BRACE–DAYKIN TYPE INEQUALIES FOR INTERSECTING FAMILIES

NORIHIDE TOKUSHIGE

ABSTRACT. Let n, k and $r \geq 8$ be positive integers. Suppose that a family $\mathscr{F} \subset \binom{[n]}{k}$ satisfies $F_1 \cap \cdots \cap F_r \neq \emptyset$ for all $F_1, \ldots, F_r \in \mathscr{F}$ and $\bigcap_{F \in \mathscr{F}} F = \emptyset$. We prove that there exist $\varepsilon_r > 0$ and n_r such that

$$|\mathscr{F}| \leq (r+1)\binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1}$$

holds for all n and k, satisfying $n > n_r$ and $|\frac{k}{n} - \frac{1}{2}| < \varepsilon_r$.

1. INTRODUCTION

Let n, r and t be positive integers. A family $\mathscr{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathscr{F}$. An r-wise 1-intersecting family is also called an r-wise intersecting family for short. An r-wise t-intersecting family $\mathscr{F}$ is called non-trivial if $|\bigcap \mathscr{F}| < t$, where $\bigcap \mathscr{F} = \bigcap_{F \in \mathscr{F}} F$.

Let $\mathscr{E}(n, r, t) = \{E \subset [n] : |E \cap [r+t]| \geq r+t-1\}$. Then $\mathscr{E}$ is a non-trivial r-wise t-intersecting family. Two families $\mathscr{G}, \mathscr{G}' \subset 2^{[n]}$ are said to be isomorphic and denoted by $\mathscr{G} \cong \mathscr{G}'$ if there exists a vertex permutation τ on $[n]$ such that $\mathscr{G}' = \{\{\tau(g) : g \in G\} : G \in \mathscr{G}\}$. Brace and Daykin proved the following.

Theorem 1 ([2]). *Suppose that $\mathscr{F} \subset 2^{[n]}$ is a non-trivial r-wise intersecting family. Then $|\mathscr{F}| \leq |\mathscr{E}(n, r, 1)|$. Moreover $\mathscr{E}(n, r)$ is the only optimal configuration (up to isomorphism) for $r \geq 3$.*

Our first result is a uniform hypergraph version of Theorem 1 (cf. [1, 3]). Let $m^*(n, k, r, t)$ be the maximal size of k-uniform non-trivial r-wise t-intersecting families on n vertices, and let $\mathscr{F}(n, k, r, t) = \mathscr{E}(n, r, t) \cap \binom{[n]}{k}$.

Theorem 2. *Let $r \geq 8$. Then there exists $\varepsilon_r > 0$ and n_r such that*

$$m^*(n, k, r, 1) = |\mathscr{F}(n, k, r, 1)| = (r+1)\binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1}$$

holds for all $n > n_r$ and k with $|\frac{k}{n} - \frac{1}{2}| < \varepsilon_r$. Moreover $\mathscr{F}(n, k, r, 1)$ is the only optimal configuration (up to isomorphism).

Date: October 11, 2006, 02:56pm.

2000 *Mathematics Subject Classification.* Primary: 05D05 Secondary: 05C65.

Key words and phrases. intersecting family; multiply intersecting hypergraph; Brace–Daykin Theorem.

The author was supported by MEXT Grant-in-Aid for Scientific Research (B) 16340027.

Our second result is an extension of Theorem 1 to a weighted version (cf. [4, 6]). Throughout this paper, p and q denote positive real numbers with $p+q=1$. For a family $\mathscr{G}\subset 2^X$ we define the p-weight of $\mathscr{G}$, denoted by $w_p(\mathscr{G}:X)$, as follows:

$$w_p(\mathscr{G}:X)=\sum_{G\in\mathscr{G}} p^{|G|}q^{|X|-|G|}=\sum_{i=0}^{|X|}\left|\mathscr{G}\cap\binom{X}{i}\right|p^iq^{|X|-i}.$$

We simply write $w_p(\mathscr{G})$ for the case $X=[n]$. Let $w^*(n,p,r,t)$ be the maximal p-weight of non-trivial r-wise t-intersecting families on n vertices.

Theorem 3. *Let $r\ge 8$. Then there exists $\varepsilon>0$ such that*

$$w^*(n,p,r,1)=w_p(\mathscr{E}(n,r,1))=(r+1)p^rq+p^{r+1}$$

holds for all $n\ge r+1$ and p with $|p-\frac{1}{2}|<\varepsilon$. Moreover $\mathscr{E}(n,r,1)$ is the only optimal configuration (up to isomorphism).

