Random walks and multiply intersecting families

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Abstract

Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. It is proved that

$$|\mathcal{F}| \leq \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even,} \\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd} \end{cases}$$

holds for $n \geq n_0$. The unique extremal configuration is determined as well.

1 Introduction

Let n, r and t be positive integers. A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. An r-wise t-intersecting family $\mathcal{F}$ is called trivial if $|\bigcap_{F \in \mathcal{F}} F| \geq t$ holds. For a real $w \in (0, 1)$ let us define the weighted size $W_w(\mathcal{F})$ of $\mathcal{F}$ by

$$W_w(\mathcal{F}) := \sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|}.$$

Some basic results concerning the maximum weighted size of multiply intersecting families can be found in [6, 7, 8]. Among others, the following is proved in [7].

Theorem 1 *Let $\mathcal{F}$ be a 3-wise 2-intersecting family. Then $W_w(\mathcal{F}) \leq w^2$ if $w < 0.5018$.*

Moreover if $W_w(\mathcal{F}) \geq 0.999w^2$, then $\mathcal{F}$ contains a certain configuration, which we will explain later (see Theorem 5 in section 4). Using this result, the following variation of the Erdős–Ko–Rado theorem [2, 1] is deduced.

Theorem 2 *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family with $k/n \leq 0.501$, $n > n_0$. Then $|\mathcal{F}| \leq \binom{n-2}{k-2}$, and equality holds only if $\mathcal{F}$ is trivial.*

For the proof of the above result, we use the "random walk method." The main tool is Theorem 4 described in the next section.

A family $\mathcal{F} \subset 2^{[n]}$ is called a Sperner family if $F \not\subset G$ holds for all distinct $F, G \in \mathcal{F}$. As an application of Theorem 2, we prove the following result.

Theorem 3 *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. Then,*

$$|\mathcal{F}| \leq \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even}, \\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd}, \end{cases}$$

holds for $n \geq n_0$. The extremal configurations are

$$\mathcal{F} = \{\{1,2\} \cup F : F \in \binom{[3,n]}{(n-2)/2}\} \qquad n \text{ even},$$
$$\mathcal{F} = \{\{1,2\} \cup F : F \in \binom{[3,n]}{(n-1)/2}\} \cup \{[n] - \{1\}\} \cup \{[n] - \{2\}\} \qquad n \text{ odd}.$$

Since $\mathcal{F} = \binom{[8]}{6}$ is 3-wise 2-intersecting Sperner and $|\mathcal{F}| = \binom{8}{6} > \binom{6}{3}$, the condition $n > n_0$ in the above theorem can not be omitted completely. It is an interesting but difficult problem to determine how small n_0 can be.

Other results concerning the maximum size of r-wise t-intersecting Sperner families can be found in [16] for the case $r = 2$, and in [3, 9, 10, 11, 12] for the case $r \geq 3$ and $t = 1$.

2 Tools

2.1 Shifting

For integers $1 \le i < j \le n$ and a family $\mathcal{F} \subset 2^{[n]}$, define the (i,j)-shift S_{ij} as follows.

$$S_{ij}(\mathcal{F}) := \{S_{ij}(F) : F \in \mathcal{F}\},$$

where

$$S_{ij}(F) := \begin{cases} (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\, j \in F,\, (F - \{j\}) \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

A family $\mathcal{F} \subset 2^{[n]}$ is called shifted if $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \le i < j \le n$. We call $\mathcal{F}$ a co-complex if $G \supset F \in \mathcal{F}$ implies $G \in \mathcal{F}$.

Let us introduce a partial order in $2^{[n]}$ by using shifting. Let $A, B \subset [n]$. Define $A \succ B$ if there exists $A' \subset [n]$ such that $A \subset A'$ and B is obtained by repeating a shifting to A'. The following fact is trivial but useful.

Fact 1 *Let $\mathcal{F} \subset 2^{[n]}$ be a shifted co-complex. If $A \in \mathcal{F}$ and $A \succ B$, then $B \in \mathcal{F}$.*

2.2 Random walk

Let $w \in (0, 2/3]$ be a fixed real number, and let $\alpha \in (0,1)$ be the root of the equation $(1-w)x^3 - x + w = 0$, more explicitly, $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$. Note that $\alpha = \alpha(w)$ is an increasing function of w and $\alpha(0) = 0$, $\alpha(2/3) = 1$. Consider the infinite random walk, starting from the origin, in which at each step we move one unit up with probability w or move one unit right with probability $1 - w$. Then the probability that we ever hit the line $y = 2x + s$ is given by α^s where s is a non-negative integer. (See [4] for details.)

Let $F \in \mathcal{F} \subset 2^{[n]}$. We define the corresponding (finite) walk to F, denoted by walk(F), in the following way. If $i \in F$ (resp. $i \notin F$) then we move one unit up (resp. one unit right) at the i-th step. Note that $F \succ G$ means walk(G) is in the area to the upper left of walk(F). The following fact shows how to use random walks to estimate the weighted size of a family.

Fact 2 *Let $\mathcal{F} \subset 2^{[n]}$, and suppose that, for all $F \in \mathcal{F}$,* walk(F) *touches the line $y = 2x + s$. Then $W_w(\mathcal{F}) \le \alpha^s$.*

Now we give a variation of the above fact for the size of a uniform family, which we will use to prove Theorem 2.

