# Weighted non-trivial multiply intersecting families

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**Abstract**

Let $n$ and $r$ be positive integers. Suppose that a family $\mathcal{F} \subset 2^{[n]}$ satisfies $F_1 \cap \cdots \cap F_r \neq \emptyset$ for all $F_1, \ldots, F_r \in \mathcal{F}$ and $\bigcap_{F \in \mathcal{F}} F = \emptyset$. We prove that there exists $\epsilon = \epsilon(r) > 0$ such that $\sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|} \leq w^r(r+1-rw)$ holds for $1/2 \leq w \leq 1/2 + \epsilon$ if $r \geq 13$.

## 1 Introduction

Let $n, r$ and $t$ be positive integers. A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called $r$-wise $t$-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. An $r$-wise 1-intersecting family is also called an $r$-wise intersecting family for short. An $r$-wise $t$-intersecting family $\mathcal{F}$ is called non-trivial if $|\bigcap_{F \in \mathcal{F}} F| < t$.

Let us define the Brace–Daykin structure as follows.

$$\mathcal{F}_{BD}^r = \{F \subset [n] : |F \cap [r+1]| \geq r\}.$$

Then $\mathcal{F}_{BD}^r$ is a non-trivial $r$-wise intersecting family. Brace and Daykin proved the following.

**Theorem 1** [1] *Suppose that $\mathcal{F} \subset 2^{[n]}$ is a non-trivial $r$-wise intersecting family. Then $|\mathcal{F}| \leq |\mathcal{F}_{BD}^r|$.*

For a real $w \in (0,1)$ let us define the weighted size (or simply weight) $W_w(\mathcal{F})$ of $\mathcal{F}$ by

$$W_w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|}.$$

Note that $W_{1/2}(\mathcal{F}) = |\mathcal{F}|/2^n$. See [3] for the maximum weighted size of intersecting families, and see [2, 4] for applications of weighted size to Erdős–Ko–Rado and Sperner type results concerning multiply intersecting families. In this note, we consider the maximum weighted size of non-trivial intersecting families and extend Theorem 1. The weight of the Brace–Daykin family is calculated as follows:

$$W_w(\mathcal{F}_{BD}^r) = (r+1)w^r(1-w) + w^{r+1} = w^r(r+1-rw).$$

Let us define

$$g_n(w,r,t) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\},$$

$$g(w,r,t) := \lim_{n\to\infty} g_n(w,r,t).$$

Then the Brace–Daykin theorem states that $g_n(1/2,r,1) = W_{1/2}(\mathcal{F}_{BD}^r)$ and thus $g(1/2,r,1) = (r+2)(1/2)^{r+1}$. Can we expect the same thing for $w = 1/2+\epsilon$? The answer is "yes" for $r \geq 13$, and "no" for $r \leq 5$.

**Theorem 2** *Let $r \geq 13$. Then there exists $\epsilon = \epsilon(r) > 0$ such that $g(w,r,1) = W_w(\mathcal{F}_{BD}^r) = w^r(r+1-rw)$ holds for $1/2 \leq w \leq 1/2+\epsilon$.*

In the last section, we shall construct non-trivial $r$-wise intersecting families with weights larger than $W_w(\mathcal{F}_{BD}^r)$ for $r \leq 5$. The cases $6 \leq r \leq 12$ remain open.

**Conjecture 1** *Theorem 2 is true for $r \geq 6$.*

## 2 Tools

In this section we summarize some results on the maximum weight of (not necessarily non-trivial) $r$-wise $t$-intersecting families. Let us define

$$f_n(w,r,t) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\},$$

$$f(w,r,t) := \lim_{n\to\infty} f_n(w,r,t).$$

If $\mathcal{F} \subset 2^{[n]}$ satisfies $f_n(w,r,t) = W_w(\mathcal{F})$ then $\mathcal{F}' := \mathcal{F} \cup \{F \cup \{n+1\} : F \in \mathcal{F}\} \subset 2^{[n+1]}$ satisfies $W_w(\mathcal{F}') = W_w(\mathcal{F}) = f_n(w,r,t)$, which implies

$f_{n+1}(w,r,t) \geq f_n(w,r,t)$. Since $\mathcal{F} = \{F \subset [n] : [t] \subset F\}$ is $r$-wise $t$-intersecting and $W_w(\mathcal{F}) = w^t$, it follows that $f(w,r,t) \geq f_n(w,r,t) \geq w^t$.