Theorem 2 and Theorem 3 are closely related. For comparison, it is natural to consider the situation $n,k\to\infty$ for fixed $p=\frac{k}{n}$ and t in the k-uniform version. Then we have

$$|\mathscr{F}(n,k,r,t)|/\binom{n}{k}=w_p(\mathscr{E}(n,r,t))+o(1).$$

See [13] for more about the relation between $m^*(n,k,r,t)/\binom{n}{k}$ and $w^*(n,p,r,t)$.

Theorem 2 fails for $2\le r\le 5$. We give a Hilton–Milner[7] type construction for the case $r=5$ below. For integers a and b, let $[a,b]$ denote the set $\{a,a+1,\dots,b\}$ if $a\le b$, and let $[a,b]=\emptyset$ if $a>b$.

Example 1. Fix $\frac{1}{2}<p\le\frac{2}{3}$ and let $p=\frac{k}{n}$. We construct a non-trivial 5-wise intersecting family $\mathscr{H}\subset\binom{[n]}{k}$ as follows:

$$\mathscr{H}=\{H_1,H_2,H_3\}\cup\{H\in\binom{[n]}{k}:[3]\subset H,\,|H\cap[4,k+1]|>\tfrac{k-2}{2}\},$$

where $H_j=[k+1]\setminus\{j\}$ for $1\le j\le 3$. Then we have $|\mathscr{H}|=3+\sum_{\ell>\frac{k-2}{2}}\binom{k-2}{\ell}\binom{n-k-1}{k-3-\ell}$. (We need $n-k-1\ge k-3-\ell$, which follows from $p\le\frac{2}{3}$.) Using standard bounds on deviations of the hypergeometric distribution (see e.g., [8]), we have $\lim_{n\to\infty}|\mathscr{H}|/\binom{n}{k}=p^3$ if $p>1/2$. On the other hand, we have $\lim_{n\to\infty}\mathscr{F}(n,k,5,1)/\binom{n}{k}=6p^5q+p^6$, which is less than p^3 if $p<\frac{1+\sqrt{21}}{10}$. Therefore we have $|\mathscr{H}|>|\mathscr{F}(n,k,5,1)|$ if $\frac{1}{2}<p<\frac{1+\sqrt{21}}{10}$ and n is sufficiently large. $\square$

Using the fact that $\binom{[m]}{\ell}$ is s-wise t-intersecting if $(s-1)m+(t-1)<s\ell$, we can extend the above construction to get a lower bound for $m^*(n,k,r,t)$ as follows.

Example 2. *Let $i\in\mathbb{N}$, $0\le i\le r-1$, and $\frac{r-i-1}{r-i}<p\le\frac{r-i}{r-i+1}$. Then, for fixed $p=\frac{k}{n}$ and i, we have* $\lim_{n\to\infty}m^*(n,k,r,t)/\binom{n}{k}\ge p^{it}$.

Proof. We will construct a non-trivial r-wise t-intersecting family $\mathscr{H}_i \subset \binom{[n]}{k}$. Let ℓ_i be the smallest integer ℓ which satisfies $(r-i-1)(k+t-it)+(t-1) < (r-i)\ell$. Then $\binom{[it+1,k+t]}{\ell}$ is $(r-i)$-wise t-intersecting for $\ell \geq \ell_i$. Let $H_j = [k+t] - [(j-1)t+1, jt]$ for $1 \leq j \leq i$, and define $\mathscr{H}_i$ as follows:

$$\mathscr{H}_i = \{H_1, \dots, H_i\} \cup \{H \in \tbinom{[n]}{k} : [it] \subset H, |H \cap [it+1, k+t]| > \ell_i\}.$$

Since $p > \frac{r-i-1}{r-i}$ we have $p(k+t-it) > \ell_i$ for n,k sufficiently large. Thus we have $\lim_{n\to\infty} |\mathscr{H}_i|/\binom{n}{k} = \lim_{n\to\infty} \sum_{\ell \geq \ell_i} \binom{k+t-it}{\ell}\binom{n-k-t}{k-it-\ell}/\binom{n}{k} = p^{it}$. □

The condition $|\frac{k}{n} - \frac{1}{2}| < \varepsilon_r$ in Theorem 2 can possibly be improved, but we need some restriction on $\frac{k}{n}$ as we will see below. Setting $t = 1$ and $i = r-1$ in Example 2, we have $\lim_{n\to\infty} m^*(n,k,r,1)/\binom{n}{k} \geq p^{r-1}$ for all fixed $p = \frac{k}{n} \leq \frac{1}{2}$ and n sufficiently large. On the other hand, simple computation shows $p^{r-1} > (r+1)p^r q + p^{r+1}$ iff $p < \frac{1}{r}$. This means $m^*(n,k,r,1) > |\mathscr{F}(n,k,r)|$ in this range, namely, Theorem 2 fails for $\frac{k}{n} < \frac{1}{r}$.

