Weighted multiply intersecting families

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April 21, 2003

Abstract

Let n and r be positive integers. Suppose that a family $\mathcal{F} \subset 2^{[n]}$ satisfies $F_1 \cap \cdots \cap F_r \neq \emptyset$ for all $F_1, \ldots, F_r \in \mathcal{F}$. We prove that if $0 < w \leq (r-1)/r$ then $\sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|} \leq w$.

1 Introduction

Let n and r be positive integers. A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise intersecting if $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. For applications it is often important to consider weighted intersecting theorems, i.e., results where instead of $|\mathcal{F}|$ some different function is maximized. For a real $w \in (0, 1)$ let us define the weighted size $W_w(\mathcal{F})$ of $\mathcal{F}$ by

$$W_w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|}.$$

Note that $W_{1/2}(\mathcal{F}) = |\mathcal{F}|/2^n$. More generally, let $\vec{v} = (v_1, v_2, \ldots, v_n)$ be a random 0-1 vector where $v_i = 1$ with probability w and $v_i = 0$ with probability $1-w$. Let $F(\vec{v})$ be the corresponding subset of $[n]$, i.e., $F(\vec{v}) = \{i : v_i = 1\}$. Now $W_w(\mathcal{F})$ is the probability that for a random 0-1 vector $\vec{v}$, $F(\vec{v}) \in \mathcal{F}$ holds. The weighted size also appears in the optimization of reliability polynomial, see [5] for details.

Finally, define

$$f_{w,r}(n) = \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise intersecting}\}.$$

Let us check

$$f_{w,r}(n) \geq w. \tag{1}$$

Set $\mathcal{F}_0 = \{F \subset [n] : 1 \in F\}$. Then $\mathcal{F}_0$ is r-wise intersecting for every r, and

$$\begin{aligned} W_w(\mathcal{F}_0) &= w \sum_{F \subset [2,n]} w^{|F|}(1-w)^{n-1-|F|} \\ &= w \sum_{i=0}^{n-1} \binom{n-1}{i} w^i (1-w)^{n-1-i} = w. \end{aligned}$$

Actually, this is the maximal weight.

Theorem 1 *$f_{w,r}(n) = w$ if $w \leq \dfrac{r-1}{r}$.*

On the other hand, we will see

$$\lim_{n\to\infty} f_{w,r}(n) = 1 \text{ if } w > \tfrac{r-1}{r}. \tag{2}$$

2 Proof of the theorem

We distinguish two cases $w = (r-1)/r$ and $w < (r-1)/r$.

For the first case, we need a preliminary result. Define a map $p : \{0, \ldots, r-1\}^n \to 2^{[n]}$ by $p(g_1, \ldots, g_n) = \{i : g_i \neq 0\}$ and set $p(\mathcal{G}) = \{p(g_1, \ldots, g_n) : (g_1, \ldots, g_n) \in \mathcal{G}\}$ for $\mathcal{G} \subset \{0, \ldots, r-1\}^{[n]}$.

Proposition 1 *Let $\mathcal{G} \subset \{0, \ldots, r-1\}^{[n]}$. If $p(\mathcal{G})$ is r-wise intersecting, then $|\mathcal{G}| \leq (\frac{r-1}{r})r^n$.*

Proof For $g = (g_1, \dots, g_n) \in \mathcal{G}$, define

$$\varphi(g) = ((g_1 + 1) \bmod r, \dots, (g_n + 1) \bmod r),$$

and $\varphi(\mathcal{G}) = \{\varphi(g) : g \in \mathcal{G}\}$. If $p(\mathcal{G})$ is r-wise intersecting, $\{g, \varphi(g), \dots, \varphi^{r-1}(g)\} \not\subset \mathcal{G}$ for any $g \in \{0, \dots, r-1\}^n$, and thus $\mathcal{G} \cap \varphi(\mathcal{G}) \cap \cdots \cap \varphi^{r-1}(\mathcal{G}) = \emptyset$. Therefore,

$$r|\mathcal{G}| = |\mathcal{G}| + |\varphi(\mathcal{G})| + \cdots + |\varphi^{r-1}(\mathcal{G})| \le (r-1)r^n,$$

or equivalently $|\mathcal{G}| \le (\frac{r-1}{r})r^n$. ∎

Now we assume that $w = (r-1)/r$ and prove the theorem in this case. Let $\mathcal{F} \subset 2^{[n]}$ be r-wise intersecting. Then,

$$W_w(\mathcal{F}) = \sum_{F \in \mathcal{F}} \left(\frac{r-1}{r}\right)^{|F|} \left(\frac{1}{r}\right)^{n-|F|} = \left(\frac{1}{r}\right)^n \sum_{F \in \mathcal{F}} (r-1)^{|F|}.$$

On the other hand, using the proposition, we have

$$\sum_{F \in \mathcal{F}} (r-1)^{|F|} = |p^{-1}(\mathcal{F})| \le (\frac{r-1}{r})r^n.$$

Therefore, $W_w(\mathcal{F}) \le \frac{r-1}{r} = w$ and $f_{w,r}(n) = w$.