Let $\alpha_{w,r} \in (1/2, 1)$ be the unique root of the equation $(1-w)x^r - x + w = 0$. The following inequality is not sharp but it is very useful (see Fact 3 on page 98 of [2]).

**Lemma 1** *$f(w,r,t) \leq \alpha_{w,r}^t$.*

For the case $t = 1$, we proved the following in [3].

**Lemma 2** *$f(w,r,1) = w$ if $w \leq \frac{r-1}{r}$, and $f(w,r,1) = 1$ if $w > \frac{r-1}{r}$.*

For the case $r = 3$, we proved the following in [2] (see Proposition 2 on page 104).

**Lemma 3** *$f(w,3,t) \leq w^2 \alpha_{w,3}^{t-2}$ if $t \geq 2$ and $w < 0.5018$.*

We also use the following simple fact.

**Lemma 4** *If $\alpha_{w,r-1}^{t+1} \leq w^t$ then $f(w,r,t) = w^t$.*

**Proof.** Suppose that $\mathcal{F}$ is an $r$-wise $t$-intersecting family with $W_w(\mathcal{F}) = f(w,r,t) \geq w^t$. If $\mathcal{F}$ has $(r-1)$ edges $F_1, \ldots, F_{r-1}$ with $|F_1 \cap \cdots \cap F_{r-1}| = t$ then all edges in $\mathcal{F}$ must contain this $t$-subset, which proves $W_w(\mathcal{F}) \leq w^t$. Thus we may assume that $\mathcal{F}$ is $(r-1)$-wise $(t+1)$-intersecting. By Lemma 1, we have $W_w(\mathcal{F}) \leq f(w, r-1, t+1) \leq \alpha_{w,r-1}^{t+1} \leq w^t$. □

Using above lemmas, we have the following.

**Lemma 5** *There exists $\epsilon = \epsilon(r)$ such that $f(w,r,t) = w^t$ holds for $1/2 \leq w \leq 1/2 + \epsilon$ in the following cases: $r = 3$ and $t \leq 2$, $r = 4$ and $t \leq 2$, $r = 5$ and $t \leq 7$.*

**Proof.** The case $t = 1$ follows from Lemma 2. The case $r = 3$ and $t = 2$ follows from Lemma 3.

Let us consider the case $r = 4$ and $t = 2$. Since $\alpha_{\frac{1}{2},3} = \frac{\sqrt{5}-1}{2} \approx 0.618$, we have $\alpha_{\frac{1}{2},3}^3 < (\frac{1}{2})^2$. Then, by the continuity, $\alpha_{\frac{1}{2}+\epsilon,3}^3 < (\frac{1}{2}+\epsilon)^2$ holds for sufficiently small $\epsilon > 0$. Thus $f(w,4,2) \leq w^2$ for $\frac{1}{2} \leq w \leq \frac{1}{2} + \epsilon$ follows from Lemma 4. One can prove the case $r = 5$ and $2 \leq t \leq 7$ similarly. □

Note also that

$$\alpha_{\frac{1}{2},r-1}^{t+1} < (\frac{1}{2} + \frac{1}{2^{r-1}})^{t+1} = (\frac{1}{2})^{t+1}(1 + \frac{1}{2^{r-2}})^{t+1} < (\frac{1}{2})^{t+1} \exp(\frac{t+1}{2^{r-2}}),$$

which is smaller than $(1/2)^t$ if $t+1 \le 2^{r-2}\log 2$. This means that $f(w,r,t) = w^t$ holds for $w = 1/2 + \epsilon(r)$ if $t \le 2^{r-2}\log 2 - 1$. We shall use the following weaker version later.

**Proposition 1** *Let $\mathcal{F} \subset 2^{[n]}$ be an $r$-wise $r$-intersecting family. If $r \ge 5$, then there exists $\epsilon = \epsilon(r) > 0$ such that $W_w(\mathcal{F}) \le w^r$ holds for $\frac{1}{2} \le w \le \frac{1}{2}+\epsilon$.*

# 3 Proof of Theorem 2

**Proof.** We prove Theorem 2 by induction on $r$. First we prove the initial step $r = 13$.

**Proposition 2** *Suppose that $\mathcal{F} \subset 2^{[n]}$ is a non-trivial 13-wise intersecting family. Then there exists $\epsilon > 0$ such that $W_w(\mathcal{F}) \le W_w(\mathcal{F}_{BD}^{13})$ holds for $\frac{1}{2} \le w \le \frac{1}{2} + \epsilon$.*

