A product version of the Erdős–Ko–Rado theorem

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Abstract

Let $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset \binom{[n]}{k}$ be r-cross t-intersecting, that is, $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$. We prove that for every $p, \mu \in (0,1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \leq t < (1/p - \mu)^{r-1}/(1-p) - 1$, there exist n_0 and ϵ so that if $n > n_0$ and $|k/n - p| < \epsilon$, then $|\mathcal{F}_1| \cdots |\mathcal{F}_r| \leq \binom{n-t}{k-t}^r$.

Keywords: Erdős–Ko–Rado Theorem, cross intersecting family, random walk

1. Introduction

Let n, k, r and t be positive integers with $t \leq k \leq n$, and let $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{G} \subset 2^{[n]}$ is called r-wise t-intersecting if $|G_1 \cap \cdots \cap G_r| \geq t$ holds for all $G_1, \ldots, G_r \in \mathcal{G}$. For example, let us consider the case $r = 2$. The following two families are both k-uniform 2-wise t-intersecting families:

$$\mathcal{A}_0 = \{A \in \tbinom{[n]}{k} : [t] \subset A\},$$
$$\mathcal{A}_1 = \{A \in \tbinom{[n]}{k} : |A \cap [t+2]| \geq t+1\}.$$

By comparing the sizes of $\mathcal{A}_0 \setminus \mathcal{A}_1$ and $\mathcal{A}_1 \setminus \mathcal{A}_0$, we see that

$$|\mathcal{A}_0| \geq |\mathcal{A}_1| \text{ iff } n \geq (t+1)(k-t+1). \tag{1}$$

Frankl [6] and Wilson [29] proved that if $n > (t+1)(k-t+1)$ then $|\mathcal{A}_0| = \binom{n-t}{k-t}$ is the maximum size of 2-wise t-intersecting families in $\binom{[n]}{k}$, and $\mathcal{A}_0$ is the only optimal family up to isomorphism. Recall that two families $\mathcal{G}, \mathcal{G}' \subset 2^{[n]}$ are said to be isomorphic, and denoted by $\mathcal{G} \cong \mathcal{G}'$, if there exists a vertex permutation τ on $[n]$ such that $\mathcal{G}' = \{\{\tau(g) : g \in G\} : G \in \mathcal{G}\}$.

Let us define a typical r-wise t-intersecting family $\mathcal{B}_\ell(n, r, t)$ and its k-uniform subfamily $\mathcal{A}_\ell(n, k, r, t)$, where $0 \leq \ell \leq \lfloor \frac{n-t}{r} \rfloor$, as follows:

$$\mathcal{B}_\ell(n, r, t) = \{B \subset [n] : |B \cap [t + r\ell]| \geq t + (r-1)\ell\},$$
$$\mathcal{A}_\ell(n, k, r, t) = \mathcal{B}_\ell(n, r, t) \cap \tbinom{[n]}{k}.$$

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Let $m(n,k,r,t)$ be the maximum size of k-uniform r-wise t-intersecting families on n vertices. The problem of determining $m(n,k,r,t)$ goes back to Erdős–Ko–Rado [4], and is still wide open. All known results, e.g., [1, 4, 5, 6, 7, 12, 22, 23, 24, 25, 26, 27, 29], suggest

$$m(n,k,r,t) = \max_{\ell} |\mathcal{A}_\ell(n,k,r,t)|. \tag{2}$$

A notable result due to Ahlswede and Khachatrian [1] says that the conjecture (2) is true for $r = 2$.

For a fixed real $p \in (0,1)$, we will consider the situation that $n,k \to \infty$ with keeping $p = k/n$. In this case, we see from (1) that

$$\lim_{n,k\to\infty} |\mathcal{A}_0(n,k,2,t)|/|\mathcal{A}_1(n,k,2,t)| \geq 1 \text{ iff } t+1 \leq 1/p.$$

In this paper, we are interested in the situation that

$$m(n,k,r,t) = \binom{n-t}{k-t}, \tag{3}$$

and the above fact suggests that the range of such t, for fixed $p = k/n$ and r, is bounded from above by a function of p. In fact, a direct computation (see Lemma 10) shows that

$$\lim_{n\to\infty} |\mathcal{A}_0(n,k,r,t)|/|\mathcal{A}_1(n,k,r,t)| \geq 1 \text{ iff } 1 \leq t \leq (p^{1-r} - p)/(1-p) - r =: t_{p,r}. \tag{4}$$

Let us see some other examples. If $n \geq 2k$, then $m(n,k,2,1) = |\mathcal{A}_0(n,k,2,1)| = \binom{n-1}{k-1}$. A 2-wise 1-intersecting family is also an r-wise 1-intersecting family for any $r \geq 2$. So, we have $m(n,k,r,1) = \binom{n-1}{k-1}$ for $r \geq 2$ and $k/n \leq 1/2$.

If $n < 2k$, then $\binom{[n]}{k} = \mathcal{A}_{\lfloor \frac{n-1}{2} \rfloor}(n,k,2,1)$ is a 2-wise 1-intersecting family. But $\binom{[n]}{k}$ is not necessarily r-wise 1-intersecting for $r \geq 3$. In fact, it is known from [5, 7, 12, 19] that $m(n,k,r,1) = \binom{n-1}{k-1}$ iff $n \geq rk/(r-1)$. Thus we have

$$m(n,k,r,1) = \tbinom{n-1}{k-1} \text{ for } r \geq (1 - \tfrac{k}{n})^{-1}.$$

In general, if $rk \geq (r-1)n + t$, then $\binom{[n]}{k}$ is r-wise t-intersecting. So, to get (3) for a fixed $p = k/n$ with $n,k \to \infty$, we need $p \leq \frac{r-1}{r}$, or equivalently, $r \geq \frac{1}{1-p}$. Namely, the range of such r is bounded from below by a function of p. The following sample result, essentially proved in [27], is a starting point of our research (cf. Corollary 1).

Theorem 1. *For every $p \in (0,1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \leq t < t_{p,r}$, there exists n_0 so that if $n > n_0$ and $k/n \leq p$, then*

$$m(n,k,r,t) = |\mathcal{A}_0(n,k,r,t)| = \tbinom{n-t}{k-t}.$$

Moreover $\mathcal{A}_0(n,k,r,t)$ is the only optimal family (up to isomorphism).

In this paper, we will extend the above result to cross intersecting families. Indeed, we will consider two types of problems related to (2); one is about **k**-uniform cross intersecting families and the other is about the **p**-weight of cross intersecting families.

A set of families $\{\mathcal{G}_1,\ldots,\mathcal{G}_r\}$, where $\mathcal{G}_1,\ldots,\mathcal{G}_r \subset 2^{[n]}$, is called r-cross t-intersecting if $|G_1 \cap \cdots \cap G_r| \geq t$ holds for all $G_1 \in \mathcal{G}_1,\ldots,G_r \in \mathcal{G}_r$. We first consider a **k**-uniform product version of the Erdős–Ko–Rado problem. Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ and let us define

$$m^\times(n, \mathbf{k}, r, t) = \max \prod_{s=1}^{r} |\mathcal{F}_s|,$$

where the maximum is taken over all r-cross t-intersecting families $\{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ with $\mathcal{F}_s \subset \binom{[n]}{k_s}$ for $1 \le s \le r$. If $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise t-intersecting, then $\{\mathcal{F}, \dots, \mathcal{F}\}$ (the set of r copies of $\mathcal{F}$) is clearly r-cross t-intersecting. This means

$$m(n, k, r, t)^r \le m^\times(n, \{k, \dots, k\}, r, t).$$

Some values of $m^\times(n, \mathbf{k}, r, t)$ are known. For example, Pyber [20], Matsumoto and the author [16], and Bey [2] proved the following:

$$m^\times(n, \{a, b\}, r = 2, t = 1) = \tbinom{n-1}{a-1}\tbinom{n-1}{b-1} \text{ for } n \ge \max\{2a, 2b\}.$$

