) \le w^4v^2\alpha$. Thus, we have

$$W(\tilde{\mathcal{F}}(1345)) \le w^4v^2(w + v\alpha).$$

We can apply the same thing to $\tilde{\mathcal{F}}(2345)$, because $\{2,3,4,5\} \cup *(8) \notin \mathcal{F}$ follows from the shiftedness of $\mathcal{F}$. Thus,

$$\sum_{|S|=4} \leq 2w^4v^2(w+v\alpha) + 3w^4v =: W_{52}.$$

(The former corresponds to 1345, 2345, and the latter corresponds to 1234, 1235, 1245.) Therefore, we have

$$W(\mathcal{F}) \leq W_5 + W_{52} + w^5 < w^2. \qquad \square$$

Case 5.3 $\{2,3\} \cup *(4) \in \mathcal{F}$ and $\{1,3,4,5\} \cup *(8) \in \mathcal{F}$.
Since $\{2,3\} \cup *(4) \succ [2,7] \cup *(10)$, and

$$(\{1,3,4,5\} \cup *(8)) \cap ([2,7] \cup *(10)) \cap (\{1,2,3,6,7,8\} \cup *(9)) = \{3\},$$

we have $\{1,2,3,6,7,8\} \cup *(9) \notin \mathcal{F}$. This implies that $f(S) \leq \alpha^6$ if $|S| = 3$. If $|S| = 2$ and $S \neq \{1,2\}$ then we have $f(S) \leq w^2\alpha^4$ by Lemma 5. Thus, we have

$$\begin{aligned} W(\mathcal{F}) \ \leq\ & w^2\alpha^{10} \cdot v^5 + 5w^2\alpha^7 \cdot wv^4 + (9w^2\alpha^4 + \alpha^6)w^2v^3 \\ & +10\alpha^6 \cdot w^3v^2 + 5w^4v + w^5 < w^2. \qquad \square \end{aligned}$$

Case 6 $\{1,3\} \cup *(4) \notin \mathcal{F}$.
By using Proposition 1, we may assume that $R := \{1,2,3\} \cup *(6) \notin \mathcal{F}$. This implies that

$$f(123), f(124), f(125) \leq \alpha^3.$$

Since $\{1,2,6,7,8\} \cup *(9) \succ R$, we have $\{1,2,6,7,8\} \cup *(9) \notin \mathcal{F}$, and thus, $f(12) \leq \alpha^6$. Therefore, (using Lemma 5) we have

$$\begin{aligned} W(\mathcal{F}) \ \leq\ & w^2\alpha^{10} \cdot v^5 + 5w^2\alpha^7 \cdot wv^4 + (9w^2\alpha^4 + \alpha^6)w^2v^3 \\ & +(7w^2\alpha + 3\alpha^3)w^3v^2 + (4+\alpha)w^4v + w^5 < w^2. \qquad \square \end{aligned}$$

This completes the proof of Proposition 2 and Theorem 2 at the same time. $\square$

4 Proof of Theorem 3

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family. This family is clearly 2-wise 2-intersecting, too. Therefore, by the Erdős–Ko–Rado theorem it follows that $|\mathcal{F}| \leq \binom{n-2}{k-2}$ if $n \geq 3(k-1)$. So we may assume that $n < 3k$.

Let $\delta > 0$ be given. We shall prove $|\mathcal{F}| < (1+\delta)\binom{n-2}{k-2}$ for sufficiently large n. Set $w := 0.5017$ and $v := 1 - w$. By Theorem 2, we must have $W_w(\mathcal{G}) \leq w^2$ for any 3-wise 2-intersecting family $\mathcal{G} \subset 2^{[n]}$. Choose $\epsilon > 0$ sufficiently small so that

$$(1+\delta/2)(1-\epsilon)^4 > 1, \tag{4}$$

$$0.501 < (1-\epsilon)w. \tag{5}$$

Define an open interval $I := ((1-\epsilon)wn, (1+\epsilon)wn)$. Choose $n_0 = n_0(\delta, \epsilon)$ sufficiently large so that

$$\sum_{i \in I} \binom{n}{i} w^i v^{n-i} > 1 - \epsilon \quad \text{for all } n > n_0, \tag{6}$$

$$\epsilon > 2/((1-\epsilon)wn) \quad \text{for all } n > n_0. \tag{7}$$

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family with $1/3 < k/n < 0.501$. Suppose that $|\mathcal{F}| = (1+\delta)\binom{n-2}{k-2}$. We shall derive a contradiction by constructing a 3-wise 2-intersecting family $\mathcal{G} \subset 2^{[n]}$ with $W_w(\mathcal{G}) > w^2$. Set $\mathcal{F}^c := \{[n] - F : F \in \mathcal{F}\}$. Define

$$\mathcal{G} := \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathcal{F}^c))^c \ \ (\subset \bigcup_{i=k}^{n} \binom{[n]}{i}).$$