Theorem 4 *Let $w \in \mathbb{R}$, $d \in \mathbb{Q}$, $s \in \mathbb{N}$ be fixed constants with $0 < d \le w \le 2/3$, and set $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$. Let $\mathcal{F} \subset \binom{[n]}{k}$ with $d = k/n$, $k > s$. Suppose that, for all $F \in \mathcal{F}$, $\mathrm{walk}(F)$ touches the line $y = 2x + s$. Then we have the following.*

(i) *For every $\epsilon > 0$, $|\mathcal{F}|/\binom{n}{k} \le (1+\epsilon)\alpha^s$ holds for $n > n_0(\epsilon)$.*

(ii) *If $w \le 0.51$ then $|\mathcal{F}|/\binom{n}{k} \le \alpha^s$ for $n > n_0$.*

Conjecture 1 *Theorem 4 (i) is true for $\epsilon = 0$ (or equivalently, (ii) is true for all $w \le 2/3$).*

2.3 Shadow

For a family $\mathcal{F} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓ-th shadow of $\mathcal{F}$, denoted by $\Delta_\ell(\mathcal{F})$, as follows.

$$\Delta_\ell(\mathcal{F}) := \{G \in \binom{[n]}{\ell} : G \subset \exists F \in \mathcal{F}\}.$$

We use the following version of the Kruskal–Katona theorems[15, 14, 5]:

Proposition 1 *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $|\mathcal{F}| \le \binom{m}{k}$. Then,*

$$|\Delta_\ell(\mathcal{F})| \ge |\mathcal{F}| \binom{m}{\ell} / \binom{m}{k}.$$

Equality holds only if $\mathcal{F} = \binom{Y}{k}$, $|Y| = m$.

We also use the following Katona's shadow theorem for t-intersecting families [13].

Proposition 2 *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is 2-wise t-intersecting, and $n \ge 2k - t$, $k > l \ge k - t$. Then,*

$$|\Delta_\ell(\mathcal{F})| \ge |\mathcal{F}| \binom{2k-t}{\ell} / \binom{2k-t}{k}.$$

Equality holds only if $\mathcal{F} = \binom{Y}{k}$, $|Y| = 2k - t$.

3 Proof of Theorem 4

If $w = 2/3$ then $\alpha = 1$ and the theorem is trivial in this case. So we assume that $w < 2/3$. Since the theorem clearly holds for $s = 0$ also, we may assume that $s \geq 1$. For each $i = 0, 1, \ldots, \lfloor\frac{k-s}{2}\rfloor$ let a_i be the number of walks of length $3i + s$, which attain the line L: $y = 2x + s$ at $(i, 2i + s)$ for the first time. Then the total number of walks from $(0, 0)$ to $(n - k, k)$ that attain L is

$$\sum_{i=0}^{\lfloor\frac{k-s}{2}\rfloor} a_i \binom{n-3i-s}{k-2i-s}. \tag{1}$$

To obtain the probability that a walk attains the line, we have to divide (1) by $\binom{n}{k}$.

Next consider a walk where each step is chosen independently and randomly with probability w for one step up and probability $1 - w$ for one step right. Then the probability for this random walk to attain the line by n steps is

$$\sum_{i=0}^{\lfloor\frac{k-s}{2}\rfloor} a_i w^{2i+s}(1-w)^i. \tag{2}$$

Recall that the above probability is less than α^s, where $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$.

Comparing (1) and (2), Theorem 4 (i) will be proved as soon as we establish the following inequality for all $0 \leq i \leq \lfloor\frac{k-s}{2}\rfloor$, $n > n_0(\epsilon)$:

$$\binom{n-3i-s}{k-2i-s} \Big/ \binom{n}{k} \leq (1+\epsilon) w^s \left\{w^2(1-w)\right\}^i.$$

This is certainly true for $i = 0$ (even if $\epsilon = 0$) because $\binom{n-s}{k-s}/\binom{n}{k} \leq (k/n)^s \leq w^s$. Note that $\binom{n-3i-s}{k-2i-s}/(\binom{n}{k}w^s)$ is a decreasing function of s. So it suffices to prove the above inequality for $s = 1$, that is,

$$\frac{k}{n} \prod_{j=0}^{i-1} \frac{(k-2j-1)(k-2j-2)(n-k-j)}{(n-3j-1)(n-3j-2)(n-3j-3)} \leq (1+\epsilon) w \left\{w^2(1-w)\right\}^i$$

for $1 \leq i \leq \lfloor\frac{k-s}{2}\rfloor$, $n > n_0(\epsilon)$. Since $d \leq w$ and $w^2(1-w)$ is an increasing function of w for $0 \leq w \leq 2/3$, we have $d(d^2(1-d))^i \leq w(w^2(1-w))^i$. Thus, it is sufficient to prove the case $d = w$, that is

$$\prod_{j=0}^{i-1} f(j) \le (1+\epsilon)\{d^2(1-d)\}^i \tag{3}$$

where

$$f(j) = \frac{(dn-2j-1)(dn-2j-2)(n-dn-j)}{(n-3j-1)(n-3j-2)(n-3j-3)}.$$

Here let us check that $f(j)$ is a decreasing function of j for $0 \le j \le i-1 \le \frac{k-1}{2} - 1 = \frac{dn-3}{2}$. Set $g(j) = f'(j)(n-3j-1)^2(n-3j-2)^2(n-3j-3)^2$, and $g'(j) = 2(n-3j-2)h(j)$. Then $h(j) = -36j^2 + O(j)$, $h(0) = (2-3d)^2(1+3d)n^3 + O(n^2) > 0$ and $h(dn/2) = (1/2)(2-3d)^3n^3 + O(n^2) > 0$. Note that $h(j)$ is a concave parabola as a function of j, and the both ends $(j = 0, dn/2)$ have positive value. This means $h(j) > 0$ and $g'(j) > 0$ for $0 \le j \le dn/2$. Then $g(\frac{dn-3}{2}) = -\frac{3}{8}(2-3d)^4n^4 + O(n^3) < 0$ implies $g(j) < 0$ and so $f'(j) < 0$ for $0 \le j \le \frac{dn-3}{2}$.

Thus, we have $\prod_{j=0}^{i-1} f(j) \le f(0)^i$. If $d \le 1/2$ then one can check $f(0) < d^2(1-d)$ for n sufficiently large, and so $\prod_{j=0}^{i-1} f(j) < (d^2(1-d))^i$ follows. This is stronger than (3). Now we may assume that $d > 1/2$.

If $j \ge \sqrt{n}$ then for $n > n_0$ we have

$$f(j) \le d^2(1-d). \tag{4}$$

In fact, for $j = \sqrt{n}$, we have

$$d^2(1-d)D - N = d(2-3d)^2n^{5/2} + O(n^2) > 0$$

where D and N stand for the denominator and the numerator of $f(j)$.