Frankl and the author [9] proved that

$$m^\times(n, \{k, \dots, k\}, r, t = 1) = \tbinom{n-1}{k-1}^r \text{ for } (r-1)n \ge rk.$$

In [28], it is proved that for all p, t with $0 < p < 0.11$ and $1 \le t \le 1/(2p)$, there exists n_0 such that for all n, k with $n > n_0$ and $k/n = p$ the following holds:

$$m^\times(n, \{k, k\}, r = 2, t) = \tbinom{n-t}{k-t}^2.$$

Our first result is a generalization of these results for the case that k/n is bounded and r is large enough. For $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{R}^r$, let $\|\mathbf{x}\| = (x_1^2 + \dots + x_r^2)^{1/2}$, and let

$$\tau_{p,r,\mu} := \left(1/p - \mu\right)^{r-1}/(1-p) - 1.$$

Theorem 2. *For all $p, p', \mu \in (0, 1)$ with $p \ge p'$, there exists r_1 such that the following holds. For all $r \ge r_1$, all t with $1 \le t \le \tau_{p,r,\mu}$, and all $\mathbf{p} = (p_1, \dots, p_r) \in [p', p]^r$, there exist positive constants ϵ, n_0 such that*

$$m^\times(n, \mathbf{k}, r, t) = \prod_{s=1}^{r} |\mathcal{A}_0(n, k_s, r, t)| = \prod_{s=1}^{r} \tbinom{n-t}{k_s-t}$$

holds for all $n > n_0$ and all $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ with $\|\frac{1}{n}\mathbf{k} - \mathbf{p}\| < \epsilon$. Moreover $\{\mathcal{A}_0(n, k_s, r, t) : 1 \le s \le r\}$ is the only set of optimal families (up to isomorphism).

By taking $p = p'$ and $k_1 = \dots = k_r = k$ in Theorem 2, we get the following statement appeared in the abstract, which is a cross intersecting extension of Theorem 1 with a weaker bound for t.

Corollary 1. *For every $p, \mu \in (0, 1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \le t \le \tau_{p,r,\mu}$, there exist n_0 and ϵ so that if $n > n_0$ and $|k/n - p| < \epsilon$, then*

$$m^\times(n, \{k, \dots, k\}, r, t) = |\mathcal{A}_0(n, k, r, t)|^r = \tbinom{n-t}{k-t}^r.$$

For fixed p, μ, it is easy to see that $\tau_{p,r,\mu} < t_{p,r}$ if r is large enough. So we will assume $\tau_{p,r,\mu} < t_{p,r}$ in the proof of Theorem 2, but the upper bound for t in Theorem 2 could be possibly replaced with $t_{p,r}$. If so, the bound $t_{p,r}$ would be tight. Theorem 1 says that Corollary 1 is true for all $1 \leq t \leq t_{p,r}$ if $\mathcal{F}_1 = \cdots = \mathcal{F}_r$.

Next we introduce a p-weight version of the Erdős–Ko–Rado problem. Throughout this paper, p and $q = 1 - p$ denote positive real numbers. For $X \subset [n]$ and a family $\mathcal{G} \subset 2^X$ we define the p-weight of $\mathcal{G}$, denoted by $w_p(\mathcal{G} : X)$, as follows:

$$w_p(\mathcal{G} : X) = \sum_{G \in \mathcal{G}} p^{|G|} q^{|X|-|G|} = \sum_{i=0}^{|X|} \left| \mathcal{G} \cap \binom{X}{i} \right| p^i q^{|X|-i}.$$

We need this definition in the proof of Lemma 3, otherwise we simply write $w_p(\mathcal{G})$ for the case $X = [n]$. For example, we have $w_p(\mathcal{B}_0(n, r, t)) = p^t$, and the p-weight version of (4) is the following:

$$w_p(\mathcal{B}_0(n, r, t)) \geq w_p(\mathcal{B}_1(n, r, t)) \text{ iff } 1 \leq t \leq t_{p,r}.$$

Let $w(n, p, r, t)$ be the maximum p-weight of r-wise t-intersecting families on n vertices. For fixed $p = k/n$, it follows that $\lim_{n\to\infty} |\mathcal{A}_\ell(n, k, r, t)|/\binom{n}{k} = w_p(\mathcal{B}_\ell(n, r, t))$. In fact, we will see that $m(n, k, r, t)/\binom{n}{k}$ and $w(n, p, r, t)$ are corresponding if $p \approx k/n$, so it might be natural to expect

$$w(n, p, r, t) = \max_{\ell} w_p(\mathcal{B}_\ell(n, r, t)).$$

For example, Theorem 1 has the following p-weight version.

Theorem 3 ([27])**.** *For all $p \in (0, 1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \leq t \leq t_{p,r}$, and all $n \geq t$, we have*

$$w(n, p, r, t) = w_p(\mathcal{B}_0(n, r, t)) = p^t.$$

Moreover, $\mathcal{B}_0(n, r, t)$ is the only optimal family for $1 \leq t < t_{p,r}$, and $\mathcal{B}_0(n, r, t)$ and $\mathcal{B}_1(n, r, t)$ are the only optimal families for $t = t_{p,r}$ (up to isomorphism).

To consider a **p**-weight product version, where $\mathbf{p} = (p_1, \ldots, p_r) \in (0, 1)^r$, let us define

$$w^\times(n, \mathbf{p}, r, t) = \max \prod_{s=1}^{r} w_{p_s}(\mathcal{G}_s),$$

where the maximum is taken over all r-cross t-intersecting families $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\}$ on n vertices. We have

$$w(n, p, r, t)^r \leq w^\times(n, \{p, \ldots, p\}, r, t).$$

Frankl studied the case $p_1 = \cdots = p_r = 1/2$ in [7, 8], and he obtained

$$w^\times(n, \{1/2, \ldots, 1/2\}, r, t) = (1/2)^{tr} \tag{5}$$

for $r \geq 3$ and $1 \leq t \leq 2^r - r - 2 = t_{1/2,r} - 1$. Our second result is an extension of (5) as follows.

Theorem 4. *For all $p, p', \mu \in (0,1)$ with $p \geq p'$, there exists r_1 such that for all $r \geq r_1$, all t with $1 \leq t \leq \tau_{p,r,\mu}$, all $\mathbf{p} = (p_1, \dots, p_r) \in [p', p]^r$, and all $n \geq t$, we have*

$$w^{\times}(n, \mathbf{p}, r, t) = \prod_{s=1}^{r} w_{p_s}(\mathcal{B}_0(n, r, t)) = \prod_{s=1}^{r} {p_s}^{t}.$$

Moreover r copies of $\mathcal{B}_0(n, r, t)$ are the only optimal families (up to isomorphism).

We conjecture that Theorem 4 is true for all $\mathbf{p} \in (0, p)^r$, all $r \geq 1/(1-p)$, and all t with $1 \leq t \leq t_{p,r}$.

Theorem 2 and Theorem 4 state essentially the same thing in a different way. It is often the case that the p-weight version (such as Theorem 3 and Theorem 4) is technically easier to handle than the corresponding k-uniform version (such as Theorem 1 Theorem 2). So the basic strategy for proofs of these results is to show the p-weight version first, and then to deduce the k-uniform version from the p-weight version. We will take this strategy. Unfortunately, Theorem 4 is not strong enough to show Theorem 2. So we will consider stronger, "stability" type results corresponding to both versions.