**Proof.** Let $\mathcal{F} \subset 2^{[n]}$ be a non-trivial 13-wise intersecting family. We assume that $\mathcal{F}$ is shifted and (size) maximal. (Recall that $\mathcal{F}$ is called shifted iff $(F - \{j\}) \cup \{i\} \in \mathcal{F}$ holds for all $1 \le i < j \le n$ and for all $F \in \mathcal{F}$ which satisfies $F \cap \{i,j\} = \{j\}$. See [2] for more about shifting.) Note also that if $F \in \mathcal{F}$ and $F \subset G$ then $G \in \mathcal{F}$ because $\mathcal{F}$ is maximal.

Let

$$k := \max\{i : \forall F \in \mathcal{F},\ |F \cap [i+1]| \ge i\}.$$

We can find such $k$, for $|F \cap [1]| \ge 0$ (i.e., the case $i = 0$) is evident. If $k \ge 13$ then $\mathcal{F} \subset \mathcal{F}_{BD}^{13}$. So we may assume that $k \le 12$. Let $t(\ell) := \max\{t : \mathcal{F}$ is $\ell$-wise $t$-intersecting$\}$. Then $1 \le t(13) < t(12) < \cdots < t(6) < \cdots$. This implies $8 \le t(6) < t(5) < t(4)$.

Since $\alpha_{1/2,4} \approx 0.543689$, the weight of 4-wise 12-intersecting family is, by Lemma 1, at most $\alpha_{1/2,4}^{12} \approx 0.000667124$. On the other hand, $W_{1/2}(\mathcal{F}_{BD}^{13}) = 15(1/2)^{14} \approx 0.000915527$. Thus for sufficiently small $\epsilon > 0$ we have $\alpha_{\frac{1}{2}+\epsilon,4}^{12} < W_{\frac{1}{2}+\epsilon}(\mathcal{F}_{BD}^{13})$, because these functions of both sides are continuous with respect to $w = \frac{1}{2} + \epsilon$. This means $W_w(\mathcal{F}) < W_w(\mathcal{F}_{BD}^{13})$ holds for $\frac{1}{2} \le w \le \frac{1}{2} + \epsilon$ if $\mathcal{F}$ is 4-wise 12-intersecting. So we may assume that $\mathcal{F}$ is not 4-wise 12-intersecting, that is, $t(4) \le 11$. Consequently we have $8 \le t(6) < t(5) < t(4) \le 11$, and so $t(6) + 1 = t(5)$ or $t(5) + 1 = t(4)$.

**Lemma 6** *If $t(\ell + 1) + 1 = t(\ell)$ then $k \ge t(\ell + 1)$.*

**Proof.** Set $t := t(\ell+1)$. If $t(\ell) = t+1$ then $\mathcal{F}$ is $\ell$-wise $(t+1)$-intersecting, but $\mathcal{F}$ is not $\ell$-wise $(t+2)$-intersecting. So there exist $F_1, \dots, F_\ell \in \mathcal{F}$ such that $|F_1 \cap \cdots \cap F_\ell| = t+1$. Since $\mathcal{F}$ is shifted, we may assume that $F_1 \cap \cdots \cap F_\ell = [t+1]$. If there exists $F \in \mathcal{F}$ such that $|F \cap [t+1]| \leq t-1$, then $|F \cap F_1 \cap \cdots \cap F_\ell| \leq t-1$ and this means $\mathcal{F}$ is not $(\ell+1)$-wise $t$-intersecting. Thus we must have $|F \cap [t+1]| \geq t$ for all $F \in \mathcal{F}$ and this proves $k \geq t = t(\ell+1)$. □

Using the lemma we have $k \geq t(6)$ if $t(6)+1 = t(5)$, or $k \geq t(5) > t(6)$ if $t(5)+1 = t(4)$. In either case we have $8 \leq t(6) \leq k \leq 12$. For $1 \leq i \leq k+1$ define

$$\mathcal{F}(i) := \{F \in \mathcal{F} : F \cap [k+1] = ([k+1] \setminus \{i\})\},$$

and for $i = 0$ define $\mathcal{F}(0) := \{F \in \mathcal{F} : [k+1] \subset F\}$, and set

$$\mathcal{G}(i) := \{F \cap [k+2, n] : F \in \mathcal{F}(i)\}$$

for $0 \leq i \leq k+1$. Since $\mathcal{F}$ is non-trivial intersecting, shifted and maximal, we have

$$\emptyset \neq \mathcal{G}(1) \subset \mathcal{G}(2) \subset \cdots \subset \mathcal{G}(k+1) \subset \mathcal{G}(0). \tag{1}$$