Since

$$\lim_{n\to\infty} \left(\frac{f(0)}{d^2(1-d)} \right)^{\sqrt{n}} = 1,$$

we have

$$\prod_{j=0}^{\sqrt{n}-1} f(j) \le f(0)^{\sqrt{n}} < (1+\epsilon)\{d^2(1-d)\}^{\sqrt{n}}. \tag{5}$$

If $i > \sqrt{n}$ then by (4) and (5) we have

$$\prod_{j=0}^{i-1} f(j) = \left(\prod_{j=0}^{\sqrt{n}-1} f(j) \right) \left(\prod_{j=\sqrt{n}}^{i-1} f(j) \right) \le (1+\epsilon)\{d^2(1-d)\}^i.$$

So we may assume that $i \leq \sqrt{n}$. Since $d > 1/2$ and $n > n_0$, we have $f(0) > d^2(1-d)$ and

$$\left(\frac{f(0)}{d^2(1-d)}\right)^i \leq \left(\frac{f(0)}{d^2(1-d)}\right)^{\sqrt{n}} < 1+\epsilon.$$

Therefore, $\prod_{j=0}^{i-1} f(j) \leq f(0)^i < (1+\epsilon)(d^2(1-d))^i$ follows. This completes the proof of (i).

Now we prove (ii). For $d \leq 1/2$, we have proved $f(0) < d^2(1-d)$ and this implies the desired inequality. So we assume $d > 1/2$. Then $f(0) > d^2(1-d)$. However, for $j \geq 1$ and $d < 0.547$, we still have $f(j) \leq d^2(1-d)$ because

$$d^2(1-d) - f(1) = \{d(15d^2 - 21d + 7)n^2 + O(n)\}/\{n^3 + O(n^2)\}.$$

In the same way, one can prove $f(0)f(1) \leq \{d^2(1-d)\}^2$ for $d < 0.529$ because

$$\{d^2(1-d)\}^2 - f(0)f(1) = \frac{d^3(1-d)(21d^2 - 30d + 10)n^5 + O(n^4)}{n^6 + O(n^5)}.$$

Therefore, we have

$$\prod_{j=0}^{i-1} f(j) \leq \{d^2(1-d)\}^i \tag{6}$$

for $i \geq 2$. Our goal is to prove

$$\sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} a_i \prod_{j=0}^{i-1} f(j) \leq \sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} a_i \{d^2(1-d)\}^i. \tag{7}$$

To deal with the case $i = 1$, we show the following for $d < 0.515$:

$$a_1 f(0) + a_2 f(0)f(1) \leq a_1 d^2(1-d) + a_2 d^4(1-d)^2. \tag{8}$$

Since $a_1 = 1$, $a_2 = 3$, the above inequality follows from the fact that RHS $-$ LHS is

$$\{3d(1-d)^2(1-d+9d^2-21d^3)n^5 + O(n^4)\}/\{n^6 + O(n^5)\}.$$

Finally (7) follows from (6) and (8). This completes the proof of (ii). $\square$

In principle, one can verify whether

$$\sum_{i=1}^{p} a_i \prod_{j=0}^{i-1} f(j) \leq \sum_{i=1}^{p} a_i \{d^2(1-d)\}^i \tag{9}$$

is true or not for any concrete p, and (8) is the case $p = 2$. The larger p we take, the better bound for d we can get if (9) is true. For example, taking $p = 42$ we can verify (9) (with the aid of computer) for $d \leq 0.6$, this shows that Conjecture 1 is true for $d \leq 0.6$.

4 Proof of Theorem 2

Let us define the following.

$$\begin{aligned} *(i) &:= \{i, i+1, i+3, i+4, i+6, i+7, \ldots\} \cap [n] \\ &= [n] - ([i-1] \cup \{i+3j+2 : 0 \leq j \leq \lfloor \frac{n-i-2}{3} \rfloor\}) \\ P_i &:= \{1,2\} \cup *(i+4). \end{aligned}$$

Note that $*(i) \cap *(i+1) \cap *(i+2) = \emptyset$, and $P_i \cap P_{i+1} \cap P_{i+2} = \{1,2\}$. In [7] the following is proved (see the first paragraph of the proof of Proposition 4 on page 111 in [7]).

Theorem 5 *Let $\mathcal{G} \subset 2^{[n]}$ be a 3-wise non-trivial 2-intersecting shifted co-complex. If $W_w(\mathcal{G}) \geq 0.999w^2$ and $w \leq 0.5015$ then, for some $i \geq 1$, $\mathcal{G}$ contains $P_0, P_1, \ldots, P_i$ but does not contain P_{i+1}.*

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family. If $\mathcal{F}$ fixes a 2-element set, then $|\mathcal{F}| \leq \binom{n-2}{k-2}$. So we may assume that $\mathcal{F}$ is non-trivial. We shall prove that $|\mathcal{F}| < \binom{n-2}{k-2}$. Suppose that $|\mathcal{F}| \geq 0.999\binom{n-2}{k-2}$, and set $w := 0.5015$. Define $\mathcal{F}^c := \{[n] - F : F \in \mathcal{F}\}$ and

$$\mathcal{G} := \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathcal{F}^c))^c \ \ (\subset \bigcup_{i=k}^{n} \binom{[n]}{i}).$$

Clearly $\mathcal{G}$ is a non-trivial 3-wise 2-intersecting family. Let us show that $W_w(\mathcal{G}) > 0.999w^2$ if n is sufficient large.