Next we consider the case $r = 8$ and $t = 1$. Fix $p = \frac{k}{n}$. By setting $i = 4$ in Example 2, we have $\lim_{n\to\infty} m^*(n,k,8,1)/\binom{n}{k} \geq p^4$ for $\frac{3}{4} < p \leq \frac{4}{5}$, while $p^4 > |\mathscr{F}(n,k,8,1)|/\binom{n}{k}$ for $p \leq 0.77$. Thus Theorem 2 fails for $\frac{3}{4} < p \leq 0.77$. For general r, by setting, e.g., $i = \frac{5r}{12}$ and $p_0 = 1 - \frac{12}{7r}$, we have $m^*(n,k,r,1) \geq p^i$ for $p > p_0$, and $\lim_{r\to\infty} p^i - ((r+1)p^r q + p^{r+1}) = \frac{7e-19}{7e^{12/7}} > 0$ at $p = p_0$. Thus we can find $\varepsilon > 0$ such that $m^*(n,k,r,1) > |\mathscr{F}(n,k,r,1)|$ if $p_0 < \frac{k}{n} < p_0 + \varepsilon$.

Theorem 3 implies Theorem 1 by setting $p = \frac{1}{2}$ for $r \geq 8$. On the other hand, similarly to Example 2, one can show that $\lim_{n\to\infty} w^*(n,p,r,t) \geq p^{it}$ if $\frac{r-i-1}{r-i} < p \leq \frac{r-i}{r-i+1}$. Thus Theorem 3 fails for $2 \leq r \leq 5$ (cf. [6]). One can also show that Theorem 3 fails for $p < \frac{1}{r}$ or $p_0 < p < p_0 + \varepsilon$.

Conjecture 1. *Theorem 2 and Theorem 3 is true for $r = 6$ and $r = 7$ as well.*

We will deduce Theorem 2 and Theorem 3 from slightly stronger results (Theorem 4 and Theorem 5 below). The reduction is based on the following simple observation.

Lemma 1. *If $\mathscr{F} \subset 2^{[n]}$ is a non-trivial r-wise t-intersecting family, then it is also a non-trivial $(r-1)$-wise $(t+1)$-intersecting family.*

Proof. If $\mathscr{F}$ is not $(r-1)$-wise $(t+1)$-intersecting, then we can find $F_1, \dots, F_{r-1} \in \mathscr{F}$ such that $|F_1 \cap \dots \cap F_{r-1}| = t$. But $\mathscr{F}$ is r-wise t-intersecting and so every $F \in \mathscr{F}$ must contain $F_1 \cap \dots \cap F_{r-1}$, which contradicts the fact that $\mathscr{F}$ is non-trivial, i.e., $|\bigcap \mathscr{F}| < t$. □

Lemma 1 gives

$$m^*(n,k,r,t) \leq m^*(n,k,r-1,t+1) \text{ and } w^*(n,p,r,t) \leq w^*(n,p,r-1,t+1).$$

Let $\mathbf{X}(n,r,t)$ be the set of non-trivial r-wise t-intersecting families $\mathscr{G} \subset 2^{[n]}$ satisfying $\mathscr{G} \not\subset \mathscr{G}'$ for any $\mathscr{G}' \cong \mathscr{E}(n,r,t) = \mathscr{E}(n,r-1,t+1)$, and let $\mathbf{Y}(n,k,r,t) = \{\mathscr{F} \subset \binom{[n]}{k} : \mathscr{F} \in$

$\mathbf{X}(n,r,t)\}$. We note that $\mathbf{X}(n,r,t) \subset \mathbf{X}(n,r-1,t+1)$ and $\mathbf{Y}(n,k,r,t) \subset \mathbf{Y}(n,k,r-1,t+1)$. Thus Theorem 2 and Theorem 3 immediately follow from the following results.