Next we assume that $w < (r-1)/r$. We use the following result proved by Frankl in [2].

Proposition 2 *If $\mathcal{G} \subset \binom{[n]}{k}$ is r-wise intersecting and $(r-1)n \ge rk$, then $|\mathcal{G}| \le \frac{k}{n}\binom{n}{k}$.*

Let $\epsilon > 0$ be a small real and set an open interval $I = ((1-\epsilon)nw, (1+\epsilon)nw)$. For any $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that $\sum_{k \notin I} \binom{n}{k} w^k (1-w)^{n-k} < \epsilon$ for $n > n_0$. Thus, choosing an optimal $\mathcal{F}$, we have

$$\begin{aligned} f_{w,r}(n) &< \sum_{k \in I} \left| \mathcal{F} \cap \binom{[n]}{k} \right| w^k (1-w)^{n-k} + \epsilon \\ &\le \sum_{k \in I} \frac{k}{n} \binom{n}{k} w^k (1-w)^{n-k} + \epsilon \quad \text{(by Prop. 2)} \\ &\le (1+\epsilon) w \sum_{k=0}^{n} \binom{n}{k} w^k (1-w)^{n-k} + \epsilon \\ &= (1+\epsilon) w + \epsilon. \end{aligned}$$

This implies

$$\lim_{n\to\infty} f_{w,r}(n) = w. \tag{3}$$

Let us choose $\mathcal{F} \subset 2^{[n]}$ with $W_w(\mathcal{F}) = f_{w,r}(n)$, and define $\mathcal{F}' \subset 2^{[n+1]}$ by $\mathcal{F}' = \mathcal{F} \cup \{F \cup \{n+1\} : F \in \mathcal{F}\}$. Then $W_w(\mathcal{F}') = f_{w,r}(n)((1-w)+w) = f_{w,r}(n)$, which means

$$f_{w,r}(n+1) \geq f_{w,r}(n). \tag{4}$$

By (1), (3) and (4), we have $f_{w,r}(n) = w$. This completes the proof of the theorem. ∎

If $w > \frac{r-1}{r}$, we can choose $\epsilon > 0$ so that $\frac{r-1}{r} < (1-\epsilon)w$. In this case $\binom{[n]}{k}$ is r-wise intersecting if $k \in I = ((1-\epsilon)nw, (1+\epsilon)nw)$. Thus, $f_{w,r}(n) \geq \sum_{k\in I}\binom{n}{k} w^k(1-w)^{n-k} \to 1$ as $n \to \infty$. This proves (2).

3 Concluding remarks

A family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. Let us define

$$f_{w,r,t}(n) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\}.$$

Set $\mathcal{F}_0 := \{F \subset [n] : [t] \subset F\}$. Then $\mathcal{F}_0$ is r-wise t-intersecting for every r, and $W_w(\mathcal{F}_0) = w^t$. This means $f_{w,r,t}(n) \geq w^t$. We have shown that $f_{w,r,1}(n) = w$ if $w \leq (r-1)/r$.

Problem 1 *Does $f_{w,r,t}(n) = w^t$ hold if $w \leq w(r,t)$ and $t \leq 2^r - r - 1$?*

In [3], the authors proved

$$f_{w,3,2}(n) = w^2 \text{ if } w < 0.5018.$$

The above result is used to prove the following, which settles a problem posed in [2].

Theorem 2 [3] *Let $\mathcal{F} \in 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. Then $|\mathcal{F}| \leq (1+o(1))\binom{n-2}{\lceil (n-2)/2\rceil}$.*

A family $\mathcal{F} \subset 2^{[n]}$ is called non-trivial if $\bigcap_{F\in\mathcal{F}} F = \emptyset$. Let us define

$$\mathcal{F}_1 = \{F \subset [n] : |F \cap [r+1]| \geq r\}.$$

Then $\mathcal{F}_1$ is a non-trivial r-wise intersecting family. Brace and Daykin proved the following.

Theorem 3 [1] *Suppose that $\mathcal{F} \subset 2^{[n]}$ is a non-trivial r-wise intersecting family. Then $|\mathcal{F}| \leq |\mathcal{F}_1|$.*

In other words, $W_{1/2}(\mathcal{F}) \leq W_{1/2}(\mathcal{F}_1)$ holds for any non-trivial r-wise intersecting family $\mathcal{F}$. Can we expect the same inequality ($W_w(\mathcal{F}) \leq W_w(\mathcal{F}_1)$) for $w = \frac{1}{2} + \epsilon$? In [4], the authors proved that the answer is "yes" for $r \geq 13$ and "no" for $r \leq 5$.

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