Choose $\epsilon > 0$ sufficiently small so that

$$0.9998(1-\epsilon)^4 > 0.999, \tag{10}$$

$$0.501 < (1-\epsilon)w. \tag{11}$$

Define an open interval $I := ((1-\epsilon)wn, (1+\epsilon)wn)$. Set $v = 1-w$ and choose $n_0 = n_0(\epsilon)$ sufficiently large so that

$$\sum_{i\in I} \binom{n}{i} w^i v^{n-i} > 1-\epsilon \quad \text{for all } n > n_0, \tag{12}$$

$$(((1-\epsilon)wn-1)/n)^2 > (1-\epsilon)^3 w^2 \quad \text{for all } n > n_0. \tag{13}$$

By our assumption on k/n and (11), we have $k \leq 0.501n < (1-\epsilon)wn$, and

$$W_w(\mathcal{G}) = \sum_{i=k}^{n} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i} \geq \sum_{i\in I} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}.$$

It follows from the Kruskal–Katona theorem that $|\Delta_{n-i}(\mathcal{F}^c)| \geq 0.9998\binom{n-2}{n-i}$ for $i \in I$. (This is Lemma 7 on page 112 in [7].) Therefore,

$$\begin{aligned}
W_w(\mathcal{G}) &\geq 0.9998 \sum_{i\in I} \binom{n-2}{n-i} w^i v^{n-i} \\
&= 0.9998 \sum_{i\in I} \frac{i}{n} \cdot \frac{i-1}{n-1} \binom{n}{i} w^i v^{n-i} \\
&> 0.9998(1-\epsilon)^3 w^2 \sum_{i\in I} \binom{n}{i} w^i v^{n-i} \quad \text{(by (13))} \\
&> 0.9998(1-\epsilon)^4 w^2 \quad \text{(by (12))} \\
&> 0.999w^2 \quad \text{(by (10))}.
\end{aligned}$$

This completes the proof of $W_w(\mathcal{G}) > 0.999w^2$.

So by Theorem 5 we may assume that $P_i \in \mathcal{G}$, $P_{i+1} \notin \mathcal{G}$, for some $i \geq 1$. Let us define the following.

$$\begin{aligned}
Q_i &:= \{1, 2, i+4\} \cup *(i+6), \\
\mathcal{F}_{12} &:= \{F \in \mathcal{F} : \{1,2\} \subset F\}, \\
\mathcal{F}_{1\bar{2}} &:= \{F \in \mathcal{F} : 1 \in F, 2 \notin F\}, \\
\mathcal{F}_{\bar{1}2} &:= \{F \in \mathcal{F} : 1 \notin F, 2 \in F\}, \\
\mathcal{F}_{\bar{1}\bar{2}} &:= \{F \in \mathcal{F} : 1 \notin F, 2 \notin F\}.
\end{aligned}$$

By definition, it follows that $P_{i+1} \succ Q_i \succ P_i$, $|\mathcal{F}| = |\mathcal{F}_{12}| + |\mathcal{F}_{1\bar{2}}| + |\mathcal{F}_{\bar{1}2}| + |\mathcal{F}_{\bar{1}\bar{2}}|$. Set $d = k/n$ ($d \leq 0.501$), and $\alpha = \frac{1}{2}(\sqrt{\frac{1+3d}{1-d}} - 1)$. (Redefine $w := d$.)

Case 1 $Q_i \notin \mathcal{G}$.
If $4i + 4 \geq n$ then we have $R = [i+2] \cup \{i+3, i+6, i+9, \ldots\} \in \mathcal{G}$ because $\mathcal{G} \ni P_i \succ R$. But this is impossible because $P_i \cap R = \{1, 2\}$ implies $\mathcal{G}$ is trivial. So we may assume that $n \geq 4i + 5$.

Observe that walk(Q_i) starts with "up, up," and $i+1$ "right," then from $(i+1, 2)$ this walk is the maximal walk which does not touch the line L: $y = 2(x - (i+1)) + 4$.

Let $F \in \mathcal{F}_{12}$, then walk(F) starts with "up, up." If walk(F) goes through the point $(i+1, 2)$, then this walk must meet the line L after passing $(i+1, 2)$. To apply Theorem 4, it is convenient to neglect the first $i+3$ moves (up, up, and then $i+1$ times right) from walk(F), in other words, we shift the origin to $(i+1, 2)$. Then the modified walk corresponding to $F - \{1, 2\} \subset \binom{[3,n]}{k-2}$, starting from the new origin, must touch the line $y = 2x + 2$. Therefore, by Theorem 4 (ii), the number of walks of this type is at most $\alpha^2 \binom{n-i-3}{k-2}$. Otherwise walk($F$) must go through one of $(0, i+3), (1, i+2), \ldots, (i, 3)$, and the number of corresponding walks is $\binom{n-2}{k-2} - \binom{n-i-3}{k-2}$. Thus, we have

$$\begin{aligned} |\mathcal{F}_{12}| &\leq \binom{n-2}{k-2} - \binom{n-i-3}{k-2} + \alpha^2 \binom{n-i-3}{k-2} \\ &= \binom{n-2}{k-2} \{1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}} (1 - \alpha^2)\}. \end{aligned}$$

To obtain an upper bound for $|\mathcal{F}_{1\bar{2}}|$, let us set

$$\begin{aligned} F_0 &:= [1, i+3] \cup \{i+6, i+9, i+12, \ldots, 4i, 4i+3\} \cup *(4i+5), \\ G &:= \{1\} \cup [3, 4i+4] \cup *(4i+6). \end{aligned}$$

Since $P_i \in \mathcal{G}$ and $P_i = \{1, 2\} \cup *(i+4) = \{1, 2\} \cup \{i+4, i+5, i+7, i+8, \ldots, 4i+1, 4i+2\} \cup *(4i+4) \succ F_0$, we have $F_0 \in \mathcal{G}$. Note that $P_i \cap F_0 \cap G = \{1\}$. Thus $G \notin \mathcal{G}$ follows from the assumption that $\mathcal{G}$ is 3-wise 2-intersecting. Now let us look at walk(G). This walk starts with "up, right," then from $(1, 1)$ this is the maximal walk which does not touch the line L: $y = 2(x-1) + (4i+4)$. Since $G \notin \mathcal{G}$, for every $F \in \mathcal{F}_{1\bar{2}}$, walk($F$) must touch the line L. To apply Theorem 4, we neglect the first two moves (up, right) from walk(F), or equivalently, we shift the origin to $(1,1)$. Then the modified walk corresponding to $F-\{1\} \subset \binom{[3,n]}{k-1}$, starting from the new origin, must touch the line $y = 2x + (4i+3)$. Then due to Theorem 4 (ii), we have

$$|\mathcal{F}_{1\bar{2}}| \le \binom{n-2}{k-1}\alpha^{4i+3} = \binom{n-2}{k-2}\frac{n-k}{k-1}\alpha^{4i+3}.$$

The same estimation is valid for $|\mathcal{F}_{\bar{1}2}|$. From now on, we will use the above trick (shifting the origin) without mentioning when we apply Theorem 4.