To explain what stability means, we start with Theorem 1. In this result, families isomorphic to $\mathcal{A}_0(n, k, r, t)$ only attain $m(n, k, r, t) = \binom{n-t}{k-t}$. Let $\mathbf{G}(n, k, r, t)$ be the set of all k-uniform r-wise t-intersecting families on n vertices. By choosing a subfamily of $\mathcal{A}_0(n, k, r, t)$, we get an $\mathcal{F} \in \mathbf{G}(n, k, r, t)$ with size $|\mathcal{F}|$ as close to $\binom{n-t}{k-t}$ as we want. But if we choose $\mathcal{F} \in \mathbf{G}(n, k, r, t)$ which is not a subfamily of a family isomorphic to $\mathcal{A}_0(n, k, r, t)$, then the size $|\mathcal{F}|$ must be much smaller. To state more precisely, let $m_0(n, k, r, t)$ be the maximum size of $\mathcal{F} \in \mathbf{G}(n, k, r, t)$ such that there is no $\mathcal{A} \approx \mathcal{A}_0(n, k, r, t)$ satisfying $\mathcal{F} \subset \mathcal{A}$. Then the stability version of Theorem 1 is as follows.

Theorem 5. *For every $p \in (0,1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \leq t < t_{p,r}$, there exist positive constants n_0 and γ so that if $n > n_0$ and $k/n \leq p$, then $m_0(n, k, r, t) < (1-\gamma)\binom{n-t}{k-t}$.*

In other words, if we have an $\mathcal{F} \in \mathbf{G}(n, k, r, t)$ with size very close to $\binom{n-t}{k-t}$, then $\mathcal{F}$ itself is close to $\mathcal{A}_0(n, k, r, t)$ (or a family isomorphic to $\mathcal{A}_0(n, k, r, t)$). Indeed, we get a family isomorphic to $\mathcal{A}_0(n, k, r, t)$ by adding edges to $\mathcal{F}$. In this case, we say that the optimal configuration for the problem is stable. (See [3, 10, 14, 17, 18, 19] for some related stability type results.) We also have a stability p-weight version corresponding to Theorem 5 in [27]. As we will see, such a stability p-weight result is strong enough to deduce the corresponding stability k-uniform result, cf. Theorem 8 in section 4.

Now we return to our r-cross t-intersecting problem. To state our main results, let us define some collections of cross intersecting families:

$$\mathbf{G}^{\times}(n, r, t) = \{\{\mathcal{G}_1, \dots, \mathcal{G}_r\} \subset 2^{2^{[n]}} : \{\mathcal{G}_1, \dots, \mathcal{G}_r\} \text{ is } r\text{-cross } t\text{-intersecting}\},$$

$$\mathbf{G}_j^{\times}(n, r, t) = \{\{\mathcal{G}_1, \dots, \mathcal{G}_r\} \in \mathbf{G}^{\times}(n, r, t) : 1 \leq \forall s \leq r, \exists \mathcal{G}'_s \cong \mathcal{B}_j(n, r, t) \text{ s.t. } \mathcal{G}_s \subset \mathcal{G}'_s\},$$

$$\mathbf{X}_\ell^{\times}(n, r, t) = \mathbf{G}^{\times}(n, r, t) \setminus \textstyle\bigcup_{0 \leq j \leq \ell} \mathbf{G}_j^{\times}(n, r, t),$$

$$\mathbf{Y}_\ell^{\times}(n, \mathbf{k}, r, t) = \{\{\mathcal{F}_1, \dots, \mathcal{F}_r\} \in \mathbf{X}_\ell^{\times}(n, r, t) : \mathcal{F}_s \subset \tbinom{[n]}{k_s} \text{ for all } 1 \leq s \leq r\},$$

where $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$. We remark that $\mathcal{G}'_1, \dots, \mathcal{G}'_r$ appeared in the definition of $\mathbf{G}_j^{\times}(n, r, t)$ are all isomorphic to $\mathcal{B}_j(n, r, t)$, but we do not require $\mathcal{G}'_1 = \cdots = \mathcal{G}'_r$. Finally let us define

$$m_\ell^\times(n,\mathbf{k},r,t)=\max\{\prod_{s=1}^r|\mathcal{F}_s|:\{\mathcal{F}_1,\dots,\mathcal{F}_r\}\in\mathbf{Y}_\ell^\times(n,\mathbf{k},r,t)\},$$
$$w_\ell^\times(n,\mathbf{p},r,t)=\max\{\prod_{s=1}^r w_{p_s}(\mathcal{G}_s):\{\mathcal{G}_1,\dots,\mathcal{G}_r\}\in\mathbf{X}_\ell^\times(n,r,t)\},$$

where $\mathbf{p}=(p_1,\dots,p_r)\in(0,1)^r$. Our main results are the following.

Theorem 6. *For all $p,p',\mu\in(0,1)$ with $p\geq p'$, there exists r_1 such that the following holds. For all $r\geq r_1$, all t with $1\leq t\leq\tau_{p,r,\mu}$, and all $\mathbf{p}=(p_1,\dots,p_r)\in[p',p]^r$, there exist positive constants γ,ϵ,n_0 such that*

$$m_0^\times(n,\mathbf{k},r,t)<(1-\gamma)\prod_{s=1}^r\binom{n-t}{k_s-t}$$

holds for all $n>n_0$ and all $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{N}^r$ with $\|\frac{1}{n}\mathbf{k}-\mathbf{p}\|<\epsilon$.

Theorem 7. *For all $p,p',\mu\in(0,1)$ with $p\geq p'$, there exists r_1 such that the following holds. For all $r\geq r_1$, all t with $1\leq t\leq\tau_{p,r,\mu}$, and all $\mathbf{p}=(p_1,\dots,p_r)\in[p',p]^r$, there exist positive constants γ,ϵ such that*

$$w_0^\times(n,\tilde{\mathbf{p}},r,t)<(1-\gamma)\prod_{s=1}^r\tilde{p}_s{}^t$$

holds for all $n\geq t$ and all $\tilde{\mathbf{p}}=(\tilde{p}_1,\dots,\tilde{p}_r)\in(0,1)^r$ with $\|\tilde{\mathbf{p}}-\mathbf{p}\|<\epsilon$.

We will prove Theorem 7 in section 3, where the main ingredient of the proof is a generalization of Frankl's random walk method (Lemma 1). Then we will deduce Theorem 6 from Theorem 7 in section 4, where the key idea is simply that the binomial distribution $B(n,p)$ is concentrated around pn. The other theorems follow from Theorem 6, Theorem 7, and some results from [26, 27]. We include these easy proofs in Appendix.

2. Tools

For integers $1\leq i<j\leq n$ and a family $\mathcal{G}\subset 2^{[n]}$, we define the (i,j)-shift σ_{ij} as follows:

$$\sigma_{ij}(\mathcal{G})=\{\sigma_{ij}(G):G\in\mathcal{G}\},$$

where

$$\sigma_{ij}(G)=\begin{cases}(G-\{j\})\cup\{i\} & \text{if } i\notin G,\ j\in G,\ (G-\{j\})\cup\{i\}\notin\mathcal{G},\\ G & \text{otherwise.}\end{cases}$$

A family $\mathcal{G}\subset 2^{[n]}$ is called *shifted* if $\sigma_{ij}(\mathcal{G})=\mathcal{G}$ for all $1\leq i<j\leq n$. If $\mathcal{G}$ is r-wise t-intersecting, then so is $\sigma_{ij}(\mathcal{G})$, see [7]. Similarly, one can verify that

$$\text{if } \{\mathcal{G}_1,\dots,\mathcal{G}_r\}\in\mathbf{G}^\times(n,r,t),\text{ then } \{\sigma_{ij}(\mathcal{G}_1),\dots,\sigma_{ij}(\mathcal{G}_r)\}\in\mathbf{G}^\times(n,r,t)$$

as well. Repeating this process one can get a shifted r-cross t-intersecting families.

