COUNTING LATTICE PATHS VIA A NEW CYCLE LEMMA

TOMOKI NAKAMIGAWA* AND NORIHIDE TOKUSHIGE†

Abstract. Let α, β, m, n be positive integers. Fix a line $L : y = \alpha x + \beta$, and a lattice point $Q = (m, n)$ on L. It is well known that the number of lattice paths from the origin to Q which touches L only at Q is given by

$$\frac{\beta}{m+n}\binom{m+n}{m}.$$

We extend the above formula in various ways, in particular, we consider the case when α and β are arbitrary positive reals. The key ingredient of our proof is a new variant of the cycle lemma originated from Dvoretzky–Motzkin [1] and Raney [8]. We also include a counting formula for lattice paths lying under a cyclically shifting boundary, which generalizes a result due to Irving and Ratten in [6], and a counting formula for lattice paths having given number of peaks, which contains the Narayana number as a special case[1].

Key words. lattice path, cycle lemma, Catalan number, Narayana number

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1. Introduction. Let α, β be positive reals, and let m, n be positive integers. Fix a line $L : y = \alpha x + \beta$, and a lattice point $Q = (m, n)$ on L, i.e., $n = \alpha m + \beta$. By a walk (or a lattice path) we will mean a path in $\mathbb{Z}^2$ with unit steps down and to the left (i.e., steps $(0, -1)$ and $(-1, 0)$, respectively). Let V be the set of walks from Q to the origin O. Clearly, we have $|V| = \binom{m+n}{m}$. Let $W \subset V$ be the set of walks which touch the line L at Q only. It is well known (for example, see Exercise 5.3.5 (b) of [2]) that if both α and β are integers, then

$$|W| = \frac{\beta}{m+n}\binom{m+n}{m}. \tag{1}$$

In particular, if $n = m + 1$ ($\alpha = \beta = 1$), then $|W| = \frac{1}{m+1}\binom{2m}{m}$ is the famous Catalan number. We will extend the above formula in various ways.

For a walk $w \in V$, we define the minimum y-distance $\delta(w)$ as follows: if w touches or crosses L after the first step, then let $\delta(w) = 0$, otherwise let $\delta(w)$ be the minimum of $\alpha m_0 + \beta - n_0$, where (m_0, n_0) runs over all lattice points on w except Q. We notice that $\delta(w) = 0$ iff $w \in V \setminus W$, so $\sum_{w\in W} \delta(w) = \sum_{w\in V} \delta(w)$. If α and β are positive integers, then $\sum_{w\in V} \delta(w)$ simply counts $|W|$ because $\delta(w) = 1$ for all $w \in W$. In this sense $\sum_{w\in V} \delta(w)$ can be viewed as a weighted sum corresponding to the number of walks.

For a real $t \geq 0$, let $W_t = \{w \in W : \delta(w) \geq t\}$. Then $|W_t|$ is a left-continuous step function of t, and it follows from the definition that

$$\int_0^1 |W_t| dt = \sum_{w\in V} \delta(w). \tag{2}$$

*Department of Information Science, Shonan Institute of Technology, 1-1-25 Tsujido-Nishikaigan, Fujisawa 251-8511, Japan (`nakami@info.shonan-it.ac.jp`), supported by JSPS KAKENHI (23540168).

† College of Education, Ryukyu University, Nishihara, Okinawa, 903-0213 Japan (`hide@edu.u-ryukyu.ac.jp`), supported by JSPS KAKENHI (20340022).

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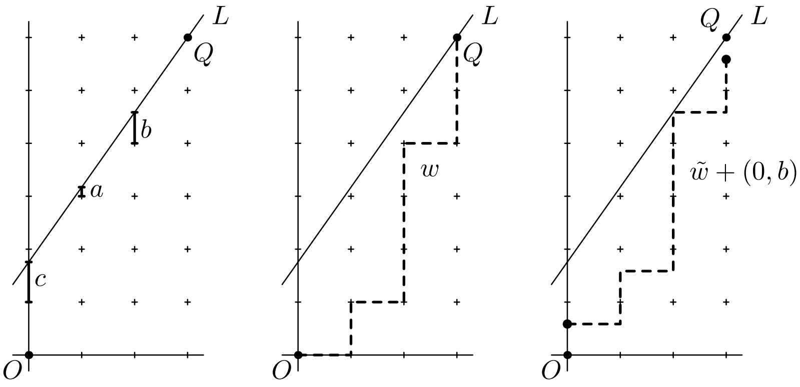