Note also that

$$W_w(\mathcal{F}) = w^k(1-w) \sum_{i=1}^{k+1} W_w(\mathcal{G}(i)) + w^{k+1} W_w(\mathcal{G}(0)). \tag{2}$$

By the definition of $k$, there exists $F \in \mathcal{F}$ such that $|F \cap [k+2]| \leq k$. Since $\mathcal{F}$ is shifted and maximal, it follows that $E_1 := [n] - \{k+1, k+2\} \in \mathcal{F}$. By shifting $E_1$, we have $E_i := [n] - \{k+i, k+i+1\} \in \mathcal{F}$ for $1 \leq i \leq n-k-1$. Set $s := r-k = 13-k$. We will only use the fact that there exist $\mathcal{F} \ni E_1, \dots, E_{2s}$ such that

$$k+i, k+i+1 \notin E_i \text{ for } i = 1, \dots, 2s.$$

Note that $E_1 \cap E_3 \cap \cdots \cap E_{2j-1} \cap [k+1, k+2j] = \emptyset$, and $E_2 \cap E_4 \cap \cdots \cap E_{2j} \cap [k+2, k+2j+1] = \emptyset$.

**Lemma 7** *$\mathcal{G}(i)$ is $(k+1-i)$-wise $2s$-intersecting for $i = 1, \dots, k-2$.*

**Proof.** Suppose, on the contrary, that $\mathcal{G}(i)$ is not $(k+1-i)$-wise $2s$-intersecting. Then we can find $G_i, G_{i+1}, \dots, G_k \in \mathcal{G}(i)$ such that $|G_i \cap \cdots \cap G_k| \leq 2s-1$. By the shiftedness, we may assume that $G_i \cap \cdots \cap G_k =$

$[k+2, k+2s]$. For $i \le j \le k$, let $F_j := ([k+1] - \{i\}) \cup G_j \in \mathcal{F}(i)$. Applying $(i,j)$-shift to $F_j$ we have

$$F'_j := (F_j \setminus \{j\}) \cup \{i\} \in \mathcal{F}(j) \text{ for } i < j \le k.$$

Set $F'_i := F_i$ and choose $F_j \in \mathcal{F}(j)$ for $j = 1, \dots, i-1$ arbitrarily. Then $F_1 \cap \cdots \cap F_{i-1} \cap F'_i \cap \cdots \cap F'_k \subset [k+2, k+2s]$ and so $F_1 \cap \cdots \cap F_{i-1} \cap F'_i \cap \cdots \cap F'_k \cap E_1 \cap E_3 \cap \cdots \cap E_{2s-1} = \emptyset$. This means that we have $k+s = r$ edges in $\mathcal{F}$ whose intersection is empty and this is a contradiction. □

**Lemma 8**
*$\mathcal{G}(k-1)$ is 3-wise $(2s-1)$-intersecting if $s \ge 1$.*
*$\mathcal{G}(k)$ is 3-wise $(2s-3)$-intersecting if $s \ge 2$.*
*$\mathcal{G}(k+1)$ is 3-wise $(2s-5)$-intersecting if $s \ge 3$.*
*$\mathcal{G}(0)$ is 3-wise $(2s-6)$-intersecting if $s \ge 4$.*

**Proof.** The proof is similar to the previous lemma. For example, suppose that $\mathcal{G}(k-1)$ is not 3-wise $(2s-1)$-intersecting. Then there exist $G_{k-1}, G_k, G_{k+1} \in \mathcal{G}(k-1)$ such that $G_{k-1} \cap G_k \cap G_{k+1} = [k+2, k+2s-1]$. Set $F'_j := ([k+1] - \{j\}) \cup G_j \in \mathcal{F}(j)$ for $j = k-1, k, k+1$ and choose $F_j \in \mathcal{F}(j)$ for $j = 1, \dots, k-2$ arbitrarily. Then $F_1 \cap \cdots \cap F_{k-2} \cap F'_{k-1} \cap F'_k \cap F'_{k+1} \cap E_2 \cap E_4 \cap \cdots \cap E_{2s-2} = \emptyset$, which is a contradiction. The remaining statements can be proved in the same way. □

Recall that $8 \le k \le 12$ and so $1 \le s \le 5$. Let us deal with the hardest case $k = 10$ $(s = 3)$ first.