Now we introduce a key lemma to prove our main result Theorem 7. Let $p\in(0,1)$ and $q=1-p$. The equation $qx^r-x+p=0$ has a unique root in $(p,1)$, which is denoted by $\alpha_{r,p}$. The next lemma enables us to bound $w^\times(n,\mathbf{p},r,t)$ in terms of $\alpha_{r,p}$ with $p\in\mathbf{p}$.

Lemma 1. $w^\times(n,\mathbf{p},r,t) \le (\alpha_{r,p_1}\cdots\alpha_{r,p_r})^t$*, where* $\mathbf{p}=(p_1,\dots,p_r)\in(0,1)^r$.

To prove Theorem 7 we need to show $w^\times(n,\mathbf{p},r,t) \le (p_1\cdots p_r)^t$. Since $\alpha_{r,p} > p$, the above lemma does not give what we want directly. But, as we will see in the proof of Theorem 7, inside an r-cross t-intersecting families, we will be able to find a nice substructure such as $(r-1)$-cross $(t+1)$-intersecting families. Then Lemma 1 will be very useful.

This result was essentially proved by Frankl in [7]. He considered the case $p_1=\cdots=p_r=1/2$, but one can extend his proof quite naturally to the general case $\mathbf{p}\in(0,1)^r$. For convenience we include a sketch of the proof here. See [7, 26] for more details. We mention that Green and Tao used this fact in [11] as one of the tools for proving Freiman's theorem in finite fields.

Proof of Lemma 1. First we show $w(n,p,r,t)\le\alpha^t_{r,p}$. For $G\subset[n]$ we define the corresponding n-step walk on $\mathbb{Z}^2$, denoted by walk(G), as follows. The walk is from $(0,0)$ to $(|G|,n-|G|)$, and the i-th step is one unit up ($\uparrow$) if $i\in G$, or one unit to the right ($\rightarrow$) if $i\notin G$. Let $\mathcal{G}\subset 2^{[n]}$ be r-wise t-intersecting. We can find a shifted r-wise t-intersecting family $\mathcal{G}^*\subset 2^{[n]}$ with $w_p(\mathcal{G})=w_p(\mathcal{G}^*)$. Then a crucial observation is the following: for each $G\in\mathcal{G}^*$, walk(G) touches the line $L: y=(r-1)x+t$. Thus we have $\mathcal{G}^*\subset\mathcal{W}_n$, where $\mathcal{W}_n=\{W\subset[n]:\text{walk}(W)\text{ touches }L\}$. We note that $\mathcal{W}_n$ is not necessarily r-wise t-intersecting.

Now consider an infinite random walk in $\mathbb{Z}^2$ starting from $(0,0)$, taking $\uparrow$ with probability p, and $\rightarrow$ with probability $q=1-p$ at each step independently. Suppose that $\mathcal{G}$ is r-cross t-intersecting with maximum p-weight. Then it follows that

$$\begin{aligned} w(n,p,r,t) &= \sum_{G\in\mathcal{G}} p^{|G|}q^{n-|G|} \le \sum_{W\in\mathcal{W}_n} p^{|W|}q^{n-|W|} \le \lim_{n\to\infty}\sum_{W\in\mathcal{W}_n} p^{|W|}q^{n-|W|} \\ &= \text{Prob (the infinite random walk touches } L) = \alpha^t_{r,p}. \end{aligned}$$

Next we consider $w^\times(n,\mathbf{p},r,t)$. Let $\{\mathcal{G}_1,\dots,\mathcal{G}_r\}\in\mathbf{G}^\times(n,r,t)$ be shifted with maximum $\mathbf{p}$-weight. For $G_s\in\mathcal{G}_s$ $(1\le s\le r)$ let us define its characteristic vector $v(G_s)=(v_1^{(s)},\dots,v_n^{(s)})\in\{0,1\}^n$ by $v_i^{(s)}=1$ iff $i\in G_s$, and define $\bigoplus_{s=1}^r G_s\in\{0,1\}^{rn}$ by

$$(v_1^{(1)},v_1^{(2)},\dots,v_1^{(r)},v_2^{(1)},v_2^{(2)},\dots,v_2^{(r)},\dots,v_n^{(1)},\dots,v_n^{(r)}).$$

Then one can check that walk$(\bigoplus G_s)$ touches the line $\tilde{L}: y=(r-1)x+rt$. We consider a new infinite random walk in $\mathbb{Z}^2$ starting from $(0,0)$, which takes $\uparrow$ with probability p_s and $\rightarrow$ with probability $q_s=1-p_s$ at the i-th step iff $i\equiv s \bmod r$. We will associate the quantity $\langle\mathbf{p},\bigoplus_{s=1}^r G_s\rangle := \prod_{s=1}^r p_s^{|G_s|}q_s^{n-|G_s|}$ with the first rn steps of the infinite random walk. Namely, we have

$$\begin{aligned} w^\times(n,\mathbf{p},r,t) &= \prod_{s=1}^r \sum_{G_s\in\mathcal{G}_s} p_s^{|G_s|}q_s^{n-|G_s|} \\ &= \sum\{\langle\mathbf{p},\textstyle\bigoplus G_s\rangle : (G_1,\dots,G_r)\in\mathcal{G}_1\times\cdots\times\mathcal{G}_r\} \\ &\le \text{Prob (the new infinite random walk touches } \tilde{L}) = (\alpha_{r,p_1}\cdots\alpha_{r,p_r})^t. \end{aligned}$$

□

Lemma 2. *Let* $\mathbf{p} = (p_1, \ldots, p_r) \in (0,1)^r$ *and let* $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\} \in \mathbf{X}_0^{\times}(n, r, t)$ *be* $\mathbf{p}$*-weight maximum, namely,* $\prod_{s=1}^{r} w_{p_s}(\mathcal{G}_s) = w_0^{\times}(n, \mathbf{p}, r, t)$*. Then we can find a shifted families* $\{\mathcal{G}'_1, \ldots, \mathcal{G}'_r\} \in \mathbf{X}_0^{\times}(n, r, t)$ *with* $w_{p_s}(\mathcal{G}'_s) = w_{p_s}(\mathcal{G}_s)$ *for all* $1 \le s \le r$*, and* $I := \bigcap\{G' : G' \in \mathcal{G}'_1 \cup \cdots \cup \mathcal{G}'_r\} = \emptyset$*.*

Proof. Since $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\} \in \mathbf{X}_0^{\times}(n, r, t)$, one of the families is non-trivial, so we may assume that $|\bigcap\{G : G \in \mathcal{G}_r\}| < t$. If there are $G_i \in \mathcal{G}_i$ for $1 \le i \le r-1$ such that $T := G_1 \cap \cdots \cap G_{r-1}$ with $|T| = t$, then the r-cross t-intersecting property forces $T \subset G$ for all $G \in \mathcal{G}_r$, which is a contradiction. This means $\{\mathcal{G}_1, \ldots, \mathcal{G}_{r-1}\} \in \mathbf{G}^{\times}(n, r-1, t+1)$. By shifting $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\}$ simultaneously we get shifted families $\{\mathcal{G}'_1, \ldots, \mathcal{G}'_r\} \in \mathbf{G}^{\times}(n, r, t)$ with $w_{p_s}(\mathcal{G}'_s) = w_{p_s}(\mathcal{G}_s)$ for all $1 \le s \le r$. Note that $\{\mathcal{G}'_1, \ldots, \mathcal{G}'_{r-1}\} \in \mathbf{G}^{\times}(n, r-1, t+1)$.