FIG. 1. *A line L and a lattice path.*

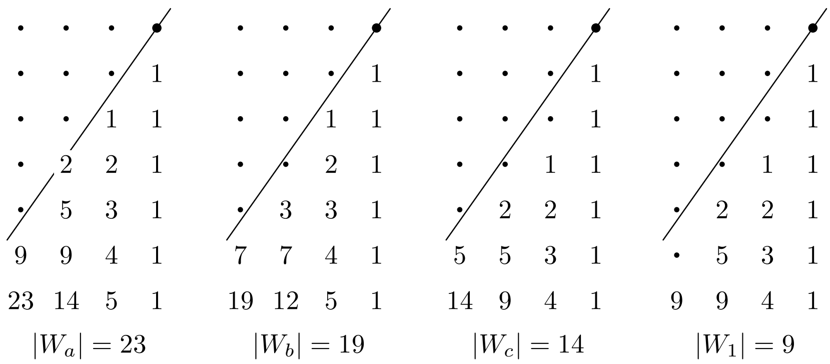

FIG. 2. *The number of partial paths in W_t from Q.*

One can count each $|W_t|$ by a recursion as in Figure 2. On the other hand, it is somewhat surprising that the weighted sum (2) has a simple closed formula, as we will soon see. Indeed one can get (2) almost effortlessly, without counting individual $|W_t|$.

EXAMPLE 1. *Let $Q = (m, n)$ be a lattice point on a line $L : y = \alpha x + \beta$, where $\alpha = \sqrt{2}, \beta = 6 - 3\sqrt{2}, m = 3$, and $n = \alpha m + \beta = 6$, see Figure 1. Then we have*

$$|W_t| = \begin{cases} 23 & \text{if } 0 < t \le a, \\ 19 & \text{if } a < t \le b, \\ 14 & \text{if } b < t \le c, \\ 9 & \text{if } c < t \le 1, \end{cases}$$

where $a = 3 - 2\sqrt{2}$, $b = 2 - \sqrt{2}$, $c = 5 - 3\sqrt{2}$ (see Figure 2), and

$$\int_0^1 |W_t| dt = 23a + 19(b - a) + 14(c - b) + 9(1 - c) = 56 - 28\sqrt{2}.$$

On the other hand, we have

$$\frac{\beta}{m + n}\binom{m + n}{m} = \frac{6 - 3\sqrt{2}}{3 + 6}\binom{3 + 6}{3} = 56 - 28\sqrt{2},$$

which verifies our main results, Theorem 1 and Corollary 1 stated below.

THEOREM 1. *Let m, n be positive integers, and let α, β be positive reals with $n = \alpha m + \beta$. Let V be the set of walks from (m, n) to the origin. Then, we have*

$$\sum_{w \in V} \delta(w) = \frac{\beta}{m+n} \binom{m+n}{m}.$$

Apparently, Theorem 1 is a generalization of (1), and it can be equivalently stated in the following integration form.

COROLLARY 1. *Under the same assumptions as in Theorem 1, we have*

$$\int_0^1 |W_t| dt = \frac{\beta}{m+n} \binom{m+n}{m}.$$

We notice that if $\alpha \in (1/\ell)\mathbb{N}$ for some $\ell \in \mathbb{N}$, then

$$\int_0^1 |W_t| dt = \frac{1}{\ell} \sum_{t \in T} |W_t|,$$

where $T = \{\delta(w) : w \in W\} = \{1/\ell, 2/\ell, \ldots, (\ell-1)/\ell, 1\}$. For the general case $\alpha \in \mathbb{R}$, the following interpretation would help to understand the LHS of the formula in the corollary intuitively. If $w \in W$, then the first step is a down step. Let $\tilde{w}$ be a walk obtained from w by omitting this down step. Namely, $\tilde{w}$ is a walk of $m+n-1$ steps, from $Q - (0,1) = (m, n-1)$ to the origin. By translating $\tilde{w}$ to the direction $(0,t)$, we get a walk $\tilde{w} + (0,t)$ from $(m, n-1+t)$ to $(0,t)$. For $0 < t \leq 1$, we see that $w \in W_t$ if and only if $\tilde{w} + (0,t)$ does not cross the line L. Thus we can think of W_t as the set of $(m+n-1)$-step walks, from $(m, n-1+t)$ to $(0,t)$, which do not cross L.

In Section 2 we first show a new variant of the cycle lemma (Lemma 1) originated from Dvoretzky–Motzkin [1] and Raney [8] (see also [4] chapter 7.5). Then we prove Theorem 1 using the lemma. It turns out this simple looking lemma is rather strong. For example, we can show a higher dimensional version of the theorem without any extra effort. In Section 3 we apply the lemma to extend the theorem in two ways: one is counting lattice paths lying under a cyclically shifting boundary (Theorem 2), and the other is counting lattice paths having given number of peaks (Theorem 3). As a special case (Corollary 2), we get the main result of [6] due to Irving and Ratten with a much simpler proof.