Next, set $H := [3, 4i+7] \cup *(4i+9)$. Since $P_i \cap F_0 \cap H = \{4i+5\}$, we have $H \notin \mathcal{G}$, which implies

$$|\mathcal{F}_{\bar{1}\bar{2}}| \le \binom{n-2}{k}\alpha^{4i+6} = \binom{n-2}{k-2}\frac{(n-k)(n-k-1)}{k(k-1)}\alpha^{4i+6}.$$

Therefore, $|\mathcal{F}| \le c\binom{n-2}{k-2}$ where

$$c = 1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha^2) + \frac{2(n-k)}{k-1}\alpha^{4i+3} + \frac{(n-k)(n-k-1)}{k(k-1)}\alpha^{4i+6}.$$

Let us check $c < 1$ for $n > n_0$. The target inequality can be rewritten to

$$2\alpha^3 + \frac{(1-d)n-1}{dn}\alpha^6 < (1-\alpha^2)\frac{dn-1}{n-2}\prod_{j=1}^{i}\frac{(1-d)n-j}{(n-j-2)\alpha^4}. \tag{14}$$

Since $d \le 0.501$ and $j \le i \le \frac{n-5}{4}$, we have $\frac{(1-d)n-j}{(n-j-2)\alpha^4} > 1$. So the RHS of (14) is minimal when $i = 1$, and to prove the inequality for $n > n_0$ it suffices to show

$$2\alpha^3 + \frac{1-d}{d}\alpha^6 < \frac{(1-\alpha^2)\,d\,(1-d)}{\alpha^4}$$

and this is true for $d \le 0.528$. (To verify this, reduce $f(d) := d\alpha^4(\text{RHS}-\text{LHS})$ by using $(1-d)\alpha^3 - \alpha + d = 0$. Then one can check that $g(d) := f(d)(1-d)^3$ has two real zeros, i.e., $d = 0$ and $d = 0.528\ldots$, and moreover $g(d) > 0$ inside this interval.)

Case 2 $Q_i \in \mathcal{G}$.
If $4i+6 \ge n$ then we have $R = [i+3] \cup \{i+5, i+8, i+11, \ldots\} \in \mathcal{G}$ because $\mathcal{G} \ni Q_i \succ R$. But this is impossible because $Q_i \cap R = \{1,2\}$ implies $\mathcal{G}$ is trivial. So we may assume that $n \ge 4i+7$.

Since $P_{i+1} \notin \mathcal{G}$, we have

$$\begin{aligned} |\mathcal{F}_{12}| &\le \binom{n-2}{k-2} - \binom{n-i-3}{k-2} + \alpha\binom{n-i-3}{k-2} \\ &= \binom{n-2}{k-2}\{1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha)\}. \end{aligned}$$

Set

$$\begin{aligned} F &:= [1, i+3] \cup \{i+5, i+8, i+11, \dots, 4i+5\} \cup *(4i+7), \\ G &:= \{1\} \cup [3, 4i+6] \cup *(4i+8). \end{aligned}$$

Since $Q_i \in \mathcal{G}$ and $Q_i = \{1,2\} \cup \{i+4, i+6, i+7, \dots, 4i+3, 4i+4\} \cup *(4i+6) \succ F$, we have $F \in \mathcal{G}$. Note that $Q_i \cap F \cap G = \{1\}$. Thus $G \notin \mathcal{G}$ follows from the assumption that $\mathcal{G}$ is 3-wise 2-intersecting. Therefore,

$$|\mathcal{F}_{1\bar{2}}| \le \binom{n-2}{k-1}\alpha^{4i+5}.$$

The same estimation is valid for $|\mathcal{F}_{\bar{1}2}|$. Set $H := [3, 4i+9] \cup *(4i+11)$. Since $Q_i \cap F \cap H = \{4i+7\}$, we have $H \notin \mathcal{G}$, which implies

$$|\mathcal{F}_{\bar{1}\bar{2}}| \le \binom{n-2}{k}\alpha^{4i+8}.$$

Therefore, $|\mathcal{F}| \le c\binom{n-2}{k-2}$ where

$$c = 1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha) + \frac{2(n-k)}{k-1}\alpha^{4i+5} + \frac{(n-k)(n-k-1)}{k(k-1)}\alpha^{4i+8}.$$

One can check that $c < 1$ for $n > n_0$. Indeed, this time it suffices to show

$$2\alpha^5 + \frac{1-d}{d}\alpha^8 < \frac{(1-\alpha)\,d\,(1-d)}{\alpha^4},$$

and this is true for $d \le 0.536$. This completes the proof of Theorem 2. □

In Case 1 and Case 2, we proved $c = |\mathcal{F}|/\binom{n-2}{k-2} < 1$. On the other hand, we can construct a series of non-trivial 3-wise 2-intersecting k-uniform families $\mathcal{F}^{(n)}$ on n vertices with $k = (\frac{1}{2}+\epsilon)n$ which satisfies $\lim_{n\to\infty} \mathcal{F}^{(n)}/\binom{n-2}{k-2} = 1$ as follows:

$$\mathcal{F}_{12}^{(n)} = \{\{1,2\} \cup G : |G \cap [3, k+2]| \ge \frac{k+2}{2}\},$$

$$\mathcal{F}^{(n)}_{1\bar{2}} = \mathcal{F}^{(n)}_{\bar{1}2} = \emptyset,\ \mathcal{F}^{(n)}_{\bar{1}\bar{2}} = \{[3, k+2]\}.$$

The maximal i such that $P_i \in \mathcal{F}^{(n)}$ is given by $i = \lfloor \frac{k}{4} \rfloor - 2$ for k odd, and $i = \lceil \frac{k}{4} \rceil - 2$ for k even.