We have to show $I = \emptyset$. If this is not the case, then we may assume that $1 \in I$ and $[2, n] \notin \mathcal{G}'_r$. Since $\prod_{s=1}^{r} w_{p_s}(\mathcal{G}_s) = \prod_{s=1}^{r} w_{p_s}(\mathcal{G}'_s)$ is maximum, adding $[2, n]$ to $\mathcal{G}'_r$ will destroy the r-cross t-intersecting property. This means we can find $G'_s \in \mathcal{G}'_s$ for $1 \le s < r$ such that $|G'_1 \cap \cdots \cap G'_{r-1} \cap [2, n]| < t$ and $|G'_1 \cap \cdots \cap G'_{r-1}| < t+1$, a contradiction. □

Lemma 3. *For any* $\ell \ge 0$*, we have* $w_\ell^{\times}(n+1, \mathbf{p}, r, t) \ge w_\ell^{\times}(n, \mathbf{p}, r, t)$*.*

Proof. Choose $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\} \in \mathbf{X}_\ell^{\times}(n, r, t)$ with $\prod_s w_p(\mathcal{G}_s) = w_\ell^{\times}(n, \mathbf{p}, r, t)$. Set $\mathcal{G}'_s := \mathcal{G}_s \cup \{G \cup \{n+1\} : G \in \mathcal{G}_s\}$ for $1 \le s \le r$. Then $\{\mathcal{G}'_1, \ldots, \mathcal{G}'_r\} \in \mathbf{X}_\ell^{\times}(n+1, r, t)$. Since $w_{p_s}(\mathcal{G}'_s : [n+1]) = w_{p_s}(\mathcal{G}_s : [n])(q+p)$, we have $\prod_s w_{p_s}(\mathcal{G}'_s : [n+1]) = \prod_s w_{p_s}(\mathcal{G}_s : [n]) = w_\ell^{\times}(n, \mathbf{p}, r, t)$, which gives the desired inequality. □

Lemma 4. *For all* $p, \epsilon \in (0,1)$ *there exists* r_0 *such that*

$$p^{-r} - (qr/p) - \epsilon < \alpha_{r,p}^{-r} < p^{-r} - (qr/p)$$

holds for all $r > r_0$*.*

Proof. We start with the representation

$$\alpha_{r,p}^{m} = \sum_{i \ge 0} \frac{m}{ri+m} \binom{ri+m}{i} p^{(r-1)i+m} q^i, \tag{6}$$

see e.g., [21]. Setting $m = -r$, this gives $\alpha_{r,p}^{-r} = p^{-r} - (qr/p) - \sum_{i \ge 1} f(i)$, where

$$f(i) = \frac{1}{i} \binom{ri}{i+1} p^{ri-i-1} q^{i+1},$$

Since $f(i) > 0$ for all $i \ge 1$, we get the upper bound.

For the lower bound, let $z = \max\{p, q\}$. Then we have

$$f(i) \le \frac{1}{i} \binom{ri}{i+1} z^{ri} < \frac{1}{i} \frac{(ri)^{i+1}}{(i+1)!} z^{ri} = \frac{r(riz^r)^i}{(i+1)!} < r(riz^r)^i (e/i)^i = r(rz^r e)^i,$$

which gives $\sum f(i) < \epsilon$ for r large enough. □

Lemma 5. *Let* $t > 0$*. Then* $\alpha_{r,p}^{t+1}/p^t$ *is an increasing function of* p *for* $0 < p < \frac{r-1}{r}$*.*

Proof. By (6) we have

$$\alpha_{r,p}^{t+1}/p^t = \sum_{i \ge 0} \frac{t+1}{ri+t+1} \binom{ri+t+1}{i} (p^{r-1}q)^i p.$$

Since $p^{r-1}q$ is an increasing function of p, the result follows. □

3. Proof of Theorem 7

Let $p, p', \mu \in (0,1)$ be given. Choose r_1 sufficiently large. More specifically, we choose r_1 so that (11), (13), and (14) below will be satisfied for all $r > r_1$. Choose $r > r_1$, $t \leq \tau_{p,r,\mu}$, and $\mathbf{p} = (p_1, \ldots, p_r) \in [p', p]^r$. Set $q = 1 - p$ and $\alpha = \alpha_{r-1,p}$. Also set $q_s = 1 - p_s$, $\alpha_s = \alpha_{r-1,p_s}$ for $1 \leq s \leq r$. By Lemma 3 we may assume that n is large enough.

Let $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\} \in \mathbf{X}_0^{\times}(n, r, t)$ be $\mathbf{p}$-weight maximum, namely, $w_1 \cdots w_r = w_0^{\times}(n, \mathbf{p}, r, t)$, where $w_s = w_{p_s}(\mathcal{G}_s)$ for $1 \leq s \leq r$. By Lemma 2 we may assume that each $\mathcal{G}_s$ is shifted and $\bigcap\{G : G \in \bigcup \mathcal{G}_s\} = \emptyset$. Thus we can find some $F \in \bigcup \mathcal{G}_s$ such that $t \notin F$, and we may assume that $F \in \mathcal{G}_r$ (by renaming families if necessary).

Let h be the maximum i such that $|H \cap [t+1]| = (t+1) - i$ holds for some $H \in \mathcal{G}_r$. Since $F \in \mathcal{G}_r$ we have $1 \leq h \leq t+1$. Then, $\{[t+1] \setminus G : G \in \mathcal{G}_r\} \subset \bigcup_{j=0}^{h} \binom{[t+1]}{j}$ implies

$$w_r \leq \sum_{j=0}^{h} \tbinom{t+1}{j} p_r^{t+1-j} q_r^j. \tag{7}$$

We may assume that $\mathcal{G}_r$ is co-complex, namely, if $G \in \mathcal{G}_r$ and $G \subset G'$ then $G' \in \mathcal{G}_r$. By the definition of h and shiftedness of $\mathcal{G}_r$, we have $H := [n] \setminus [t+2-h, t+1] \in \mathcal{G}_r$.

On the other hand, we have $\{\mathcal{G}_1, \ldots, \mathcal{G}_{r-1}\} \in \mathbf{G}^{\times}(n, r-1, t+h)$. In fact, if this is not the case, then we can find $G_s \in \mathcal{G}_s$ $(1 \leq s < r)$ such that $G_1 \cap \cdots \cap G_{r-1} = [t+h-1]$. By choosing $G_r = F \in \mathcal{G}_r$ if $h = 1$, or $G_r = H \in \mathcal{G}_r$ if $h \geq 2$, we have $|G_1 \cap \cdots \cap G_r| < t$, a contradiction. Thus Lemma 1 implies

$$w_1 \cdots w_{r-1} \leq (\alpha_1 \cdots \alpha_{r-1})^{t+h}. \tag{8}$$

By (7) and (8) we have $w_1 \cdots w_r \leq f_h(\mathbf{p})$, where

$$f_h(\mathbf{p}) := (\alpha_1 \cdots \alpha_{r-1})^{t+h} \sum_{j=0}^{h} \tbinom{t+1}{j} p_r^{t+1-j} q_r^j.$$

Thus we have $w_0^{\times}(n, \mathbf{p}, r, t) \leq \max\{f_h(\mathbf{p}) : 1 \leq h \leq t+1\}$. We will show

$$\max_h \{f_h(\mathbf{p})\} = f_1(\mathbf{p}) < (p_1 \cdots p_r)^t. \tag{9}$$

Since $f_1(\mathbf{p})$ is a continuous function of $\mathbf{p}$, Theorem 7 follows from (9) with the following simple observation.