Before closing the section, we remark that the formula (1) is proved and generalized in [3] by using the reflection method instead of the cycle method. Generalizations of the formula (1) are also seen in [5].

2. Proofs. We start with a variant of the cycle lemma. Let $z = (z_1, z_2, \ldots, z_k)$ be a sequence of reals. The i-th partial sum will mean $z_1 + z_2 + \cdots + z_i$, where $1 \leq i \leq k$. The case $i = k$ is called the total sum. We define the weight $\theta(z)$ of z as follows: if every partial sum of z is positive, then let $\theta(z)$ be the minimum partial sum, otherwise let $\theta(z) = 0$. Let $z^{\langle j \rangle} = (z_{1+j}, z_{2+j}, \ldots, z_{k+j})$ denote the j-th shift of z, where the indices are read modulo k.

For example, if $z = (1, 3, -1)$, then $z^{\langle 1 \rangle} = (3, -1, 1)$ with partial sums $\{3, 2, 3\}$ etc., and we get $\theta(z^{\langle 0 \rangle}) + \theta(z^{\langle 1 \rangle}) + \theta(z^{\langle 2 \rangle}) = 1 + 2 + 0 = 3$, which coincides with the total sum of z. This is not just a coincidence but a consequence of the following new

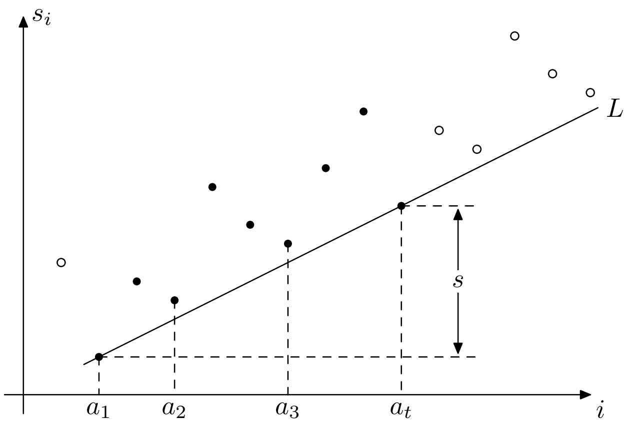

FIG. 3. *Partial sums of a cyclically infinite sequence. The points of Y are shown by black dots. There are four minimal points in Y.*

cycle lemma.

LEMMA 1. *Let $z = (z_1, \ldots, z_k)$ be a sequence of reals with total sum $s > 0$. Then we have*

$$\sum_{0 \le j < k} \theta(z^{\langle j \rangle}) = s.$$

Proof. We extend the sequence z cyclically to obtain the infinite sequence $u = (u_1, u_2, \ldots)$, where $u_i = z_j$ for $j \equiv i \bmod k$. Let $s_i = u_1 + \cdots + u_i$ be the i-th partial sum of u, and plot (i, s_i) for $i \ge 1$ in the plane, see Figure 3. Let $X = \{(i, s_i) : i \ge 1\}$. For each i, the line passing (i, s_i) and $(i + k, s_{i+k})$ has slope s/k. We get (at most) k lines in this way, and let L be the line in the bottom. Note that all the points (i, s_i) are above the line L.

Define a partial order on X by $(i, s_i) \succ (j, s_j)$ iff $i < j$ and $s_i \ge s_j$. Geometrically, a point x in this poset X is minimal iff X contains no point to the right and (weakly) below from x. Now the crucial observation is as follows.

CLAIM 1. *Let $i \ge 1$. A point $(i, s_i) \in X$ is one of the minimal points iff partial sums of $z^{\langle i \rangle}$ are all positive.*

Proof. Let $x = (i, s_i) \in X$. Suppose that some partial sum of $z^{\langle i \rangle}$ is non-positive. Then there exists an integer j with $i < j$ such that $u_{i+1} + u_{i+2} + \cdots + u_j = s_j - s_i \le 0$. Hence, we have $s_j \le s_i$. Therefore, we have $x \succ (j, s_j)$, and x is not a minimal point of X.

Conversely, suppose that x is not a minimal point. Then there exists a point (j, s_j) such that $i < j$ and $s_i \ge s_j$. Hence, we have a non-positive partial sum $u_{i+1} + u_{i+2} + \cdots + u_j$ of $z^{\langle i \rangle}$. □

Choose a point (a, s_a) on L. Note that both (a, s_a) and $(a + k, s_{a+k})$ are minimal points. We will look at the set of $k + 1$ points $Y = \{(i, s_i) : a \le i \le a + k\}$. Let t be the number of minimal points of Y, and let $\{(a_1, s_{a_1}), (a_2, s_{a_2}), \ldots, (a_t, s_{a_t})\}$ be the minimal points, where $a = a_1 < a_2 < \cdots < a_t = a + k$.