Theorem 4. *Let $r \geq 7$. Then there exist positive constants γ, ε, n_0 such that the following (i) and (ii) are true for all $n > n_0$ and k with $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$.*

(i) $m^*(n,k,r,2) = |\mathscr{F}(n,k,r,2)| = (r+2)\binom{n-r-2}{k-r-1} + \binom{n-r-2}{k-r-2}$.
(ii) *If $\mathscr{F} \in \mathbf{Y}(n,k,r,2)$ then $|\mathscr{F}| < (1-\gamma)\, m^*(n,k,r,2)$.*

Theorem 5. *Let $r \geq 7$. Then there exist positive constants γ, ε such that the following (i) and (ii) are true for all $n \geq r+2$ and p with $|p - \frac{1}{2}| < \varepsilon$.*

(i) $w^*(n,p,r,2) = w_p(\mathscr{E}(n,r,2)) = (r+2)p^{r+1}q + p^{r+2}$.
(ii) *If $\mathscr{G} \in \mathbf{X}(n,r,2)$ then $w_p(\mathscr{G}) < (1-\gamma)\, w^*(n,p,r,2)$.*

In Section 2, we prepare some tools for the proofs. We prove Theorem 5 in Section 3. In the last section we deduce Theorem 4 from Theorem 5.

2. Tools

Here we list some known results to prove the theorems. Let $m(n,k,r,t)$ be the maximal size of k-uniform r-wise t-intersecting families on n vertices and let $w(n,p,r,t)$ be the maximal p-weight of r-wise t-intersecting families on n vertices. Trivial t-intersecting families give that $m(n,k,r,t) \geq \binom{n-t}{k-t}$ and $w(n,p,r,t) \geq p^t$.

Lemma 2 ([4]). *$w(n,p,r,1) = p$ holds for $p \leq \frac{r-1}{r}$.*

Lemma 3 ([5]). *We have $w(n,p,3,2) = p^2$ for $p < 0.501$ and n sufficiently large.*

Lemma 4 ([12]). *For $1 \leq t \leq 7$, there exists ε and n_0 such that $m(n,k,4,t) = \binom{n-t}{k-t}$ holds for $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_0$.*

Lemma 5 ([13]). *Let r,t and p_0 be fixed constants. Then (M) implies (W).*

(M) There exist $\varepsilon > 0$ and n_0 such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for all $n > n_0$ and k with $|\frac{k}{n} - p_0| < \varepsilon$.
(W) There exists $\varepsilon > 0$ such that $w(n,p,r,t) = p^t$ holds for all $n \geq t$ and p with $|p - p_0| < \varepsilon$.

For integers $1 \leq i < j \leq n$ and a family $\mathscr{G} \subset 2^{[n]}$, we define the (i,j)-shift σ_{ij} as follows:

$$\sigma_{ij}(\mathscr{G}) = \{\sigma_{ij}(G) : G \in \mathscr{G}\},$$

where

$$\sigma_{ij}(G) = \begin{cases} (G - \{j\}) \cup \{i\} & \text{if } i \notin G,\ j \in G,\ (G-\{j\}) \cup \{i\} \notin \mathscr{G}, \\ G & \text{otherwise.} \end{cases}$$

A family $\mathscr{G} \subset 2^{[n]}$ is called *shifted* if $\sigma_{ij}(\mathscr{G}) = \mathscr{G}$ for all $1 \leq i < j \leq n$, and $\mathscr{G}$ is called *tame* if it is shifted and $\bigcap \mathscr{G} = \emptyset$. If $\mathscr{G}$ is r-wise t-intersecting, then so is $\sigma_{ij}(\mathscr{G})$. Thus, starting from any r-wise t-intersecting family $\mathscr{G}$, one can get a shifted r-wise t-intersecting family $\mathscr{G}'$ with $|\mathscr{G}'| = |\mathscr{G}|$. For the non-trivial intersecting case, we have the following.

Lemma 6. *Let $\mathscr{G} \subset 2^{[n]}$ be a non-trivial r-wise t-intersecting family with maximal p-weight. Then we can find a tame r-wise t-intersecting family $\mathscr{G}' \subset 2^{[n]}$ with $w_p(\mathscr{G}') = w_p(\mathscr{G})$.*

Proof. By Lemma 1, $\mathscr{G}$ is $(r-1)$-wise $(t+1)$-intersecting. We apply all possible shifting operations to $\mathscr{G}$ to get a shifted $(r-1)$-wise $(t+1)$-intersecting family $\mathscr{G}'$.