5 Proof of Theorem 3

For a family $\mathcal{F} \subset 2^{[n]}$, set $\mathcal{F}_i := \mathcal{F} \cap \binom{[n]}{i}$. First we prove the following inequality.

Proposition 3 *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family with $k/n \le 0.501$, $n > n_0$. Then $\sum_{i=1}^{k} |\mathcal{F}_i| / \binom{n-2}{i-2} \le 1$.*

Proof. We prove $\sum_{i=1}^{k} |\mathcal{F}_i| / \binom{n-2}{i-2} \le 1$ for $n > n_0$ by induction on the number of nonzero $|\mathcal{F}_i|$'s.

If this number is one then the inequality follows from Theorem 2. If it is not the case then let p be the smallest and r the second-smallest index for which $|\mathcal{F}_i| \ne 0$. Set $\mathcal{F}_p^c := \{[n] - F : F \in \mathcal{F}_p\} \subset \binom{[n]}{n-p}$. Since $\mathcal{F}_p$ is 3-wise 2-intersecting, it follows from Theorem 2 that $|\mathcal{F}_p| = |\mathcal{F}_p^c| \le \binom{n-2}{p-2} = \binom{n-2}{n-p}$. Then by Proposition 1, we have

$$\frac{|\Delta_{n-r}(\mathcal{F}_p^c)|}{|\mathcal{F}_p^c|} \ge \frac{\binom{n-2}{n-r}}{\binom{n-2}{n-p}} = \frac{\binom{n-2}{r-2}}{\binom{n-2}{p-2}}. \tag{15}$$

Set $\mathcal{G}_r := \{G \in \binom{[n]}{r} : G \supset \exists F \in \mathcal{F}_p\}$. Due to (15) and the fact $\mathcal{G}_r = (\Delta_{n-r}(\mathcal{F}_p^c))^c$, we have $|\mathcal{G}_r| / \binom{n-2}{r-2} \ge |\mathcal{F}_p| / \binom{n-2}{p-2}$. Since $\mathcal{F}$ is Sperner, $\mathcal{F}_r \cap \mathcal{G}_r = \emptyset$ and $\mathcal{H} := (\mathcal{F} - \mathcal{F}_p) \cup \mathcal{G}_r$ is a 3-wise 2-intersecting Sperner family. Moreover, the number of nonzero $|\mathcal{H}_i|$'s is one less than that of $|\mathcal{F}_i|$'s. Therefore, by the induction hypothesis and the fact that $\mathcal{F} \triangle \mathcal{H} = \mathcal{F}_p \cup \mathcal{G}_r$, we have

$$\sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{i-2}} \le \sum_{i=1}^{k} \frac{|\mathcal{H}_i|}{\binom{n-2}{i-2}} \le 1,$$

which completes the proof of the proposition. □

By (15), we have $|\Delta_{n-r}(\mathcal{F}_p^c)| \ge |\mathcal{F}_p^c|$ (and so $|\mathcal{F}| \le |\mathcal{H}|$) if $n \ge p + r - 2$. Replace $\mathcal{F}$ by $\mathcal{H}$ (and find new p and r) and continue the same procedure as long as $n \ge p + r - 2$. In the end, we have at most one index $p < \lceil \frac{n+2}{2} \rceil$ such that $\mathcal{F}_p \neq \emptyset$. If we have such p, then set $r := \lceil \frac{n+2}{2} \rceil$ even though $\mathcal{F}_r = \emptyset$ may happen only in this last step, and replace $\mathcal{F}_p$ by $\mathcal{G}_r$ and obtain $\mathcal{H}$ from $\mathcal{F}$. In this way, we can construct a 3-wise 2-intersecting Sperner family $\mathcal{H}$ with $|\mathcal{H}| \geq |\mathcal{F}|$ and $\mathcal{H}_i = \emptyset$ for all $i < \lceil \frac{n+2}{2} \rceil$. In this process, $|\mathcal{H}| = |\mathcal{F}|$ happens only if $n = p + r - 2$ and $\mathcal{F}_p^c = \binom{Y}{n-p}$, $|Y| = n - 2$ (cf. Proposition 1), that is,

$$\mathcal{F}_p \cong \{\{a, b\} \cup G : G \in \tbinom{Y}{p-2}\}.$$

But then we can find $A, B \in \mathcal{F}_p$ with $A \cap B = \{a, b\}$ because $|Y| = n - 2 = (p-2) + (r-2) \geq 2(p-2)$. In this case, all members in $\mathcal{F}$ must contain $\{a, b\}$ and we can easily verify Theorem 3. Therefore, for the proof of Theorem 3, we may assume that $\mathcal{F}_i = \emptyset$ for $i < \lceil \frac{n+2}{2} \rceil$ from the beginning (otherwise replace $\mathcal{F}$ by $\mathcal{H}$). This remark is needed because we claim the uniqueness of the extremal configuration.

Let us now prove Theorem 3. Suppose that $\mathcal{F} \subset 2^{[n]}$ is a 3-wise 2-intersecting Sperner family of maximal size. We may assume that $\mathcal{F}_i = \emptyset$ for all $i < \lceil \frac{n+2}{2} \rceil =: m$. Set $k = \lfloor 0.501n \rfloor$ and $r_i = |\mathcal{F}_i| / \binom{n-2}{i-2}$ ($r_1 = \cdots = r_{m-1} = 0$).