By the minimality, we have $s_{a_i} < s_{a_{i+1}}$, and so $s_{a_1} < s_{a_2} < \cdots < s_{a_t}$. Let $h > a_i$. If (h, s_h) is minimal, then $s_h \ge s_{a_{i+1}}$. If (h, s_h) is not minimal, then there is a minimal point (g, s_g) with $h < g$ and $s_h \ge s_g \ge s_{a_{i+1}}$. Consequently, we have

$\min\{s_h : h > a_i\} = s_{a_{i+1}}$.

CLAIM 2. *For $a \leq j < a+k$, we have*

$$\theta(z^{\langle j \rangle}) = \begin{cases} 0 & \text{if } j \notin \{a_1, a_2, \ldots, a_t\}, \\ s_{a_{i+1}} - s_{a_i} & \text{if } j = a_i \text{ and } 1 \leq i < t. \end{cases}$$

Proof. If $j \notin \{a_1, a_2, \ldots, a_t\}$, then we have $\theta(z^{\langle j \rangle}) = 0$ by Claim 1.

Let $j = a_i$ for some i with $1 \leq i < t$. Since $\theta(z^{\langle j \rangle}) = \min\{s_h - s_j : j < h\}$, it suffices to show that $s_{a_{i+1}} \leq s_h$ for all integers h with $a_i < h$, which we have just shown above. □

By Claim 2, we have

$$\sum_{0 \leq j < k} \theta(z^{\langle j \rangle}) = \sum_{a \leq j < a+k} \theta(z^{\langle j \rangle}) = \sum_{1 \leq i < t} (s_{a_{i+1}} - s_{a_i}) = s_{a_t} - s_{a_1} = s_{a+k} - s_a = s.$$

This completes the proof of Lemma 1. □

Proof of Theorem 1. For a walk $w \in V$, let w_i be the i-th step, which is one step down or to the left. For each $w \in V$, we assign a sequence $\operatorname{seq}(w) = (z_1, \ldots, z_{m+n}) \in \mathbb{R}^{m+n}$ by $z_i = 1$ if w_i is a down step, and $z_i = -\alpha$ if w_i is a left step. Finally, set $\theta(w) = \theta(\operatorname{seq}(w))$.

CLAIM 3. *$\theta(w) = \delta(w)$.*

Proof. Suppose that the first $i+j$ steps of w consist of j down steps and i left steps. Then after $i+j$ steps, we are at $(m,n) - j(0,1) - i(1,0) = (m-i, n-j)$. The y-distance from here to the line $L : y = \alpha x + \beta$ is $\alpha(m-i) + \beta - (n-j) = 1 \cdot j - \alpha \cdot i$, where we used $\alpha m + \beta - n = 0$. This y-distance coincides with the $(i+j)$-th partial sum of $\operatorname{seq}(w)$. So, $\theta(w)$ is the minimum y-distance, and the desired result follows. □

By Claim 3, we have

$$\sum_{w \in V} \delta(w) = \sum_{w \in V} \theta(w).$$

For $w = (w_1, \ldots, w_{m+n}) \in V$ and $0 \leq j < m+n$, let $w^{\langle j \rangle} = (w_{1+j}, \ldots, w_{m+n+j})$, where indices are read modulo $m+n$. Then, $\operatorname{seq}(w^{\langle j \rangle}) = (\operatorname{seq}(w))^{\langle j \rangle}$. Notice that $w^{\langle j \rangle} \in V$ and $(w^{\langle -j \rangle})^{\langle j \rangle} = w$. Thus each walk $w \in V$ appears $m+n$ times in a multiset $\{w^{\langle j \rangle} : w \in V, 0 \leq j < m+n\}$ of cardinality $|V|(m+n)$. This gives

$$\sum_{w \in V} \theta(w) = \sum_{w \in V} \frac{1}{m+n} \sum_{0 \leq j < m+n} \theta(w^{\langle j \rangle}).$$

For $w \in V$, the total sum of $\operatorname{seq}(w)$ is $n - \alpha m = \beta$. Thus, by Lemma 1, we have

$$\sum_{w \in V} \frac{1}{m+n} \sum_{0 \leq j < m+n} \theta(w^{\langle j \rangle}) = \sum_{w \in V} \frac{\beta}{m+n} = \frac{\beta}{m+n}|V|,$$

which finishes the proof of Theorem 1. □

The proof can be extended verbatim to higher dimensions. Namely, fix a hyperplane $L : x_d = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{d-1} x_{d-1} + \beta$ in $\mathbb{R}^d$, and a lattice point $Q = (m_1, \ldots, m_d) \in \mathbb{N}^d$ on L, and consider lattice paths in $\mathbb{Z}^d$ with unit steps of d types

$$e_1 = (-1, 0, \ldots, 0), e_2 = (0, -1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, -1).$$