**Case 1** $k = 10$ $(s = 3)$.

By Lemma 7 and Lemma 8, we get a table representing the $\ell$-wise $t$-intersecting property for $\mathcal{G}(i)$ as follows:

| $\mathcal{G}(i)$ | $\mathcal{G}(6)$ | $\mathcal{G}(7)$ | $\mathcal{G}(8)$ | $\mathcal{G}(9)$ | $\mathcal{G}(10)$ | $\mathcal{G}(11)$ |
|---|---|---|---|---|---|---|
| $\ell$-wise | 5 | 4 | 3 | 3 | 3 | 3 |
| $t$-int. | 6 | 6 | 6 | 5 | 3 | 1 |

By Lemma 5 we have $W_w(\mathcal{G}(6)) \le w^6$. Using (1) we have

$$W_w(\mathcal{G}(1)) + \cdots + W_w(\mathcal{G}(6)) \le 6W_w(\mathcal{G}(6)) \le 6w^6.$$

By Lemma 1, $W_w(\mathcal{G}(7)) \le \alpha_{w,4}^6$ follows. By Lemma 3 we have

$$W_w(\mathcal{G}(8)) + W_w(\mathcal{G}(9)) + W_w(\mathcal{G}(10)) \le w^2\alpha_{w,3}^4 + w^2\alpha_{w,3}^3 + w^2\alpha_{w,3}.$$

By Lemma 2, $W_w(\mathcal{G}(11)) \le w$. For $\mathcal{G}(0)$ we use the trivial bound $W_w(\mathcal{G}(0)) \le 1$. Therefore using (2) we have

$$W_w(\mathcal{F}) \le w^{10}(1-w)\{6w^6+\alpha_{w,4}^6+w^2\alpha_{w,3}^4+w^2\alpha_{w,3}^3+w^2\alpha_{w,3}+w\}+w^{11}. \quad (3)$$

Since $\alpha_{1/2,3} \approx 0.618033$ and $\alpha_{1/2,4} \approx 0.543689$, we have

$$W_{1/2}(\mathcal{F}) \le 0.00091288 < W_{1/2}(\mathcal{F}_{BD}^{13}) \approx 0.000915527.$$

So we can conclude that $W_w(\mathcal{F}) < W_w(\mathcal{F}_{BD}^{13})$ for $w = 1/2 + \epsilon$ because both the RHS of (3) and $W_w(\mathcal{F}_{BD}^{13}) = (14-13w)w^{13}$ are continuous with respect to $w$.

The proof for the cases $k = 12, 11, 9, 8$ is similar (and easier). We give a sketchy proof here.

**Case 2** $k = 12$ $(s = 1)$.

By Lemma 7 and Lemma 8, we have the following table.

| $\mathcal{G}(i)$ | $\mathcal{G}(10)$ | $\mathcal{G}(11)$ |
|---|---|---|
| $\ell$-wise | 3 | 3 |
| $t$-int. | 2 | 1 |

Therefore, we have

$$W_w(\mathcal{F}) \le w^{12}(1-w)\{10w^2 + w + 1 + 1\} + w^{13},$$

and $W_{1/2}(\mathcal{F}) \le 0.000732422 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

**Case 3** $k = 11$ $(s = 2)$.

By Lemma 7 and Lemma 8, we have the following table.

| $\mathcal{G}(i)$ | $\mathcal{G}(9)$ | $\mathcal{G}(10)$ | $\mathcal{G}(11)$ |
|---|---|---|---|
| $\ell$-wise | 3 | 3 | 3 |
| $t$-int. | 4 | 3 | 1 |

Therefore, we have

$$W_w(\mathcal{F}) \le w^{11}(1-w)\{9w^2\alpha_{w,3}^2 + w^2\alpha_{w,3} + w + 1\} + w^{12},$$

and $W_{1/2}(\mathcal{F}) \le 0.000857893 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

**Case 4** $k = 9$ $(s = 4)$.

By Lemma 7 and Lemma 8, we have the following table.