Lemma 6. *Let r and t be fixed. Let $\mathbf{p} = (p_1, \ldots, p_r) \in (0,1)^r$. Suppose that $\prod_s w_{p_s}(\mathcal{G}_s) \leq f(\mathbf{p}) < (p_1 \cdots p_r)^t$ holds for some continuous function $f(\mathbf{x})$. Then there exist $\gamma, \epsilon > 0$ such that $\prod_s w_{\tilde{p}_s}(\mathcal{G}_s) < (1-\gamma)(\tilde{p}_1 \cdots \tilde{p}_r)^t$ for all $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_r) \in (0,1)^r$ with $\|\tilde{\mathbf{p}} - \mathbf{p}\| < \epsilon$.*

Fix $\mathbf{p}$ and let $f_h := f_h(\mathbf{p})$. The following two lemmas can be proved using standard calculus only, and then (9) follows immediately.

Lemma 7. $\max_h\{f_h\} = f_1 > f_2 > \cdots > f_{t+1}$.

Proof. Let $h \geq 1$ and we will show that $f_h > f_{h+1}$, or equivalently,

$$1 + \frac{\binom{t+1}{h+1}(\frac{q_r}{p_r})^{h+1}}{\sum_{j=0}^{h}\binom{t+1}{j}(\frac{q_r}{p_r})^j} < (\alpha_1 \cdots \alpha_{r-1})^{-1}.$$

The LHS is less than

$$1 + \frac{\binom{t+1}{h+1}(\frac{q_r}{p_r})^{h+1}}{\binom{t+1}{h}(\frac{q_r}{p_r})^h} = 1 + \frac{(t+1-h)q_r}{(h+1)p_r} \leq 1 + \frac{tq_r}{2p_r} < 1 + \frac{1}{2p'}(\frac{1}{p} - \mu)^{r-1},$$

where we used $p' \leq p_r$ and $t \leq \tau_{p,r,\mu}$ in the last inequality. On the other hand, applying Lemma 4 (with $r-1$ instead of r, and $\epsilon = 1$), we have

$$p^{1-r} - q(r-1)/p - 1 < \alpha^{1-r} \leq (\alpha_1 \cdots \alpha_{r-1})^{-1}. \tag{10}$$

Thus it suffices to show that

$$1 + \frac{1}{2p'}(\frac{1}{p} - \mu)^{r-1} < p^{1-r} - q(r-1)/p - 1, \tag{11}$$

which is true for sufficiently large r. □

Lemma 8. $f_1 < (p_1 \cdots p_r)^t$.

Proof. We have to show $(\alpha_1 \cdots \alpha_{r-1})^{t+1}(p_r^{t+1} + (t+1)p_r^t q_r) < (p_1 \cdots p_r)^t$, that is, $tq_r + 1 < (p_1 \cdots p_{r-1})^t/(\alpha_1 \cdots \alpha_{r-1})^{t+1}$. By Lemma 5 the RHS is at least $(p^t/\alpha^{t+1})^{r-1}$. Thus it suffices to show $t+1 < (p^t/\alpha^{t+1})^{r-1}$, or equivalently,

$$\alpha < p\left(\frac{p^{1-r}}{t+1}\right)^{\frac{1}{(t+1)(r-1)}}.$$

Using (10) it suffices to show

$$\left(p^{1-r} - \frac{q}{p}(r-1) - 1\right)^{-\frac{1}{r-1}} < p\left(\frac{p^{1-r}}{t+1}\right)^{\frac{1}{(t+1)(r-1)}},$$

that is,

$$1 < \left(1 - p^{r-1}(\frac{q}{p}(r-1) + 1)\right)\left(\frac{p^{1-r}}{t+1}\right)^{\frac{1}{t+1}}. \tag{12}$$

Since $t \leq \tau_{p,r,\mu}$, we have $a := \frac{1}{t+1} \geq b := q(\frac{1}{p} - \mu)^{1-r} > 0$. Using $(p^{1-r}a)^a \geq (p^{1-r}b)^b$, we see that the RHS of (12) is at least

$$A := \left(1 - p^{r-1}(\frac{q}{p}(r-1) + 1)\right)\left(p^{1-r}q(\frac{1}{p} - \mu)^{1-r}\right)^{q(\frac{1}{p}-\mu)^{1-r}}.$$

So it suffices to show that $\log A > 0$. Now we generously use $\log(1-x) > -2x$ for, say, $0 < x < 1/2$. Since r is large enough, we have

$$x := p^{r-1}\left(\frac{q}{p}(r-1) + 1\right) < 1/2 \tag{13}$$

and

$$\log A > -2p^{r-1}\big(\frac{q}{p}(r-1)+1\big) + q(\frac{1}{p}-\mu)^{1-r}\log q(1-p\mu)^{1-r}.$$

Consequently we need to show

$$q(\frac{1}{p}-\mu)^{1-r}(r-1)\log\big(\frac{q^{\frac{1}{r-1}}}{1-p\mu}\big) > 2p^{r-1}(\frac{q}{p}(r-1)+1),$$

or equivalently,

$$\frac{q}{(1-p\mu)^{r-1}}\log\big(\frac{q^{\frac{1}{r-1}}}{1-p\mu}\big) > 2\big(\frac{q}{p}+\frac{1}{r-1}\big), \tag{14}$$

which is certainly true for r sufficiently large. □

4. Proof of Theorem 6

We derive Theorem 6 from Theorem 7. This can be done by setting $\ell = 0$ in the next theorem. (We formally define $\mathcal{A}_{-1}(n,k,r,t) = \mathcal{B}_{-1}(n,r,t) = \emptyset$.)

Theorem 8. *Let $r,t,\ell \in \mathbb{N}$ and $\mathbf{p} = (p_1,\dots,p_r) \in (0,1)^r$ be given. Then (W) implies (M).*

(W) There exist positive constants $\gamma_0, \epsilon_0, n_0$ such that

$$w_\ell^\times(n,\tilde{\mathbf{p}},r,t) < (1-\gamma_0)\prod_{s=1}^{r}\max\{w_{\tilde{p}_s}(\mathcal{B}_{\ell-1}(n,r,t)), w_{\tilde{p}_s}(\mathcal{B}_\ell(n,r,t))\}$$

holds for all $\tilde{\mathbf{p}} = (\tilde{p}_1,\dots,\tilde{p}_r) \in (0,1)^r$ with $\|\tilde{\mathbf{p}}-\mathbf{p}\| < \epsilon_0$ and all n with $n \ge n_0$.

(M) There exist positive constants γ, ϵ, N such that

$$m_\ell^\times(n,\mathbf{k},r,t) < (1-\gamma)\prod_{s=1}^{r}\max\{|\mathcal{A}_{\ell-1}(n,k_s,r,t)|, |\mathcal{A}_\ell(n,k_s,r,t)|\}$$

holds for all $n > N$ and all $\mathbf{k} = (k_1,\dots,k_r) \in \mathbb{N}^r$ with $\|\frac{1}{n}\mathbf{k}-\mathbf{p}\| < \epsilon$.

For reals $0 < \epsilon < p$ we write $p \pm \epsilon$ to mean the open interval $(p-\epsilon, p+\epsilon)$, and for $n \in \mathbb{N}$, $n(p\pm\epsilon)$ means $((p-\epsilon)n, (p+\epsilon)n) \cap \mathbb{N}$. For $\mathbf{p} = (p_1,\dots,p_r) \in (0,1)^r$ let $\mathbf{p} \pm \epsilon = \prod_{s=1}^{r}(p_s \pm \epsilon) \subset \mathbb{R}^r$.

Proof. Let r,t,ℓ and $\mathbf{p}$ be given. To show (W) $\to$ (M), we prove that (W)$\wedge\neg$(M) is false. Namely, we assume both (W) and the negation of (M), then we will derive a contradiction by constructing a counterexample to (W).

