The game of n-times nim

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Abstract

The following game is considered. The first player can take any number of stones, but not all the stones, from a single pile of stones. After that, each player can take at most n-times as many as the previous one. The player first unable to move loses and his opponent wins.

Let f_1, f_2, $\ldots$ be an initial sequence of stones in increasing order, such that the second player has a winning strategy when play begins from a pile of size f_i. It is proved that there exist constants $c = c(n)$ and $k_0 = k_0(n)$ such that $f_{k+1} = f_k + f_{k-c}$ for all $k > k_0$, and $\lim_{n\to\infty} c(n)/(n \log n) = 1$.

Key words: fibonacci sequence, nim

Let us consider the following game which we call *n-times nim*. The first player can take any number of stones, but not all the stones, from a single pile of stones. After that, each player can take at most n-times as many as the previous one. The player first unable to move loses and his opponent wins. Usually this game is considered for a positive integer n, but throughout this paper we only assume that $n \geq 1$, i.e., n can be a real number.

Let $F(n) := \{f_1, f_2, \ldots\}$ be the sequence of initial numbers of stones in increasing order such that the second player has a winning strategy when the first player begins moving from a pile of size f_i. Clearly $f_1 = 1$, since the first player has no move, so the second wins. Then, obviously,

$$f_i = i \text{ holds for } i \leq \lfloor n+1 \rfloor, \tag{1}$$

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also $f_{\lfloor n+2\rfloor} = \lfloor n+3\rfloor$.

Whinihan (who ascribes "Fibonacci nim" to R.E. Gaskell, see [2]) found that 2-times nim satisfies $f_{k+1} = f_k + f_{k-1}$, that is, $F(2)$ is the Fibonacci sequence. Then Schwenk [1] proved that in n-times nim there exist constants $c = c(n)$ and $k_0 = k_0(n)$ such that $f_{k+1} = f_k + f_{k-c}$ for all $k > k_0$. He asked to determine the behavior of $c = c(n)$. We are going to prove the result of Schwenk in a different way, and answer his question.

Theorem 1 *Let n be a fixed positive real at least 1.*

(i) *For every $k \geq 1$ there exists an $r = r(k)$ such that $f_{k+1} = f_k + f_r$ holds.*
(ii) *$r(k)$ can be computed by $r(k) = \min\{r : nf_r \geq f_k\}$.*
(iii) *$(\frac{n+1}{n})f_k \leq f_{k+1}$ holds for all $k \geq 1$.*
(iv) *$r(k) \leq r(k+1) \leq r(k)+1$, i.e., the function $r(k)$ is continuous in the discrete sense.*
(v) *There is a constant $c = c(n)$ such that $r(k) = k - c$ holds for all $k > k_0$.*

PROOF. We prove all these statements simultaneously, applying induction on k. The cases $k \leq n$ are trivial, with $r(k) = 1$.

Suppose now that all statements are proved for $k' < k$ and consider k. Let $r(k)$ be defined via (ii). We first prove that the first player has a winning strategy for s stones as long as

$$f_k < s < f_k + f_{r(k)}. \tag{2}$$

If $n(s - f_k) < f_k$ holds then the first player can remove $s - f_k$ stones and win, as f_k is a second player win. From now on, we suppose

$$n(s - f_k) \geq f_k, \quad \text{i.e.,} \quad s \geq \frac{n+1}{n} f_k. \tag{3}$$

Let us show that

$$s - f_k \text{ is a first player win.}$$

Suppose the contrary, then $s - f_k = f_q$ holds for some q. Since $f_q = s - f_k < f_{r(k)}$ by (2), the definition of $r(k)$ implies $nf_q < f_k$, and $n(s - f_k) = nf_q < f_k$, contradicting (3).

Now let the first player play according to the winning strategy for $s - f_k$ stones. This will enable him a finite number of moves to reduce the number of remaining stones to exactly f_k.

For convenience, we make this strategy even more clear, by requiring him to remove all the "extra stones," i.e., reduce the number of remaining stones to

f_k only if he has no other winning moves for his "mind game" of $s - f_k$ stones. This makes sure that when he reduces the number of stones to exactly f_k, the number of stones, say t, that he is taking is a *second player win*. That is

$$t = f_l \quad \text{for some} \quad l < r(k), \tag{4}$$

implying, by the definition of $r(k)$ that

$$nt < f_k, \tag{5}$$

and thus completing the proof that this is a winning strategy for the first player.

Now, to complete the proof of (i), we must show that

$$f_k + f_{r(k)} \text{ is a second player win.}$$

If the first player removes $f_{r(k)}$ or more stones, then the second can remove all the remaining and win. Otherwise let the second player play the "mind game" for $f_{r(k)}$ stones, by delaying his ultimate move (as above) as long as he can. Then, the number of stones (say t) which he removes finally to reduce the number of remaining stones to f_k will satisfy (4) and thus (5) too, proving that this is a correct winning strategy. This concludes the proof of (i) and (ii). Then (iii) follows directly from (i) and (ii).

The proof of (iv). From (ii), $r(k) \leq r(k+1)$ is clear. Using (i) and (ii), $nf_{r(k)+1} \geq n(f_{r(k)} + f_{r(r(k))}) \geq f_k + f_{r(k)} = f_{k+1}$ follows, proving $r(k+1) \leq r(k) + 1$.