| $\mathcal{G}(i)$ | $\mathcal{G}(9)$ | $\mathcal{G}(10)$ | $\mathcal{G}(0)$ |
|---|---|---|---|
| $\ell$-wise | 3 | 3 | 3 |
| $t$-int. | 5 | 3 | 2 |

Therefore, we have

$$W_w(\mathcal{F}) \leq w^9(1-w)\{9w^2\alpha_{w,3}^3 + w^2\alpha_{w,3}\} + w^{10} \cdot w^2,$$

and $W_{1/2}(\mathcal{F}) \leq 0.000913729 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

**Case 5** $k = 8$ $(s = 5)$.

By Lemma 7 and Lemma 8, we have the following table.

| $\mathcal{G}(i)$ | $\mathcal{G}(8)$ | $\mathcal{G}(9)$ | $\mathcal{G}(0)$ |
|---|---|---|---|
| $\ell$-wise | 3 | 3 | 3 |
| $t$-int. | 7 | 5 | 4 |

Therefore, we have

$$W_w(\mathcal{F}) \leq w^8(1-w)\{8w^2\alpha_{w,3}^5 + w^2\alpha_{w,3}^3\} + w^9 \cdot w^2\alpha_{w,3}^2,$$

and $W_{1/2}(\mathcal{F}) \leq 0.000653997 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

This completes the proof of Proposition 2. $\square$

Now we are going back to the proof of the theorem. Let $\mathcal{F}$ be a non-trivial $r$-wise intersecting family. To apply induction, we suppose $r > 13$. We also suppose that $\mathcal{F}$ is shifted and maximal. Let us define

$$\mathcal{F}(1) := \{F - \{1\} : 1 \in F \in \mathcal{F}\}, \quad \mathcal{F}(\bar{1}) := \{F \in \mathcal{F} : 1 \notin F\}.$$

Since $\mathcal{F}$ is non-trivial intersecting and maximal, we have $[2, n] \in \mathcal{F}(\bar{1})$. By shifting $[2, n]$, we have $[n] - \{i\} \in \mathcal{F}$ for $1 \leq i \leq n$. Thus $\bigcap_{F \in \mathcal{F}(1)} F = \emptyset$. Since $\mathcal{F}$ is $r$-wise intersecting and $[2, n] \in \mathcal{F}$, it follows that $\mathcal{F}(1)$ is a non-trivial $(r-1)$-wise intersecting family. Thus using the induction hypothesis we have $W_w(\mathcal{F}(1)) \leq W_w(\mathcal{F}_{BD}^{r-1}) = w^{r-1}(r - (r-1)w)$.

On the other hand, $\mathcal{F}(\bar{1})$ is $r$-wise $r$-intersecting. To see this fact, suppose on the contrary that there exist $F_1, \ldots, F_r \in \mathcal{F}(\bar{1})$ such that $|F_1 \cap \cdots \cap F_r| < r$. Since $\mathcal{F}$ is shifted, we may assume that $F_1 \cap \cdots \cap F_r = [2, r]$. Then $F_i' := (F_i - \{i\} \cup \{1\}) \in \mathcal{F}$ for $2 \le i \le r$, and $F_1 \cap F_2' \cap \cdots \cap F_r' = \emptyset$, a contradiction. Therefore $\mathcal{F}(\bar{1})$ is $r$-wise $r$-intersecting and using Proposition 1 we have $W_w(\mathcal{F}(\bar{1})) \le w^r$. Consequently it follows that

$$\begin{aligned} W_w(\mathcal{F}) &= wW_w(\mathcal{F}(1)) + (1-w)W_w(\mathcal{F}(\bar{1})) \\ &\le w(w^{r-1}(r-(r-1)w)) + (1-w)w^r \\ &= w^r(r+1-rw) = W_w(\mathcal{F}^r_{BD}). \end{aligned}$$

This completes the proof of Theorem 2. □

# 4 Constructions

First we check that Theorem 2 fails if $r = 5$. Recall that $W_w(\mathcal{F}^r_{BD}) = (r+1-rw)w^r$.

**Example 1** We construct a non-trivial 5-wise intersecting family $\mathcal{F} \subset 2^{[n]}$ as follows:

$$\mathcal{F} := \{\{1,2,3\} \cup G : G \subset [4,n], |G| \ge \lceil \tfrac{n-2}{2} \rceil\} \cup \{F_1, F_2, F_3\},$$

$$\text{where } F_i = [n] \setminus \{i\}.$$

Then $\lim_{n\to\infty} W_w(\mathcal{F}) = w^3$ for $w > 1/2$. This implies $g(w,5,1) \ge w^3 > W_w(\mathcal{F}^5_{BD}) = (6-5w)w^5$ for $1/2 < w < \frac{1+\sqrt{21}}{10}$.