If $n \ge t + r\ell$, then $w_p(\mathcal{B}_\ell(n,r,t))$ is independent of n. So, let $n \ge t+r\ell$ and let $g_\ell(p,r,t) = w_p(\mathcal{B}_\ell(n,r,t))$. Now (W) provides some γ_0, ϵ_0 and $n_0 \ge t + r\ell$ such that $w_\ell^\times(n,\tilde{\mathbf{p}},r,t) < (1-\gamma_0)f(\tilde{\mathbf{p}})$ holds for all $\tilde{\mathbf{p}}$ with $\|\tilde{\mathbf{p}}-\mathbf{p}\| < \epsilon_0$ and all $n \ge n_0$, where

$$f(\tilde{\mathbf{p}}) := \prod_{s=1}^{r}\max\{g_{\ell-1}(\tilde{p}_s,r,t), g_\ell(\tilde{p}_s,r,t)\}.$$

Let $\epsilon = \frac{\epsilon_0}{2}$, $\gamma = \frac{\gamma_0}{4}$, and $\tilde{\mathbf{I}} = \mathbf{p} \pm \epsilon$. We are going to define N. Since $f(\tilde{\mathbf{p}})$ is a uniformly continuous function of $\tilde{\mathbf{p}} = (\tilde{p}_1, \dots, \tilde{p}_r)$ on $\mathbf{p} \pm \epsilon_0$, we can choose $\epsilon_1 \ll \epsilon/2$ so that

$$(1-3\gamma) f(\tilde{\mathbf{p}}) > (1-4\gamma) f(\tilde{\mathbf{p}} + \delta) \tag{15}$$

holds for all $\tilde{\mathbf{p}} \in \tilde{\mathbf{I}}$ and all $0 < \delta \le \epsilon_1$. As the binomial distribution $B(n,p)$ is concentrated around pn, we can choose n_1 so that

$$\prod_{s=1}^{r} \sum_{j \in J_s} \binom{n}{j} (p'_s)^j (1-p'_s)^{n-j} > (1-3\gamma)/(1-2\gamma) \tag{16}$$

holds for all $n > n_1$ and all $\mathbf{p}' = (p'_1, \dots, p'_r) \in \mathbf{I} := \mathbf{p} \pm \frac{3\epsilon}{2}$, where $J_s = n(p'_s \pm \epsilon_1)$. We note that $\tilde{\mathbf{I}} \pm \epsilon_1 \subset \tilde{\mathbf{I}} \pm \frac{\epsilon}{2} = \mathbf{I}$. For fixed $p = k/n$, we have $|\mathcal{A}_\ell(n,k,r,t)|/\binom{n}{k} \to g_\ell(p,r,t)$ as $n \to \infty$. With this in mind, a little computation shows that we can choose n_2 so that

$$(1-\gamma) \prod_{s=1}^{r} \max\{|\mathcal{A}_{\ell-1}(n,k_s,r,t)|, |\mathcal{A}_\ell(n,k_s,r,t)|\} > (1-2\gamma) f(\tfrac{1}{n}\mathbf{k}) \prod_{s=1}^{r} \binom{n}{k_s} \tag{17}$$

holds for all $n > n_2$ and all $\mathbf{k} = (k_1, \dots, k_r)$ with $\frac{1}{n}\mathbf{k} \in \tilde{\mathbf{I}}$. Finally set $N = \max\{n_0, n_1, n_2\}$.

For our choice of ϵ, γ and N, the negation of (M) gives us some $n, \mathbf{k}$ and $\{\mathcal{F}_1, \dots, \mathcal{F}_r\} \in \mathbf{Y}^{\times}_\ell(n, \mathbf{k}, r, t)$ such that

$$\prod_{s=1}^{r} |\mathcal{F}_s| \ge (1-\gamma) \prod_{s=1}^{r} \max\{|\mathcal{A}_{\ell-1}(n,k_s,r,t)|, |\mathcal{A}_\ell(n,k_s,r,t)|\}, \tag{18}$$

where $n > N$ and $\frac{1}{n}\mathbf{k} \in \tilde{\mathbf{I}}$. We fix these $n, \mathbf{k}$ and $\{\mathcal{F}_1, \dots, \mathcal{F}_r\}$, and set $\tilde{\mathbf{p}} = \frac{1}{n}\mathbf{k}$. By (17) and (18) we have $\prod_{s=1}^{r} |\mathcal{F}_s| > (1-2\gamma) f(\tilde{\mathbf{p}}) \prod_{s=1}^{r} \binom{n}{k_s}$, or equivalently,

$$\prod_{s=1}^{r} c_s > (1-2\gamma) f(\tilde{\mathbf{p}}) \tag{19}$$

where $c_s = |\mathcal{F}_s|/\binom{n}{k_s}$. For $1 \le s \le r$, let $\mathcal{G}_s = \bigcup_{k \le j \le n} (\nabla_j(\mathcal{F}_s))$ be the collection of all upper shadows of $\mathcal{F}_s$, where $\nabla_j(\mathcal{F}_s) = \{H \in \binom{[n]}{j} : H \supset \exists F \in \mathcal{F}_s\}$. Then, $\{\mathcal{G}_1, \dots, \mathcal{G}_r\} \in \mathbf{X}^{\times}_\ell(n,r,t)$. Let $\mathbf{p}' = \tilde{\mathbf{p}} + \epsilon_1 \in \mathbf{I}$, and $J_s = n(p'_s \pm \epsilon_1) = n((\tilde{p}_s + \epsilon_1) \pm \epsilon_1) = n((\frac{k_s}{n} + \epsilon_1) \pm \epsilon_1) = (k_s, k_s + 2\epsilon_1 n) \cap \mathbb{N}$ for $1 \le s \le r$.

Lemma 9. $|\nabla_j(\mathcal{F}_s)| \ge c_s \binom{n}{j}$ *for* $j \in J_s$.

Proof. Choose a real $x \le n$ so that $c_s\binom{n}{k_s} = \binom{x}{n-k_s}$. Since $|\mathcal{F}_s| = c_s\binom{n}{k_s} = \binom{x}{n-k_s}$ and $j \ge k_s$, the Kruskal–Katona Theorem [15, 13] implies that $|\nabla_j(\mathcal{F}_s)| \ge \binom{x}{n-j}$. Thus it suffices to show that $\binom{x}{n-j} \ge c_s\binom{n}{j}$, or equivalently,

$$\frac{\binom{x}{n-j}}{\binom{x}{n-k_s}} \ge \frac{c_s\binom{n}{j}}{c_s\binom{n}{k_s}}.$$

Using $j \ge k_s$ this is equivalent to $j \cdots (k_s+1) \ge (x-n+j) \cdots (x-n+k_s+1)$, which follows from $x \le n$. □

By Lemma 9 we have

$$w_{p'_s}(\mathcal{G}_s) \geq \sum_{j \in J_s} |\nabla_j(\mathcal{F}_s)| \, (p'_s)^j (1-p'_s)^{n-j} \geq c_s \sum_{j \in J_s} \tbinom{n}{j} (p'_s)^j (1-p'_s)^{n-j}. \tag{20}$$

Therefore we have

$$\begin{aligned}\prod_{s=1}^{r} w_{p'_s}(\mathcal{G}_s) &\underset{(20)}{>} \prod_{s=1}^{r} \Big(c_s \sum_{j \in J_s} \tbinom{n}{j} (p'_s)^j (1-p'_s)^{n-j}\Big) \\ &\underset{(19),(16)}{>} (1-2\gamma) f(\tilde{\mathbf{p}}) \times (1-3\gamma)/(1-2\gamma) = (1-3\gamma) f(\tilde{\mathbf{p}}) \\ &\underset{(15)}{>} (1-4\gamma) f(\tilde{\mathbf{p}} + \epsilon_1) = (1-\gamma_0) f(\mathbf{p}'),\end{aligned}$$

which contradicts (W) because $\mathbf{p}' \in \mathbf{I} = \mathbf{p} \pm \frac{3\epsilon}{2} = \mathbf{p} \pm \frac{3\epsilon_0}{4} \subset \mathbf{p} \pm \epsilon_0$. ☐

We only need the case $\ell = 0$ in Theorem 8 to show Theorem 6. But Theorem 8 would be useful to prove (2) (if true) or its product version for the cases $\ell \geq 1$ as well. In fact, this technique was used in [25, 26, 27] to get partial results of (2).