We have to show that $\bigcap \mathscr{G}' = \emptyset$. Otherwise we may assume that $1 \in \bigcap \mathscr{G}'$ and $H = [2,n] \notin \mathscr{G}'$. Since $\mathscr{G}'$ is p-weight maximal we can find $G_1, \ldots, G_{r-1} \in \mathscr{G}'$ such that $|G_1 \cap \cdots \cap G_{r-1} \cap H| < t$. Then we have $|G_1 \cap \cdots \cap G_{r-1}| < t+1$, which is a contradiction. $\square$

Lemma 7. *Let p, r, t_0, c be fixed constants, and let $\alpha \in (p,1)$ be the root of the equation $X = p + qX^r$. Suppose that $w(n,p,r,t_0) \leq c$ holds for all $n \geq t_0$. Then we have $w(n,p,r,t) \leq c\alpha^{t-t_0}$ for all $t \geq t_0$ and $n \geq t$.*

Proof. If $\mathscr{G} \subset 2^{[n]}$ is trivial r-wise t_0-intersecting, i.e., $|\bigcap \mathscr{G}| \geq t_0$, then we have $\mathscr{G} \subset \{G \subset [n] : [t_0] \subset G\}$ and $w_p(\mathscr{G}) \leq p^{t_0}$. Thus we may assume that $c \geq p^{t_0}$. Note also that $p < \alpha$.

We prove the result by double induction on $s = n - t$ and t. One of the initial steps for $t = t_0$ follows from our assumption. For the other initial step for s, we prove the result for the cases $0 \leq s \leq r-1$, or equivalently, $t \leq n \leq t+r-1$. Suppose that $\mathscr{G} \subset 2^{[n]}$ satisfies $w_p(\mathscr{G}) = w(n,p,r,t)$. We may assume that $\mathscr{G}$ is shifted and size maximal. If $\mathscr{G}$ is trivial, i.e., $|\bigcap \mathscr{G}| \geq t$, then we have $w_p(\mathscr{G}) \leq p^t = p^{t_0} p^{t-t_0} < c\alpha^{t-t_0}$ and we are done. Otherwise we have $G \in \mathscr{G}$ such that $[t] \not\subset G$, and we may assume that $G_t = [n] - \{t\} \in \mathscr{G}$ because $\mathscr{G}$ is shifted and maximal. Then again by the shiftedness we have $G_i = [n] - \{i\} \in \mathscr{G}$ for all $t \leq i \leq n$. This implies $|\bigcap_{i=t}^{n} G_i| = t-1$. But this is impossible because $\mathscr{G}$ is r-wise t-intersecting and $n - t + 1 \leq r$.

Next we show the induction step. Let $s \geq r$ and $t > t_0$. We show the case (s,t). We assume that the result holds for $\{(s,b) : b < t\} \cup \{(a,b) : a < s, b \geq t_0\}$. In particular, we can apply induction hypothesis to the case $(s, t-1)$ and $(s-r, t+r-1)$.

Let $\mathscr{G} \subset 2^{[n]}$ be r-wise t-intersecting. Define $\mathscr{G}_1, \mathscr{G}_{\bar{1}} \subset 2^{[2,n]}$ as follows:

$$\mathscr{G}_1 = \{G - \{1\} : 1 \in G \in \mathscr{G}\}, \quad \mathscr{G}_{\bar{1}} = \{G : 1 \notin G \in \mathscr{G}\}.$$

Then $\mathscr{G}_1$ is clearly r-wise $(t-1)$-intersecting. On the other hand, $\mathscr{G}_{\bar{1}}$ is r-wise $(t+r-1)$-intersecting. To see this fact suppose, on the contrary, that there exist $G_2 \ldots G_{r+1} \in \mathscr{G}_{\bar{1}}$ such that $\bigcap_{i=2}^{r+1} G_i = [2, t+r-1]$. By the shiftedness we have $G_i' = \{1\} \cup (G_i - \{i\}) \in \mathscr{G}$ for all $2 \leq i \leq r+1$. But then we have $\bigcap_{i=2}^{r+1} G_i' = [t+r-1] - [2, r+1]$, which contradicts r-wise t-intersecting property of $\mathscr{G}$.