Case 1 $n = 2m - 2$.
By Proposition 3, we have $\sum_{1 \leq i \leq k} r_i = \sum_{m \leq i \leq k} r_i \leq 1$. Thus,

$$\begin{aligned} \sum_{m \leq i \leq k} |\mathcal{F}_i| &= \sum_{m \leq i \leq k} r_i \binom{n-2}{i-2} \leq r_m \binom{n-2}{m-2} + (1 - r_m) \binom{n-2}{m-1} \\ &= \binom{n-2}{m-2} (1 - \frac{1 - r_m}{m-1}). \end{aligned}$$

On the other hand, by the LYM inequality, we have

$$1 \geq \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \geq \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{k+1}}.$$

Therefore, we have

$$|\mathcal{F}| \leq \binom{n-2}{m-2} - \frac{1 - r_m}{m-1} \binom{n-2}{m-2} + \binom{n}{\lfloor 0.501n \rfloor + 1}. \tag{16}$$

If $\mathcal{F}_m$ is 2-wise 3-intersecting, then $\mathcal{F}_m^c \subset \binom{[2m-2]}{m-2}$ is 2-wise 1-intersecting. By Proposition 2, we have $|\Delta_{m-3}(\mathcal{F}_m^c)| \geq |\mathcal{F}_m^c| = |\mathcal{F}_m|$. So we replace $\mathcal{F}$ by

$(\mathcal{F}-\mathcal{F}_m)\cup(\Delta_{m-3}(\mathcal{F}_m^c))^c$, and we may assume that $\mathcal{F}_m=\emptyset$, i.e., $r_m=0$. Then it follows $|\mathcal{F}|<\binom{n-2}{m-2}$ from (16) for $n>n_0$.

If $\mathcal{F}_m$ is not 2-wise 3-intersecting, then there exist F,F' with $|F\cap F'|=2$. Then all members in $\mathcal{F}$ contain $F\cap F'$ and we are done.

Case 2 $n=2m-3$.
By Proposition 3, we have $\sum_{m\le i\le k} r_i\le 1$. Thus,

$$\begin{aligned}\sum_{m\le i\le k}|\mathcal{F}_i| &= \sum_{m\le i\le k} r_i\binom{n-2}{i-2}\le r_m\binom{n-2}{m-2}+(1-r_m)\binom{n-2}{m-1}\\ &= \binom{n-2}{m-2}\Big(1-\frac{2(1-r_m)}{m-1}\Big).\end{aligned}$$

For $\mathcal{F}_i$, $i>k$, we use the LYM inequality. Then we have

$$|\mathcal{F}|\le\binom{n-2}{m-2}-\frac{2(1-r_m)}{m-1}\binom{n-2}{m-2}+\binom{n}{\lfloor 0.501n\rfloor+1}. \tag{17}$$

Now we look at $\mathcal{F}_m$ in detail.

Lemma 1 *If $\mathcal{F}_m$ is non-trivial, then $|\mathcal{F}_m|<0.999\binom{n-2}{m-2}$ holds for $n>n_0$.*

Proof. Here we only assume that $\mathcal{F}_m\subset\binom{[2m-3]}{m}$ is shifted, non-trivial 3-wise 2-intersecting and we do not use the other $\mathcal{F}_i$, $i\ne m$. We follow the proof of Theorem 2. Suppose that $|\mathcal{F}_m|\ge 0.999\binom{n-2}{m-2}$ and define $\mathcal{G}$ as in the proof of Theorem 2. Then, using Theorem 5, we can conclude that $P_i\in\mathcal{G}$ and $P_{i+1}\notin\mathcal{G}$ for some $i\ge 1$. First we deal with the case $Q_i\notin\mathcal{G}$. We use the same estimation for the sizes of $\mathcal{F}_{1\bar{2}},\mathcal{F}_{\bar{1}2},\mathcal{F}_{\bar{1}\bar{2}}$ as in Case 1 of the proof of Theorem 2. Noting that $n=2m-3$ and $k=m$, we have

$$|\mathcal{F}_{1\bar{2}}|,|\mathcal{F}_{\bar{1}2}|\le\binom{n-2}{k-2}\frac{n-k}{k-1}\alpha^{4i+3}=\binom{n-2}{m-2}\frac{m-3}{m-1}\alpha^{4i+3}, \tag{18}$$

$$|\mathcal{F}_{\bar{1}\bar{2}}|\le\binom{n-2}{k-2}\frac{(n-k)(n-k-1)}{k(k-1)}\alpha^{4i+6}=\binom{n-2}{m-2}\frac{(m-3)(m-4)}{m(m-1)}\alpha^{4i+6}. \tag{19}$$

Let $\mathcal{A}=\{F\cap[3,m+1]:F\in\mathcal{F}_{12}\}$. Since $\mathcal{F}_m$ is shifted and non-trivial we may assume that $\{1\}\cup[3,m+1]\in\mathcal{F}$. So $\mathcal{A}$ is 2-wise 1-intersecting. Let $\mathcal{A}_i$ be the i-uniform subfamily of $\mathcal{A}$. Clearly $|\mathcal{A}_i|\le\binom{m-1}{i}$ and if $2i\le m-1$

then $|\mathcal{A}_i| \le \binom{m-2}{i-1}$ follows from the Erdős–Ko–Rado theorem [2]. Thus we have

$$\begin{aligned} |\mathcal{F}_{12}| &\le \sum_{i=1}^{m-2} |\mathcal{A}_i| \binom{n-(m+1)}{m-i-2} \\ &\le \sum_{i \le \lfloor \frac{m-1}{2} \rfloor} \binom{m-2}{i-1}\binom{m-4}{m-i-2} + \sum_{i > \lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{i}\binom{m-4}{m-i-2}. \end{aligned}$$

Set $f(i) = \binom{m-1}{i}\binom{m-4}{m-i-2}$ and $h = \lfloor \frac{m-1}{2} \rfloor$. Then, using $\binom{m-2}{i-1} = \frac{i}{m-1}\binom{m-1}{i} \le \frac{1}{2}\binom{m-1}{i}$ for $i \le h$, we have $|\mathcal{F}_{12}| \le \frac{1}{2}\sum_{i \le h} f(i) + \sum_{i > h} f(i)$. Note also that $\binom{n-2}{m-2} = \sum_{i=0}^{m-2} f(i) = \sum_{i \le h} f(i) + \sum_{h > i} f(i)$, and $\lim_{m\to\infty}(\sum_{i \le h} f(i))/(\sum_{h>i} f(i)) = 1$. Therefore, we have

$$|\mathcal{F}_{12}| \le \left(\frac{3}{4} + \epsilon\right)\binom{n-2}{m-2} \tag{20}$$

for any $\epsilon > 0$ if $n > n_0(\epsilon)$. By (18), (19), (20) we have $|\mathcal{F}| \le 0.76\binom{n-2}{m-2}$ for n sufficiently large. This contradicts our assumption $|\mathcal{F}_m| \ge 0.999\binom{n-2}{m-2}$.