Appendix

The following result is an immediate consequence of Theorem 3 in [27] and Theorem 11 in [26].

Theorem 9. *For every $p \in (0,1)$ there exists r_0 such that for all $r > r_0$, all t with $1 \leq t \leq t_{p,r}$, there exist positive constants n_0, ϵ and γ so that if $n > n_0$ and $k/n < p + \epsilon$, then $m_0(n,k,r,t) = (1-\gamma)\max\{|\mathcal{A}_0(n,k,r,t)|, \, |\mathcal{A}_1(n,k,r,t)|\}$.*

We need one more easy computation.

Lemma 10. *Let $p \in (0,1) \cap \mathbb{Q}$ and $r \in \mathbb{N}$ be fixed, and let $1 \leq t < t_{p,r}$. Then there exists n_0 such that for all $n > n_0$ and k with $p = k/n$, we have $|\mathcal{A}_0(n,k,r,t)| > |\mathcal{A}_1(n,k,r,t)|$.*

Proof. We can rewrite $|\mathcal{A}_0(n,k,r,t)| > |\mathcal{A}_1(n,k,r,t)|$ as

$$\binom{n-t}{k-t} > (t+r)\binom{n-t-r}{k-t-r+1} + \binom{n-t-r}{k-t-r},$$

that is,

$$1 > \frac{(k-t)\cdots(k-t-r+1)}{(n-t)\cdots(n-t-r+1)}\left(\frac{(t+r)(n-k)}{k-t-r+1} + 1\right),$$

or equivalently,

$$(n-t)\cdots(n-t-r+1) > (k-t)\cdots(k-t-r+2)\{(t+r)(n-k)+k-t-r+1\}. \tag{21}$$

Both sides of (21) are polynomials in the variable n of degree r. We fix p and r, and consider the situation that $n, k \to \infty$ with keeping $k = pn$. By comparing the coefficients of n^r of (21), we can conclude that (21) holds if

$$1 > p^{r-1}((t+r)(1-p)+p),$$

that is $t < t_{p,r}$. ☐

Now Theorem 5 and Theorem 1 follow from Theorem 9 and Lemma 10.

Proof of Theorem 4. Choose $\{\mathcal{G}_1, \dots, \mathcal{G}_r\} \in \mathbf{G}^\times(n, r, t)$ with $w^\times(n, \mathbf{p}, r, t) = \prod_{s=1}^r w_{p_s}(\mathcal{G}_s)$. By Theorem 7, if $w^\times(n, \mathbf{p}, r, t) \geq (1-\gamma)p_s^t$, then $\{\mathcal{G}_1, \dots, \mathcal{G}_r\} \in \mathbf{G}_0^\times(n, r, t)$, namely, each $\mathcal{G}_s$ is a subfamily of a family isomorphic to $\mathcal{B}_0(n, r, t)$. Thus we have

$$w^\times(n, \mathbf{p}, r, t) \leq \prod_{s=1}^r w_{p_s}(\mathcal{B}_0(n, r, t)) = \prod_{s=1}^r p_s^t.$$

Moreover, if equality holds in the above inequality, then $\mathcal{G}_s \cong \mathcal{B}_0(n, r, t)$ for all $1 \leq s \leq r$. In this case, $\mathcal{G}_1 = \cdots = \mathcal{G}_r$ must hold by the r-cross t-intersecting property. □

In the same way, one can prove Theorem 2 using Theorem 6.

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References

[1] R. Ahlswede, L.H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18:125–136, 1997.
[2] C. Bey. On cross-intersecting families of sets. *Graphs Combin.*, 21:161–168, 2005.
[3] I. Dinur, E. Friedgut. Intersecting families are essentially contained in juntas. *Combin. Probab. Comput.* 18 (2009), 107–122.
[4] P. Erdős, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.
[5] P. Frankl. On Sperner families satisfying an additional condition. *J. Combin. Theory (A)*, 20:1–11, 1976.
[6] P. Frankl. The Erdős–Ko–Rado theorem is true for $n = ckt$. *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthey, 1976), Vol. I*, 365–375, Colloq. math. Soc. János Bolyai, 18, North–Holland, 1978.
[7] P. Frankl. The shifting technique in extremal set theory. "Surveys in Combinatorics 1987" (C. Whitehead, Ed. LMS Lecture Note Series 123), 81–110, Cambridge Univ. Press, 1987.
[8] P. Frankl. Multiply-intersecting families. *J. Combin. Theory (B)*, 53:195–234, 1991.
[9] P. Frankl, N. Tokushige. On r-cross intersecting families of sets. *submitted.*
[10] E. Friedgut. On the measure of intersecting families, uniqueness and stability. *Combinatorica* 28 (2008) 503–528.
[11] B, Green, T .Tao. Freiman's theorem in finite fields via extremal set theory. *Combinatorics, Probability, and Computing*, vol 18 (2009) 335–355.
[12] H.-D.O.F. Gronau. On Sperner families in which no k sets have an empty intersection III. *Combinatorica*, 2:25–36, 1982.
[13] G.O.H. Katona. A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, 1966 (Akademiai Kiadó, 1968) 187–207.
[14] P. Keevash, D. Mubayi. Set systems without a simplex or a cluster. *Combinatorica* 30 (2010) 573–606.
[15] J.B. Kruskal. The number of simplices in a complex, in: Math. Opt. Techniques (Univ. of Calif. Press, 1963) 251–278.
[16] M. Matsumoto, N. Tokushige. The exact bound in the Erdős–Ko–Rado theorem for cross-intersecting families. *J. Combin. Theory (A)*, 52:90–97, 1989.
[17] D. Mubayi. An intersection theorem for four sets. *Adv. Math.* 215 (2007) 601–615.
[18] D. Mubayi, R. Ramadurai. Simplex stability. *Combin. Probab. Comput.* 18 (2009) 441–454.
[19] D. Mubayi, J. Verstraëte. Proof of a conjecture of Erdős on triangles in set-systems. *Combinatorica* 25 2005 599–614.

[20] L. Pyber. A new generalization of the Erdős–Ko–Rado theorem. *J. Combin. Theory (A)*, 43:85–90, 1986.

[21] N. Tokushige. A frog's random jump and the Pólya identity. *Ryukyu Math. Journal*, 17:89–103, 2004.

[22] N. Tokushige. The maximum size of 3-wise t-intersecting families. *European J. Comb.*, 28:152–166, 2007.

[23] N. Tokushige. EKR type inequalities for 4-wise intersecting families. *J. Combin. Theory (A)*, 114:575–596, 2007.

[24] N. Tokushige. Brace–Daykin type inequalities for intersecting families. *European J. Combin*, 29:273-285, 2008.

[25] N. Tokushige. Multiply-intersecting families revisited. *J. Combin. Theory (B)*, 97:929–948, 2007.

[26] N. Tokushige. The random walk method for intersecting families, in: Horizons of combinatorics, *Bolyai society mathematical studies* 17:215–224, 2008.

[27] N. Tokushige. A multiply intersecting Erdős–Ko–Rado theorem — the principal case. *Discrete Math.*, vol 310 (2009) 453–460.

[28] N. Tokushige. On cross t-intersecting families of sets. *J. Combin. Theory (A)*, vol 117 (2010) 1167–1177.

[29] R.M. Wilson. The exact bound in the Erdős–Ko–Rado theorem. *Combinatorica*, 4:247–257, 1984.