Let V be the set of walks from Q to the origin. Then, we have $|V| = \frac{(m_1+\cdots+m_d)!}{m_1!m_2!\cdots m_d!}$. Let $W \subset V$ be the set of walks which touch the hyperplane L at Q only. For a lattice point $P = (m'_1, \ldots, m'_d)$ below L, let $d(P)$ be $\alpha_1 m'_1 + \cdots + \alpha_{d-1} m'_{d-1} + \beta - m'_d$, which is called the x_d-distance from P to L. For a walk $w \in V$, we define the minimum x_d-distance $\delta(w)$ as follows: if w touches or crosses L after the first step, then let $\delta(w) = 0$, otherwise let $\delta(w)$ be the minimum of $d(P)$, where P runs over lattice points on w except Q. For a walk $w = (w_1, w_2, \ldots, w_{m_1+\cdots+m_d}) \in V$, where w_i is the i-th step of w, let us assign $\operatorname{seq}(w) = (z_1, \ldots, z_{m_1+\cdots+m_d})$ to w, where $z_i = 1$ if $w_i = e_d$, and $z_i = -\alpha_j$ if $w_i = e_j$ with $j \neq d$. If P is a lattice point which is s steps away from Q along the walk w, then we have $d(P) = z_1 + \cdots + z_s$ because by a unit step e_j of w, the x_d-distance increases by 1 for $j = d$, and decreases by α_j for $j \neq d$. Hence, we have $\delta(w) = \theta(\operatorname{seq}(w))$. Since $\operatorname{seq}(w)$ has a total sum β for all $w \in V$, in the same manner as in the proof of Theorem 1, we have

$$\sum_{w \in W} \delta(w) = \int_0^1 |W_t| dt = \frac{\beta}{m_1 + \cdots + m_d} |V|.$$

3. Applications.

We extend Theorem 1 in two ways.

3.1. Lattice paths lying under a cyclically shifting boundary.

We will count the number of lattice paths lying under a cyclically shifting piecewise linear boundary of varying slope. Let $Q = (m, n) \in \mathbb{N}^2$ be a lattice point, and let $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$ be reals with $n = \alpha_1 + \alpha_2 + \cdots + \alpha_m + \beta$. Let $Q_0 = (0, \beta), Q_1 = (1, \beta + \alpha_m), Q_2 = (2, \beta + \alpha_m + \alpha_{m-1}), \ldots,$ $Q_i = (i, \beta + \sum_{0 \le j < i} \alpha_{m-j}) = (i, n - \alpha_1 - \cdots - \alpha_{m-i}), \ldots, Q_m = Q$. For $a = (\alpha_1, \ldots, \alpha_m)$, a boundary ∂a consists of line segments connecting $Q_0, Q_1, \ldots, Q_{m-1}, Q_m$ in this order.

Fix a lattice point $P = (m - m', n - n') \in \mathbb{N}^2$ below ∂a, that is, the y-coordinate of P is at most that of $Q_{m-m'}$. Let V' be the set of walks from Q to P, so $|V'| = \binom{m'+n'}{m'}$. For a walk $w = (w_1, \ldots, w_{m'+n'}) \in V'$, let us define the minimum y-distance of w with respect to ∂a, denoted by $\delta(w, \partial a)$, in the same manner as in Section 1: if w touches or crosses ∂a after the first step, then $\delta(w, \partial a) = 0$, otherwise $\delta(w, \partial a)$ is the minimum of the difference between Q_{m_0} and (m_0, n_0), where (m_0, n_0) runs over lattice points on w with $(m_0, n_0) \neq Q$. Also, for each $w \in V'$, we assign the corresponding sequence $\operatorname{seq}(w, \partial a) = (z_1, \ldots, z_{m'+n'})$ as follows: if w_i is a down step, then let $z_i = 1$; if w_i is the j-th left step, then let $z_i = -\alpha_j$. Finally, we define the weight of w with respect to ∂a by $\theta(w, \partial a) = \theta(\operatorname{seq}(w, \partial a))$. (See the definition of θ before Lemma 1.) Recall that $a^{\langle t \rangle} = (\alpha_{1+t}, \alpha_{2+t}, \ldots, \alpha_{m+t})$ where the indices are read modulo m. Similarly to Claim 3, we have

$$\theta(w, \partial a^{\langle t \rangle}) = \delta(w, \partial a^{\langle t \rangle}) \tag{3}$$

for all $w \in V'$ and $0 \le t < m$.