Then,

$$\begin{aligned} W_w(\mathcal{G}) &= \sum_{\ell=0}^{n-k} |\Delta_\ell(\mathcal{F}^c)| w^{n-\ell} v^\ell \\ &= \sum_{i=k}^{n} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}. \end{aligned}$$

Since $k < 0.501n < (1-\epsilon)wn$ by (5) and $I \subset [k, n]$, we have

$$W_w(\mathcal{G}) \geq \sum_{i \in I} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}.$$

Lemma 6 $|\Delta_{n-i}(\mathcal{F}^c)| \geq (1+\frac{\delta}{2})\binom{n-2}{n-i}$ *for* $i \in I$.

Proof. Let x ($x \leq n-3$) be a real satisfying $\binom{x}{n-k-1} = \delta\binom{n-2}{n-k}$. Then, $|\mathcal{F}^c| = |\mathcal{F}| = (1+\delta)\binom{n-2}{k-2} = \binom{n-2}{n-k} + \binom{x}{n-k-1}$. By the Kruskal–Katona theorem, we have $|\Delta_{n-i}(\mathcal{F}^c)| \geq \binom{n-2}{n-i} + \binom{x}{n-i-1}$. To prove $\binom{x}{n-i-1} \geq \frac{\delta}{2}\binom{n-2}{n-i}$, it is sufficient to show

$$\frac{\binom{x}{n-i-1}}{\binom{x}{n-k-1}} \geq \frac{\frac{\delta}{2}\binom{n-2}{n-i}}{\delta\binom{n-2}{n-k}},$$

or equivalently,

$$\frac{(i-2)\cdots(k-1)}{(x-n+i+1)\cdots(x-n+k+2)} \geq \frac{n-k}{2(n-i)}.$$

Let us check that LHS $> 1 >$ RHS. Since LHS $\geq (\frac{i-2}{x-n+i+1})^{i-k}$ and $x \leq n-3$, we have LHS > 1. On the other hand, $1 >$ RHS is equivalent to $(n+k)/2 \geq i$. Using $n < 3k$ and (5), we certainly have $(n+k)/2 \geq (n+n/3)/2 = 2n/3 \geq (1+\epsilon)wn \geq i$. This completes the proof of the lemma. ∎

By the lemma, we have

$$\begin{aligned} W_w(\mathcal{G}) &\geq \sum_{i\in I}(1+\frac{\delta}{2})\binom{n-2}{n-i}w^i v^{n-i} \\ &= \sum_{i\in I}(1+\frac{\delta}{2})\frac{i}{n}\cdot\frac{i-1}{n-1}\binom{n}{i}w^i v^{n-i}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{i}{n}\cdot\frac{i-1}{n-1} &\geq (\frac{i-1}{n})^2 \geq \frac{((1-\epsilon)wn-1)^2}{n^2} \geq (1-\epsilon)^2w^2 - \frac{2(1-\epsilon)w}{n} \\ &= (1-\epsilon)^2w^2(1-\frac{2}{(1-\epsilon)wn}) > (1-\epsilon)^3w^2 \quad \text{(by (7))}. \end{aligned}$$

Therefore,

$$\begin{aligned} W_w(\mathcal{G}) &\geq (1+\frac{\delta}{2})(1-\epsilon)^3w^2\sum_{i\in I}\binom{n}{i}w^i v^{n-i} \\ &> (1+\frac{\delta}{2})(1-\epsilon)^4w^2 \quad \text{(by (6))} \\ &> w^2 \quad \text{(by (4))}, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 3. ∎

5 Proof of Theorem 4

For a family $\mathcal{F} \subset 2^{[n]}$, set $\mathcal{F}_i := \mathcal{F} \cap \binom{[n]}{i}$. First we prove the following version of the Erdős–Ko–Rado theorem. (See [3] for 3-wise 1-intersecting families.)