Finally, we prove (v). From (iv) it follows that $k - r(k)$ is a monotone increasing, integer-valued function. Therefore, it is sufficient to prove that it is bounded from above. Actually, we shall see that

$$k - r(k) < (n+1) \log n. \tag{6}$$

To show (6), suppose the contrary. Then, using (iii), we have

$$\begin{aligned} f_{r(k)} = f_k \cdot \frac{f_{k-1}}{f_k} \cdot \frac{f_{k-2}}{f_{k-1}} \cdots \frac{f_{r(k)}}{f_{r(k)+1}} \quad &\leq \quad f_k (\frac{n}{n+1})^{k-r(k)} \\ \leq f_k ((1 - \frac{1}{n+1})^{n+1})^{\log n} \quad < \quad f_k e^{-\log n} \quad &= \quad \frac{f_k}{n}, \end{aligned}$$

contradicting the definition of $r(k)$.

Thus the proof is complete. $\square$

Theorem 2 *Let n be a fixed positive real at least 1.*

(vi) $(\frac{n}{n-1})f_k > f_{k+1}$ *holds for all* $k > k_0$.

(vii) $\left\lfloor \dfrac{\log n}{\log n - \log(n-1)} \right\rfloor \le c(n) \le \left\lceil \dfrac{\log n}{\log(n+1) - \log n} \right\rceil$.

(viii) $\displaystyle\lim_{n\to\infty} \frac{c(n)}{n \log n} = 1$.

PROOF. By (i) and (v), we have

$$f_{k+1} = f_k + f_{k-c} \tag{7}$$

for $k > k_0$. On the other hand, (ii) implies

$$nf_{(k+1)-c} \ge f_{k+1} > nf_{k-c}. \tag{8}$$

By (7) and (8), we have $f_{k+1} > nf_{k-c} = n(f_{k+1} - f_k)$, i.e., $nf_k > (n-1)f_{k+1}$, which proves (vi).

Set $U := \left\lceil \frac{\log n}{\log(n+1)-\log n} \right\rceil$, then $(\frac{n}{n+1})^U \le \frac{1}{n}$. To show $c(n) \le U$, suppose the contrary. Then using (iii), we have

$$\begin{aligned} f_{r(k)} = f_k \cdot \frac{f_{k-1}}{f_k} \cdot \frac{f_{k-2}}{f_{k-1}} \cdots \frac{f_{r(k)}}{f_{r(k)+1}} \quad &\le \quad f_k(\frac{n}{n+1})^{k-r(k)} \\ = f_k(\frac{n}{n+1})^{c(n)} \quad < \quad f_k(\frac{n}{n+1})^U \quad &\le \quad \frac{f_k}{n}, \end{aligned}$$

contradicting (ii).

Set $L := \left\lfloor \frac{\log n}{\log n - \log(n-1)} \right\rfloor$, then $(\frac{n-1}{n})^L \ge \frac{1}{n}$. To show $c(n) \ge L$, suppose on the contrary that $c(n) + 1 \le L$. Then using (vi), we have

$$\begin{aligned} f_{r(k)-1} = f_k \cdot \frac{f_{k-1}}{f_k} \cdot \frac{f_{k-2}}{f_{k-1}} \cdots \frac{f_{r(k)-1}}{f_{r(k)}} \quad &\ge \quad f_k(\frac{n-1}{n})^{k-r(k)+1} \\ = f_k(\frac{n}{n+1})^{c(n)+1} \quad \ge \quad f_k(\frac{n-1}{n})^L \quad &\ge \quad \frac{f_k}{n}, \end{aligned}$$

contradicting (ii).

(viii) follows immediately from (vii). □

Here are some numerical data concerning $c(n)$.

n	L	$c(n)$	U	$\lfloor n \log n \rfloor$
2	1	1	1	1
3	2	3	3	3
4	4	5	6	5
5	7	7	8	8
6	9	10	11	10
7	12	13	14	13
8	15	16	17	16
9	18	19	20	19
10	21	22	24	23
11	25	25	27	26
12	28	29	31	29
13	32	32	34	33
14	35	37	38	36

n			$c(n)$
1	$\leq n <$	2	0
2	$\leq n <$	5/2	1
5/2	$\leq n <$	3	2
3	$\leq n <$	7/2	3
7/2	$\leq n <$	43/11	4
43/11	$\leq n <$	13/3	5
13/3	$\leq n <$	14/3	6
14/3	$\leq n <$	51/10	7

It is worth noting that $c(n) = c(n')$ does not necessarily imply $F(n) = F(n')$. For example, $c(n) = 4$ for $7/2 \leq n < 43/11$, but there are two winning sequences for the second player, that is,

$$F(n) = \{1, 2, 3, 4, 6, 8, 11, 15, 21, 27, 35, 46, \ldots\} \quad \text{for} \quad 7/2 \leq n < 11/3,$$
$$F(n) = \{1, 2, 3, 4, 6, 8, 11, 14, 18, 24, 32, 43, \ldots\} \quad \text{for} \quad 11/3 \leq n < 43/11.$$

References

[1] A.J. Schwenk. Take-away games. *The Fibonacci Quarterly*, 8:225–234,241, 1970.

[2] M.J. Whinihan. Fibonacci Nim. *The Fibonacci Quarterly*, 1:9, 1963.