EXAMPLE 2. *Let $m = m' = 4$, $n = n' = 5$, $a = (2, -\sqrt{2}, 3, 0)$ and $\beta = \sqrt{2}$. In this case, $\delta(w, \partial a^{\langle t \rangle})$ is one of 0, $d_1 = \sqrt{2} - 1$, $d_2 = 1$, $d_3 = \sqrt{2}$ for $0 \le t < 4$ and for*

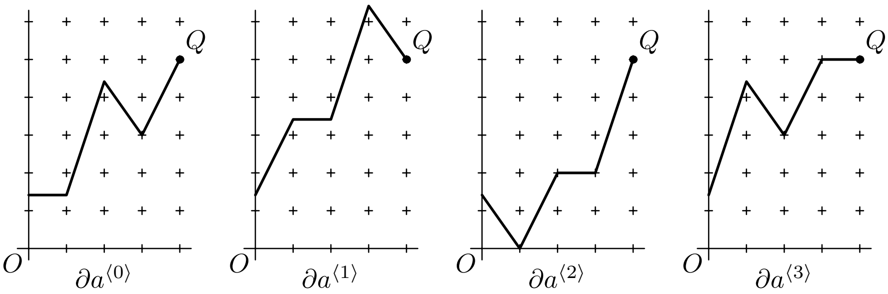

FIG. 4. *Cyclically shifting boundaries with* $a = (2, -\sqrt{2}, 3, 0)$, $\beta = \sqrt{2}$.

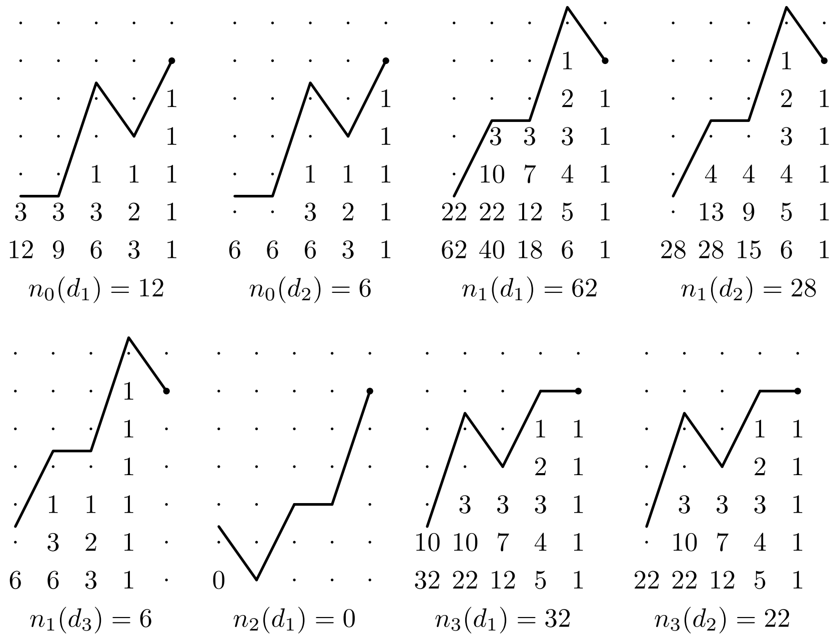

FIG. 5. $n_t(s) = \#\{w \in V' : \delta(w, \partial a^{\langle t\rangle}) \geq s\}$.

$w \in V'$. *(See Figure 4 and Figure 5.) Then we have*

$$\sum_{w\in V'} \delta(w, \partial a^{\langle 0\rangle}) = (12-6)d_1 + 6d_2 = 6\sqrt{2},$$
$$\sum_{w\in V'} \delta(w, \partial a^{\langle 1\rangle}) = (62-28)d_1 + (28-6)d_2 + 6d_3 = 40\sqrt{2} - 12,$$
$$\sum_{w\in V'} \delta(w, \partial a^{\langle 2\rangle}) = 0,$$
$$\sum_{w\in V'} \delta(w, \partial a^{\langle 3\rangle}) = (32-22)d_1 + 22d_2 = 10\sqrt{2} + 12.$$

Hence, we have

$$\sum_{0\leq t<4} \sum_{w\in V'} \delta(w, \partial a^{\langle t\rangle}) = 56\sqrt{2} = \frac{4\cdot 5 - (5-\sqrt{2})4}{4+5}\binom{4+5}{4},$$

which verifies Theorem 2.