Proposition 3 *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family with $k/n < 0.501$. Then $\sum_{i=1}^{k} |\mathcal{F}_i| \binom{n-2}{i-2}^{-1} \leq 1 + o(1)$.*

Proof. Let $\delta > 0$ be given. We prove $\sum_{i=1}^{k} |\mathcal{F}_i| \binom{n-2}{i-2}^{-1} \leq 1 + \delta$ for $n > n_0(\delta)$ by induction on the number of nonzero $|\mathcal{F}_i|$'s.

If this number is one then the inequality follows from Theorem 3. If it is not the case then let p be the smallest and r the second-smallest index for which $|\mathcal{F}_i| \neq 0$. Set $\mathcal{F}_p^c := \{[n] - F : F \in \mathcal{F}_p\}$. Then $\mathcal{F}_p^c \subset \binom{[n]}{n-p}$ is (2-wise) $(n-2p+2)$-intersecting. By the Katona's shadow theorem for intersecting family (see section 2.3), we have

$$\frac{|\Delta_{n-r}(\mathcal{F}_p^c)|}{|\mathcal{F}_p^c|} \geq \frac{\binom{2(n-p)-(n-2p+2)}{n-r}}{\binom{2(n-p)-(n-2p+2)}{n-p}} = \frac{\binom{n-2}{r-2}}{\binom{n-2}{p-2}}.$$

Set $\mathcal{G} := \{G \in \binom{[n]}{r} : G \supset \exists F \in \mathcal{F}_p\}$. Since $\mathcal{G} = (\Delta_{n-r}(\mathcal{F}_p^c))^c$, we have $|\mathcal{G}| \binom{n-2}{r-2}^{-1} \geq |\mathcal{F}_p| \binom{n-2}{p-2}^{-1}$. Note that $\mathcal{H} := (\mathcal{F} - \mathcal{F}_p) \cup \mathcal{G}$ is also 3-wise 2-intersecting Sperner family, and the number of nonzero $|\mathcal{H}_i|$'s is one less. Therefore, by the induction hypothesis we have

$$\sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{i-2}} \leq \sum_{i=1}^{k} \frac{|\mathcal{H}_i|}{\binom{n-2}{i-2}} \leq 1 + \delta,$$

which completes the proof of the proposition. $\square$

Let us now prove Theorem 4. Let $\delta > 0$ be given. Suppose that $\mathcal{F} \subset 2^{[n]}$ is a 3-wise 2-intersecting Sperner family. We show $|\mathcal{F}| < (1+\delta)\binom{n-2}{\lceil (n-2)/2 \rceil}$ for $n > n_0(\delta)$. Set $k := \lfloor 0.501n \rfloor$. By Proposition 3, we have

$$1 + \frac{\delta}{2} > \sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{i-2}} \geq \sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{\lceil (n-2)/2 \rceil}}.$$

On the other hand, by the LYM inequality, we have

$$1 \geq \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \geq \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{k+1}}.$$

Therefore, we have

$$|\mathcal{F}| \leq (1+\frac{\delta}{2})\binom{n-2}{\lceil (n-2)/2 \rceil} + \binom{n}{\lfloor 0.501n \rfloor + 1} < (1+\delta)\binom{n-2}{\lceil (n-2)/2 \rceil}$$

for sufficiently large n. $\square$

6 Proof of Theorem 5

An r-wise t-intersecting family $\mathcal{F} \subset 2^{[n]}$ is called non-trivial if $|\bigcap_{F \in \mathcal{F}} F| < t$. Define

$$g_{w,r,t}(n) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\}.$$

Proposition 4 *$g_{w,4,2}(n) \leq 0.999w^2$ if $w \leq 0.5015$.*

Proof. Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting family. In the proof of Theorem 2, we checked the inequality $W_w(\mathcal{F}) \leq w^2$. In the exactly the same way, we can check

$$W_w(\mathcal{F}) < 0.999w^2 \ \text{ for } \ w \leq 0.5015$$

in all cases but Case 2.