Note that s for $\mathscr{G}_1$ is $(n-1) - (t-1) = s$ and s for $\mathscr{G}_{\bar{1}}$ is $(n-1) - (t+r-1) = s - r$. Therefore using the induction hypothesis, we have

$$\begin{aligned} w_p(\mathscr{G}) &= p w_p(\mathscr{G}_1 : [2,n]) + q w_p(\mathscr{G}_{\bar{1}} : [2,n]) \leq pc\alpha^{t-t_0-1} + qc\alpha^{t+r-t_0-1} \\ &= c\alpha^{t-t_0-1}(p + q\alpha^r) = c\alpha^{t-t_0}. \quad \square \end{aligned}$$

Let $\alpha_{p,r} \in (p,1)$ be the root of the equation $X = p + qX^r$. For later use, we record $\alpha_{\frac{1}{2},3} = \frac{\sqrt{5}-1}{2} \approx 0.618$ and $\alpha_{\frac{1}{2},4} \approx 0.543689$.

Lemma 8. *Let $1 \le s \le 2$ and $1 \le t \le 7$. Then there exists some $\delta > 0$ such that*

$$w(n,p,3,s) = p^s \text{ and } w(n,k,4,t) = p^t$$

hold for $|p - \frac{1}{2}| < \delta$ and $n \ge s$ (resp. $n \ge t$). For the case $s > 2$ or $t > 7$ we have

$$w(n,p,3,s) \le p^2 \alpha_{p,3}^{s-2} \text{ and } w(n,k,4,t) \le p^7 \alpha_{p,4}^{t-7}$$

for $|p - \frac{1}{2}| < \delta$ and $n \ge s$ (resp. $n \ge t$).

Proof. Let $1 \le t \le 7$. By Lemma 4 and Lemma 5, there exists some $\delta > 0$ such that $w(n,p,4,t) = p^t$ holds for $|p - \frac{1}{2}| < \delta$. In particular we have $w(n,p,4,7) = p^7$. This together with Lemma 7 gives $w(n,p,4,t) \le p^7 \alpha_{p,4}^{t-7}$ for $t \ge 7$. One can prove the inequalities for the case $r = 3$ similarly using Lemma 2 and Lemma 3. □

Lemma 9 ([11])**.** *Let positive integers r and t be given. Let $p \in (0,1)$ be a fixed rational number which satisfies $p < \frac{r-2}{r}$ and*

$$(1-p)p^{\frac{t}{t+1}(r-1)} - p^{\frac{t}{t+1}} + p < 0.$$

Then $m(n,k,r,t) = \binom{n-t}{k-t}$ if $\frac{k}{n} = p$ and n is sufficiently large.

Lemma 10. *Let $r \ge 5$ and t be positive integers with $r \le t+1 \le 2^{r-2}\log 2$. Then there exist $\varepsilon > 0$ and n_0 such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_0$.*

Proof. Set $p = 1/2$. By Lemma 9 it suffices to show that

$$(1-p)p^{\frac{t}{t+1}(r-1)} - p^{\frac{t}{t+1}} + p < 0, \tag{1}$$

or equivalently, $\frac{1}{2} + \frac{1}{2}(\frac{1}{2})^{\frac{t}{t+1}(r-1)} < (\frac{1}{2})^{\frac{t}{t+1}}$ and so

$$\left(1 + (\tfrac{1}{2})^{\frac{t}{t+1}(r-1)}\right)^{t+1} < 2.$$

Since $r \le t+1$ we have $\frac{t}{t+1}(r-1) > r-2$ and $(\frac{1}{2})^{\frac{t}{t+1}(r-1)} < (\frac{1}{2})^{r-2} \le \frac{\log 2}{t+1}$. Thus we have

$$\left(1 + (\tfrac{1}{2})^{\frac{t}{t+1}(r-1)}\right)^{t+1} < \left(1 + \frac{\log 2}{t+1}\right)^{t+1} < 2,$$

which is the desired inequality. Since the LHS of (1) is a continuous function of p, we can find $\varepsilon > 0$ so that (1) holds for $|p - \frac{1}{2}| < \varepsilon$. □

Lemma 5 and Lemma 10 give the following.