Theorem 2. *Let m, n be positive integers and let $\alpha_1, \ldots, \alpha_m, \beta$ be (possibly negative) reals with $n = \alpha_1+\alpha_2+\cdots+\alpha_m+\beta$. Fix a lattice point $P = (m-m', n-n') \in \mathbb{N}^2$ below ∂a, where $a = (\alpha_1, \ldots, \alpha_m)$, and let V' be the set of walks from (m, n) to P. If P is below $\partial a^{\langle t\rangle}$ for all $0 \le t < m$, then we have*

$$\sum_{0\le t<m} \sum_{w\in V'} \delta(w, \partial a^{\langle t\rangle}) = \frac{mn' - (n-\beta)m'}{m'+n'} \binom{m'+n'}{m'}.$$

It is worth noting that the RHS of the above formula is independent of the decomposition of $a = (\alpha_1, \ldots, \alpha_m)$, and it depends only on the sum $\alpha_1 + \cdots + \alpha_m$. We can deduce Theorem 1 from Theorem 2 by setting $\alpha = \alpha_1 = \cdots = \alpha_m$, $m' = m$, and $n' = n$.

Proof. Let $0 \le t < m$ and $w \in V'$. Recall that $w^{\langle j\rangle} \in V'$ is a cyclic shift (modulo $m' + n'$) of w starting from $(j+1)$-th step. Since w consists of n' down steps and m' left steps, the total sum of $\operatorname{seq}(w, \partial a^{\langle t\rangle})$ is $n' - (\alpha_{1+t} + \cdots + \alpha_{m'+t})$, where the indices are read modulo m. Thus, by (3) and Lemma 1, we have

$$\sum_{0\le j<m'+n'} \delta(w^{\langle j\rangle}, \partial a^{\langle t\rangle}) = \sum_{0\le j<m'+n'} \theta(w^{\langle j\rangle}, \partial a^{\langle t\rangle}) = n' - (\alpha_{1+t} + \cdots + \alpha_{m'+t}).$$

Using

$$\sum_{0\le t<m} (\alpha_{1+t} + \cdots + \alpha_{m'+t}) = \sum_{1\le i\le m'} \sum_{0\le t<m} \alpha_{i+t} = \sum_{1\le i\le m'} (n-\beta) = m'(n-\beta),$$

we have

$$\sum_{0\le t<m} \sum_{0\le j<m'+n'} \delta(w^{\langle j\rangle}, \partial a^{\langle t\rangle}) = \sum_{0\le t<m} (n' - (\alpha_{1+t} + \cdots + \alpha_{m'+t})) = mn' - (n-\beta)m'.$$

Since each walk $w \in V'$ appears $m' + n'$ times in a multiset $\{w^{\langle j\rangle} : w \in V', 0 \le j < m' + n'\}$, we have

$$(m'+n') \sum_{w\in V'} \delta(w, \partial a^{\langle t\rangle}) = \sum_{w\in V'} \sum_{0\le j<m'+n'} \delta(w^{\langle j\rangle}, \partial a^{\langle t\rangle}).$$

Therefore, we have

$$\begin{aligned}(m'+n') \sum_{0\le t<m} \sum_{w\in V'} \delta(w, \partial a^{\langle t\rangle}) &= \sum_{w\in V'} \sum_{0\le t<m} \sum_{0\le j<m'+n'} \delta(w^{\langle j\rangle}, \partial a^{\langle t\rangle}) \\ &= \sum_{w\in V'} (mn' - (n-\beta)m') = (mn' - (n-\beta)m')|V'|,\end{aligned}$$

which completes the proof of Theorem 2. ◻

Next we consider the case when $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$ are (possibly negative) integers. For an integer t, $0 \le t < m$, let U_t be the set of walks in V' which touch the shifted boundary $\partial a^{\langle t\rangle}$ at Q only. By definition, we have $\delta(w, \partial a^{\langle t\rangle}) = 1$ if $w \in U_t$, and $\delta(w, \partial a^{\langle t\rangle}) = 0$ otherwise. This gives $|U_t| = \sum_{w\in V'} \delta(w, \partial a^{\langle t\rangle})$. Then Theorem 2 implies the following.

Corollary 2. *Under the same assumptions as in Theorem 2, if $\alpha_1, \ldots, \alpha_m, \beta$ are (possibly negative) integers, then we have*

$$\sum_{0\le t<m} |U_t| = \frac{mn' - (n-\beta)m'}{m'+n'} \binom{m'+n'}{m'}.$$

We get the main result of Irving–Rattan, Theorem 1 of [6], from Corollary 2 by setting our parameters (m, n, m', n', β) to $(m, n+1, \ell, k+1, 1)$. (They proved the case that $\beta = 1$ and $\alpha_1, \ldots, \alpha_m$ are all non-negative.) For comparison, we remark that the roles of x-axis and y-axis in Corollary 1 are the opposite of those in their result, and our condition "P is below $\partial a^{\langle t\rangle}$ for all t" is equivalent to their condition "$t' = (k+1, \ell)$ lies weakly to the right of $\partial a^{\langle j\rangle}$ for all j."