Now let $\mathcal{F} \subset 2^{[n]}$ be a non-trivial 4-wise 2-intersecting family. We follow the proof of Theorem 2 and all we have to deal with is only Case 2. Suppose that there exist F_1, F_2, F_2 such that $F_1 \cap F_2 \cap F_3 = \{1, 2\}$. Then every $F \in \mathcal{F}$ must contain $\{1, 2\}$, which is not possible because $\mathcal{F}$ is non-trivial. Thus we may assume that $\{F \setminus \{1, 2\} : \{1, 2\} \subset F \in \mathcal{F}\}$ is 3-wise 1-intersecting. Then, by Theorem 1, we have

$$W_w(\mathcal{F}_{12}) \leq w^3 \ \text{ for } \ w \leq 2/3.$$

For $\mathcal{F}_{1\bar{2}}, \mathcal{F}_{\bar{1}2}, \mathcal{F}_{\bar{1}\bar{2}}$ we use the same estimation in Case 2 of proof of Theorem 2, but this time we redefine $\alpha \in (0, 1)$ as the unique root (in the interval) of the equation $(1-w)x^4 - x + w = 0$. (cf. $\alpha \approx 0.543689$ if $w = 1/2$.) Then one can check in inequalities (2) and (3) that

$$W_w(\mathcal{F}) < 0.93w^2 \quad \text{for } w \leq 2/3.$$

This completes the proof. (Note that one can construct (see [8]) a non-trivial 4-wise 2-intersecting family $\mathcal{F} \subset 2^{[n]}$ with $\lim_{n\to\infty} W_w(\mathcal{F}) = w^2$ if $w > 2/3$.) $\square$

Proposition 5 *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 4-wise 2-intersecting family with $k/n < 0.501$, $n > n_0$. Then $|\mathcal{F}| \leq \binom{n-2}{k-2}$. Moreover if $\mathcal{F}$ is non-trivial then $|\mathcal{F}| < 0.9999\binom{n-2}{k-2}$.*

Proof. The proof is similar to the proof of Theorem 3, and we give a sketch here. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 4-wise 2-intersecting family. If $\mathcal{F}$ fix 2-element set, then $|\mathcal{F}| \leq \binom{n-2}{k-2}$. So we may assume that $\mathcal{F}$ is non-trivial. Suppose that $|\mathcal{F}| \geq 0.9999\binom{n-2}{k-2}$, and set $w := 0.501$, $v := 1-w$. We shall derive a contradiction by constructing a non-trivial 4-wise 2-intersecting family $\mathcal{G} \subset 2^{[n]}$ with $W_w(\mathcal{G}) > 0.999w^2$.

Choose $\epsilon > 0$ sufficiently small so that

$$0.9998(1-\epsilon)^4 > 0.999, \tag{8}$$

$$0.501 < (1-\epsilon)w. \tag{9}$$

Define an open interval $I := ((1-\epsilon)wn, (1+\epsilon)wn)$. Choose $n_0 = n_0(\delta, \epsilon)$ sufficiently large so that

$$\sum_{i\in I}\binom{n}{i} w^i v^{n-i} > 1-\epsilon \quad \text{for all } n > n_0, \tag{10}$$

$$\epsilon > 2/((1-\epsilon)wn) \quad \text{for all } n > n_0. \tag{11}$$

Set $\mathcal{F}^c := \{[n] - F : F \in \mathcal{F}\}$ and define

$$\mathcal{G} := \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathcal{F}^c))^c \;\; \Big(\subset \bigcup_{i=k}^{n} \binom{[n]}{i}\Big).$$

Then $\mathcal{G}$ is a non-trivial 4-wise 2-intersecting family, and since $k < 0.501n < (1-\epsilon)wn$ by (9), we have

$$W_w(\mathcal{G}) \geq \sum_{i\in I} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}.$$

Lemma 7 $|\Delta_{n-i}(\mathcal{F}^c)| \geq 0.9998\binom{n-2}{n-i}$ *for* $i \in I$.

Proof. Let x ($x < n-2$) be a real satisfying $|\mathcal{F}| \geq 0.9999\binom{n-2}{k-2} = \binom{x}{n-k}$. Then, by the Kruskal–Katona theorem, we have $|\Delta_{n-i}(\mathcal{F}^c)| \geq \binom{x}{n-i}$. To prove $\binom{x}{n-i} \geq 0.9998\binom{n-2}{n-i}$, it is sufficient to show

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}} \geq \frac{0.9998\binom{n-2}{n-i}}{0.9999\binom{n-2}{n-k}},$$

or equivalently,

$$\frac{(i-2)\cdots(k-1)}{(x-n+i)\cdots(x-n+k+1)} \geq \frac{0.9998}{0.9999}.$$

This is true, because LHS $\geq (\frac{i-2}{x-n+i})^{i-k} > 1 >$ LHS > 1. This completes the proof of the lemma. $\square$