Lemma 11. *Let $r \ge 5$ and t be positive integers with $r \le t+1 \le 2^{r-2}\log 2$. Then there exists $\varepsilon > 0$ such that $w(n,p,r,t) = p^t$ holds for all $n \ge t$ and $|p - \frac{1}{2}| < \varepsilon$. In particular, we have $w(n,p,r,r+1) = p^{r+1}$ for all $r \ge 6$, $n \ge r+1$ and $|p - \frac{1}{2}| < \varepsilon$.*

3. Proof of Theorem 5

3.1. **Proof of (i).** We prove (i) of Theorem 5 in a slightly stronger form, which we will use in the proof of (ii). Let $r \geq 7$ and let $\mathscr{F} \subset 2^{[n]}$ be a non-trivial r-wise 2-intersecting family. We may suppose that $\mathscr{F}$ is p-weight maximal and tame by Lemma 6. If $\mathscr{F} \subset \mathscr{E}(n,r,2)$ then there is nothing to prove. So we assume that $\mathscr{F} \not\subset \mathscr{E}(n,r,2)$, and we shall prove the following stronger inequality by induction on r.

Lemma 12. *Let $r \geq 7$ and let $\mathscr{F} \subset 2^{[n]}$ be a tame r-wise 2-intersecting family with $\mathscr{F} \not\subset \mathscr{E}(n,r,2)$. Then there exist $\gamma, \varepsilon > 0$ such that $w_p(\mathscr{F}) < (1-\gamma) w_p(\mathscr{E}(n,r,2))$ holds for all $n \geq r+2$ and p with $|p - 1/2| < \varepsilon$.*

Proof. First we prove the initial step $r = 7$. Let u be the maximal i such that $|F \cap [i+1]| \geq i$ holds for all $F \in \mathscr{F}$. If $u \geq 8$ then $\mathscr{F} \subset \mathscr{E}(n,7,2)$. So we may assume that $u \leq 7$. Let $t(\ell)$ be the maximal t such that $\mathscr{F}$ is ℓ-wise t-intersecting. Then we have $4 \leq t(5) < t(4)$ by Lemma 1. Set $h(p) = w_p(\mathscr{E}(n,7,2)) = 9p^8 q + p^9$. We compare the p-weight of $\mathscr{F}$ with $h(p)$. Note that $h(1/2) = 10/2^9 > 0.0195$. We will use the following fact.

Claim 1. *Suppose that $w_p(\mathscr{F}) \leq f(p)$ holds for some continuous function $f(p)$, and suppose further that $f(1/2) < h(1/2)$. Then there exist $\gamma, \varepsilon > 0$ such that $w_p(\mathscr{F}) < (1-\gamma) w_p(\mathscr{E}(n,7,2))$ holds for all p with $|p - \frac{1}{2}| < \varepsilon$.*

If $\mathscr{F}$ is 4-wise 6-intersecting then it follows from Lemma 8 that $w_p(\mathscr{F}) \leq p^6$ if p is sufficiently close to $1/2$. Since $p^6 < h(p)$ at $p = 1/2$, we are done in this case by the previous claim. Thus we may assume that $\mathscr{F}$ is not 4-wise 6-intersecting, i.e., $t(4) \leq 5$. This together with $4 \leq t(5) < t(4)$ gives $t(5) = 4$ and $t(4) = 5$.

Claim 2. $u \geq 4$.

Proof. Since $\mathscr{F}$ is shifted and $t(4) = 5$, there exist $F_1, \ldots, F_4 \in \mathscr{F}$ such that $F_1 \cap \cdots \cap F_4 = [5]$. If there exists $F \in \mathscr{F}$ such that $|F \cap [5]| \leq 3$, then $|F \cap F_1 \cap \cdots \cap F_4| \leq 3$ and this contradicts $t(5) = 4$. Thus we must have $|F \cap [5]| \geq 4$ for all $F \in \mathscr{F}$ and this means $u \geq 4$. $\square$

Consequently we may assume that $4 \leq u \leq 7$. For $1 \leq i \leq u+1$ define

$$\mathscr{F}(i) = \{F \in \mathscr{F} : F \cap [u+1] = ([u+1] \setminus \{i\})\},$$

and for $i = 0$ define $\mathscr{F}(0) = \{F \in \mathscr{F} : [u+1] \subset F\}$, and set

$$\mathscr{G}(i) = \{F \cap [u+2, n] : F \in \mathscr{F}(i)\}$$

for $0 \leq i \leq u+1$. Since $\mathscr{F}$ is non-trivial intersecting, shifted and maximal, we have

$$\emptyset \neq \mathscr{G}(1) \subset \mathscr{G}(2) \subset \cdots \subset \mathscr{G}(u+1), \tag{2}$$

and

$$w_p(\mathscr{F}) = p^u q \sum_{i=1}^{u+1} v_p(\mathscr{G}(i)) + p^{u+1} v_p(\mathscr{G}(0)), \tag{3}$$

where $v_p(\mathscr{G}) = w_p(\mathscr{G} : [u+2,n])$. By the definition of u, there exists $F \in \mathscr{F}$ such that $|F \cap [u+2]| \le u$. Since $\mathscr{F}$ is shifted and maximal, it follows that

$$E_{u+1} = [n] - \{u+1, u+2\} \in \mathscr{F}. \tag{4}$$

By shifting E_{u+1}, we have $E_{u+i} = [n] - \{u+i, u+i+1\} \in \mathscr{F}$ for $1 \le i \le n-u-1$.