3.2. Walks having k peaks. We give a refinement of Theorem 1, cf. Theorem 8 of [6]. Let α, β be positive reals, and let m, n be positive integers. Fix a line $L : y = \alpha x + \beta$, and a lattice point $Q = (m, n)$ on L. We use the notation V, W, $\delta(w)$ for $w \in V$, as the same meaning as in Section 1. For a walk $w \in V$, a down step followed by a left step in w is called a peak of w, and a left step followed by a down step in w is called a valley of w. For example, a walk described in Figure 1 has three peaks and two valleys. Let $V(k)$ (resp. $W(k)$) be the set of walks $w \in V$ (resp. $w \in W$) having k peaks for a non-negative integer k.

In the case α is a positive integer and $\beta = 1$, the following result is given as Theorem 3.4.3 of [7] (see also Theorem 7 of [3]).

Theorem 3. *Let m, n be positive integers, and let α, β be positive reals with $n = \alpha m + \beta$. Let V be the set of walks from (m, n) to the origin. Then, we have*

$$\sum_{w\in V(k)} \delta(w) = \frac{\beta}{k}\binom{m-1}{k-1}\binom{n-1}{k-1}.$$

If $n = m+1$ ($\alpha = \beta = 1$), then the RHS becomes $\frac{1}{m}\binom{m}{k}\binom{m}{k-1}$, which is called the Narayana number, for example, see Exercise 6.36 of [9]. We notice that Theorem 1 is derived from Theorem 3. Indeed, by taking sum over $k \geq 1$, we have

$$\begin{aligned}\sum_{w\in V} \delta(w) &= \sum_{k\geq 1}\sum_{w\in V(k)} \delta(w) \\ &= \sum_{k\geq 1} \frac{\beta}{k}\binom{m-1}{k-1}\binom{n-1}{k-1} = \sum_{k\geq 1}\frac{\beta}{m}\binom{m}{k}\binom{n-1}{n-k} \\ &= \frac{\beta}{m}\binom{m+n-1}{n} = \frac{\beta}{m+n}\binom{m+n}{m}.\end{aligned}$$

For a non-negative integer s, the subset of integers $\{1, 2, \ldots, s\}$ is denoted by $[1, s]$. For a set X, the family of all k-element subsets of X is denoted by $\binom{X}{k}$.

Proof of Theorem 3. Let $U(k) \subset V(k)$ be the set of walks in which the first step is a down step and the last step is a left step. For $\{x_1, \ldots, x_{k-1}\} \in \binom{[m-1]}{k-1}$ ($x_1 < \cdots < x_{k-1} < x_k := m$) and $\{y_1, \ldots, y_{k-1}\} \in \binom{[n-1]}{k-1}$ ($y_0 := 0 < y_1 < \cdots < y_{k-1}$), we can associate a path $w \in U(k)$ with peaks at (x_i, y_{i-1}) ($1 \leq i \leq k$). This gives

$$|U(k)| = \binom{m-1}{k-1}\binom{n-1}{k-1}.$$

For all $w \in V$, the total sum of $\mathrm{seq}(w)$ is $n - \alpha m = \beta$. Thus, by Claim 3 and Lemma 1, we have $\sum_{0\leq j<m+n} \delta(w^{\langle j\rangle}) = \sum_{0\leq j<m+n} \theta(w^{\langle j\rangle}) = \beta$. Hence, we have

$$\sum_{w\in U(k)}\sum_{0\leq j<m+n} \delta(w^{\langle j\rangle}) = \beta|U(k)|.$$

For a walk $u \in U(k)$ and for $0 \leq j < m+n$, we notice that if $\delta(u^{\langle j\rangle}) > 0$ then $u^{\langle j\rangle}$ also has exactly k peaks, because $u^{\langle j\rangle}$ starts with a down step. Thus we have $\delta(u^{\langle j\rangle}) > 0$ iff $u^{\langle j\rangle} \in W(k)$. On the other hand, for a given walk $w \in W(k)$, there exist exactly k pairs (u, j) with $u \in U(k)$, $0 \leq j < m+n$ such that $w = u^{\langle j\rangle}$. In fact, if $w \in W(k) \cap U(k)$, then w has exactly $k-1$ valleys from which we get walks $u \in U(k)$ with $w = u^{\langle j\rangle}$ for some j; if $w \in W(k) \setminus U(k)$, then w has exactly k valleys from which we get walks satisfying the same property. Therefore, we have

$$k \sum_{w \in V(k)} \delta(w) = k \sum_{w \in W(k)} \delta(w) = \sum_{u \in U(k)} \sum_{0 \leq j < m+n} \delta(u^{\langle j\rangle}) = \beta |U(k)|,$$

as desired. □

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