Therefore,

$$\begin{aligned}
W_w(\mathcal{G}) &\geq 0.9998\sum_{i\in I}\binom{n-2}{n-i}w^i v^{n-i} \quad \text{(by the lemma)}\\
&= 0.9998\sum_{i\in I}\frac{i}{n}\cdot\frac{i-1}{n-1}\binom{n}{i}w^i v^{n-i}.\\
&\geq 0.9998(1-\epsilon)^3 w^2\sum_{i\in I}\binom{n}{i}w^i v^{n-i} \quad \text{(by (11))}\\
&> 0.9998(1-\epsilon)^4 w^2 \quad \text{(by (10))}\\
&> 0.999w^2 \quad \text{(by (8))},
\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 5. $\square$

For a family $\mathcal{F} \subset 2^{[n]}$, set $\mathcal{F}_i := \mathcal{F} \cap \binom{[n]}{i}$. One can prove the next proposition in the same way we proved Theorem 4. (The only difference is to use Proposition 5 instead of Theorem 3.)

Proposition 6 *Let* $\mathcal{F} \subset 2^{[n]}$ *be a* 4*-wise* 2*-intersecting Sperner family with* $k/n < 0.501$, $n > n_0$. *Then* $\sum_{i=1}^{k}|\mathcal{F}_i|\binom{n-2}{i-2}^{-1} \leq 1$. *Moreover if* $\mathcal{F}$ *is non-trivial then* $\sum_{i=1}^{k}|\mathcal{F}_i|\binom{n-2}{i-2}^{-1} < 0.9999$.

Let us now prove Theorem 5. Let $\mathcal{F} \subset 2^{[n]}$ be a 4-wise 2-intersecting Sperner family. First suppose that $\mathcal{F}$ fix 2-element set, say $\{1,2\}$. Then $\mathcal{G} := \{F \setminus \{1,2\} : F \in \mathcal{F}\} \subset 2^{[3,n]}$ is a Sperner family. Thus we have

$$|\mathcal{F}| = |\mathcal{G}| \le \binom{n-2}{\lceil (n-2)/2 \rceil}.$$

Next suppose that $\mathcal{F}$ is non-trivial. Set $k := \lfloor 0.501n \rfloor$. By Proposition 6, we have

$$0.9999 > \sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{i-2}} \ge \sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{\lceil (n-2)/2 \rceil}}.$$

On the other hand, by the LYM inequality, we have

$$1 \ge \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \ge \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{k+1}}.$$

Therefore, we have

$$|\mathcal{F}| \le 0.9999 \binom{n-2}{\lceil (n-2)/2 \rceil} + \binom{n}{\lfloor 0.501n \rfloor + 1} < \binom{n-2}{\lceil (n-2)/2 \rceil}$$

for sufficiently large n. This completes the proof of Theorem 5 $\square$

As for 3-wise case, compared to Proposition 6, we have a following difficulty.

Example 1 *Let* $w = \frac{1}{2} + \epsilon$, $k = \lfloor wn \rfloor$, *and set* $A = [3, k+2]$. *Define a non-trivial 3-wise 2-intersecting family* $\mathcal{F}_n \subset \binom{[n]}{k}$ *as follows.*

$$\mathcal{F}_n := \left\{ \{1,2\} \cup G : |G \cap A| \ge \frac{k+2}{2}, G \in \binom{[3,n]}{k-2} \right\} \cup \{A\}.$$

Then one has $\lim_{n\to\infty} |\mathcal{F}_n| / \binom{n-2}{k-2} = 1$.

If we take all superset of $F \in \mathcal{F}_n$, that is, $\mathcal{G}_n := \{G \subset [n] : G \supset \exists F \in \mathcal{F}_n\}$, then this family is clearly non-trivial 3-wise 2-intersecting. One can check that $\lim_{n\to\infty} W_w(\mathcal{G}_n) = w^2$ for fixed $w = \frac{1}{2} + \epsilon$. Thus, Proposition 4 fails for 3-wise 2-intersecting family. However, we may still expect to refine Theorem 3 as follows:

Conjecture 2 *Let* $\mathcal{F} \subset \binom{[n]}{k}$ *be a 3-wise 2-intersecting family with* $k/n < 0.501$, $n > n_0$. *Then* $|\mathcal{F}| \le \binom{n-2}{k-2}$.

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