Using the fact that $\binom{[n]}{k}$ is $r$-wise $t$-intersecting if $(r-1)n + (t-1) < rk$, we can extend the above construction to get a slightly general lower bound for $g(w,r,t)$ as follows.

**Proposition 3** *If* $\frac{r-(i+1)}{r-i} < w$ *then* $g(w,r,t) \ge w^{it}$*, where* $i$ *is a non-negative integer.*

**Proof.** For sufficiently small $\epsilon > 0$, we may assume that $\frac{r-(i+1)}{r-i} < (1-\epsilon)w$. Moreover, for sufficiently large $n$, we may assume that $\frac{r-(i+1)}{r-i} + \frac{t-1}{(r-i)(n-it)} < (1-\epsilon)w$. Set an open interval $I = ((1-\epsilon)wn, (1+\epsilon)wn)$ and choose an integer $k \in I$, then $(1-\epsilon)w < k/n < k/(n-it)$. Thus, $\frac{r-(i+1)}{r-i} + \frac{t-1}{(r-i)(n-it)} < \frac{k}{n-it}$, or equivalently, $(r-(i+1))(n-it)+(t-1) < (r-i)k$. This means that $\binom{[it+1,n]}{k}$ is a non-trivial $(r-i)$-wise $t$-intersecting family. Therefore, the family

$$\mathcal{F} := \{[it] \cup G : G \in \binom{[it+1,n]}{k}, k \in I\} \cup \{[n] - [jt+1,(j+1)t] : 0 \leq j < i\}$$

is non-trivial $r$-wise $t$-intersecting, and

$$g_n(w,r,t) \geq W_w(\mathcal{F}) = w^{it} \sum_{k \in I} \binom{n-it}{k} w^k (1-w)^{n-it-k} + i(1-w)^t w^{n-t} \to w^{it}$$

as $n \to \infty$. □

Using the above proposition, Theorem 1 and Lemma 2, we have the following.

**Example 2**
$f(w,r,t) = g(w,r,t) = 1$ if $w > (r-1)/r$.

$$g(w,3,1) = \begin{cases} 5/16 & \text{if} \quad w = 1/2 \\ w & \text{if} \quad 1/2 < w \leq 2/3 \\ 1 & \text{if} \quad 2/3 < w \leq 1. \end{cases}$$

$$g(w,4,1) = \begin{cases} 3/16 & \text{if} \quad w = 1/2 \\ \geq w^2 & \text{if} \quad 1/2 < w \leq \frac{1+\sqrt{17}}{8} \\ \geq (5-4w)w^4 & \text{if} \quad \frac{1+\sqrt{17}}{8} \leq w \leq 2/3 \\ w & \text{if} \quad 2/3 < w \leq 3/4 \\ 1 & \text{if} \quad 3/4 < w \leq 1. \end{cases}$$

$$g(w,5,1) = \begin{cases} 7/64 & \text{if} \quad w = 1/2 \\ \geq w^3 & \text{if} \quad 1/2 < w \leq \frac{1+\sqrt{21}}{10} \\ \geq (6-5w)w^5 & \text{if} \quad \frac{1+\sqrt{21}}{10} \leq w \leq 2/3 \\ \geq w^2 & \text{if} \quad 2/3 < w \leq 3/4 \\ w & \text{if} \quad 3/4 < w \leq 4/5 \\ 1 & \text{if} \quad 4/ < w \leq 1. \end{cases}$$

# References


[1] A. Brace and D. E. Daykin. A finite set covering theorem. I,II,III,IV. *Bull. Austral. Math. Soc.*, 5:197–202, 1971, 6:19–24, 417–433, 1972, *Infinite and finite sets*, I:199–203, 1975.

[2] P. Frankl, N. Tokushige. Weighted 3-wise 2-intersecting families. *J. Combin. Theory (A)* 100:94-115, 2002.

[3] P. Frankl, N. Tokushige. Weighted multiply intersecting families. *Studia Sci. Math. Hungarica* 40:287–291, 2003.

[4] P. Frankl, N. Tokushige. Random walks and multiply intersecting families. *J. Combin. Theory (A)*, 109:121-134, 2005.