We have one more case, that is, the case $Q_i \in \mathcal{G}$. But in this case, compared to the previous case, we can put better bounds for $\mathcal{F}_{1\bar{2}}, \mathcal{F}_{\bar{1}2}, \mathcal{F}_{\bar{1}\bar{2}}$, and the same bound for $\mathcal{F}_{12}$. This completes the proof of Lemma 1. □

If $r_m < 0.999$ then $|\mathcal{F}| < \binom{n-2}{m-2}$ follows from (17). So we may assume that $|\mathcal{F}_m| \ge 0.999\binom{n-2}{m-2}$. Then Lemma 1 implies that $\mathcal{F}_m$ is trivial, i.e., all members of $\mathcal{F}_m$ contain $\{1,2\}$.

Lemma 2 *For every j ($3 \le j \le n$) we can find $F, F' \in \mathcal{F}_m$ such that $F \cap F' = \{1,2,j\}$.*

Proof. It suffices to prove the result for $j = n$. Suppose, on the contrary, that $\mathcal{C} := \{F - \{1,2,n\} : \{1,2,n\} \subset F \in \mathcal{F}_m\}$ is 2-wise 1-intersecting. There are $\binom{2m-6}{m-3}$ sets in $\binom{[3,n-1]}{m-3}$ and at most half of them can be in $\mathcal{C}$. This implies $|\mathcal{F}_m| \le \binom{n-2}{m-2} - \frac{1}{2}\binom{2m-6}{m-3} = (1 - \frac{m-2}{2(2m-5)})\binom{n-2}{m-2}$. But this is impossible because $|\mathcal{F}_m| \ge 0.999\binom{n-2}{m-2}$. □

Let $i > m$ and suppose $C \in \mathcal{F}_i$. If $C \not\supset \{1,2\}$ then, by Lemma 2, we have only two choices of C, that is, $C_1 = [n] - \{1\}$ or $C_2 = [n] - \{2\}$. Except C_1 and C_2, all the other edges in $\mathcal{F}$ contain $\{1,2\}$. Let $\mathcal{D} := \{D - \{1,2\} : \{1,2\} \subset$

$D \in \mathcal{F}\} \subset \bigcup_{j=m}^{n} \binom{[3,n]}{j-2}$. Clearly, $\mathcal{D}$ is a Sperner family. By the Sperner theorem [17] we have $|\mathcal{D}| \leq |\binom{[3,n]}{m-2}|$. Equality holds only if $\mathcal{D} = \binom{[3,n]}{m-2}$ or $\mathcal{D} = \binom{[3,n]}{m-3}$, but the latter case is impossible because we have assumed $\mathcal{F}_j = \emptyset$ for $j < m$. This proves that the unique maximal configuration in Case 2 is $\mathcal{F} = \mathcal{F}_m \cup \{C_1, C_2\}$ where $\mathcal{F}_m = \{\{1,2\} \cup D : D \in \binom{[3,n]}{m-2}\}$. This completes the proof of Case 2 and so the proof of Theorem 3. □

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References

[1] R. Ahlswede, L.H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18:125–136, 1997.

[2] P. Erdős, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.

[3] P. Frankl. On Sperner families satisfying an additional condition. *J. Combin. Theory (A)*, 20:1–11, 1976.

[4] P. Frankl. The shifting technique in extremal set theory. "Surveys in Combinatorics 1987" (C. Whitehead, Ed. LMS Lecture Note Series 123), 81–110, Cambridge Univ. Press, 1987.

[5] P. Frankl, N. Tokushige. The Kruskal–Katona Theorem, some of its analogues and applications. Conference on extremal problems for finite sets, 1991, Visegrád, Hungary, 92–108.

[6] P. Frankl, N. Tokushige. Weighted multiply intersecting families. *Studia Sci. Math. Hungarica* 40:287–291 (2003).

[7] P. Frankl, N. Tokushige. Weighted 3-wise 2-intersecting families. *J. Combin. Theory (A)* 100:94-115 (2002).

[8] P. Frankl, N. Tokushige. Weighted non-trivial multiply intersecting families. *to appear in Combinatorica.*

[9] H.-D.O.F. Gronau. On Sperner families in which no 3-sets have an empty intersection. *Acta Cybernet.*, 4:213–220, 1978.

[10] H.-D.O.F. Gronau. On Sperner families in which no k-sets have an empty intersection. *J. Combin. Theory (A)*, 28:54–63, 1980.

[11] H.-D.O.F. Gronau. On Sperner families in which no k-sets have an empty intersection II. *J. Combin. Theory (A)*, 30:298–316, 1981.

[12] H.-D.O.F. Gronau. On Sperner families in which no k-sets have an empty intersection III. *Combinatorica*, 2:25–36, 1982.

[13] G.O.H. Katona. Intersection theorems for systems of finite sets. *Acta Math. Acad. Sci. Hung.*, 15:329–337, 1964.

[14] G.O.H. Katona. A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, 1966 (Akademiai Kiadó, 1968) 187–207, MR 45 #76.

[15] J.B. Kruskal. The number of simplices in a complex, in: Math. Opt. Techniques (Univ. of Calif. Press, 1963) 251–278, MR 27 #4771.

[16] E.C. Milner. A combinatorial theorem on systems of sets. *J. London Math. Soc.*, 43:204–206, 1968.

[17] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Zeitschrift*, 27:544–548, 1928.