Claim 3. *$\mathscr{G}(i)$ is 3-wise $(14-u-i)$-intersecting for $u-2 \le i \le \min\{u+1, 6\}$.*

Proof. Suppose, on the contrary, that $\mathscr{G}(i)$ is not 3-wise $(14-u-i)$-intersecting. Then we can find $G_i, G_{i+1}, G_{i+2} \in \mathscr{G}(i)$ such that $|G_i \cap G_{i+1} \cap G_{i+2}| \le 13-u-i$. By the shiftedness, we may assume that $G_i \cap G_{i+1} \cap G_{i+2} = [u+2, 14-i]$. For $i \le j \le i+2$, let $F'_j = ([u+1] - \{i\}) \cup G_j \in \mathscr{F}(i)$. Since $\mathscr{F}$ is shifted we have $F_j := (F'_j - \{j\}) \cup \{i\} \in \mathscr{F}$ for $i < j \le i+2$. Set $F_i = F'_i$ and choose $F_j \in \mathscr{F}(j)$ for $2 \le j < i$ arbitrarily. Then we have $\bigcap_{j=2}^{i+2} F_j \subset \{1\} \cup [i+3, 14-i]$. We also note that $(6-i)$ edges $E_{i+3}, E_{i+5}, \dots, E_{13-i}$ satisfy $\left(\bigcap_{j=1}^{6-i} E_{i+2j+1}\right) \cap [i+3, 14-i] = \emptyset$. Namely we have $(i+1)+(6-i) = 7$ edges

$$F_2, F_3, \dots, F_{i+2}, E_{i+3}, E_{i+5}, \dots, E_{13-i}$$

of $\mathscr{F}$ whose intersection is $\{1\}$. This contradicts that $\mathscr{F}$ is 7-wise 2-intersecting. $\square$

Claim 4. *$\mathscr{G}(i)$ is 4-wise $(13-u-i)$-intersecting for $u-3 \le i \le \min\{u+1, 5\}$.*

Proof. One can prove this claim similarly to the previous claim, and we only show the case $u = 5$ and $i = 2$ here. Suppose that $\mathscr{G}(2)$ is not 4-wise 6-intersecting. Then we can find $G_2, G_3, G_4, G_5 \in \mathscr{G}(2)$ such that $G_2 \cap G_3 \cap G_4 \cap G_5 = [7, 11]$. For $2 \le j \le 5$ let $F'_j = ([6] - \{2\}) \cup G_j \in \mathscr{F}(2)$. Set $F_2 = F'_2$ and for $3 \le j \le 5$ let $F_j = (F'_j - \{j\}) \cup \{2\} \in \mathscr{F}$. Then we have $F_2 \cap F_3 \cap F_4 \cap F_5 \cap E_6 \cap E_8 \cap E_{10} = \{1\}$, a contradiction $\square$

Recall that $4 \le u \le 7$. We deal with the hardest case $u = 5$ first.

Case 1. $u = 5$.

Subcase 1.1. *$G \cap [7, 9] \ne \emptyset$ holds for all $G \in \mathscr{G}(0)$.*

By Claim 3 (for $\mathscr{G}(4)$, $\mathscr{G}(5)$, $\mathscr{G}(6)$) and Claim 4 (for $\mathscr{G}(2)$ and $\mathscr{G}(3)$), we get the following table representing the ℓ-wise t-intersecting property of $\mathscr{G}(i)$.

$\mathscr{G}(i)$	$\mathscr{G}(2)$	$\mathscr{G}(3)$	$\mathscr{G}(4)$	$\mathscr{G}(5)$	$\mathscr{G}(6)$
ℓ-wise	4	4	3	3	3
t-int.	6	5	5	4	3

Since $\mathscr{G}(2) \subset 2^{[7,n]}$ is 4-wise 6-intersecting, it follows Lemma 8 that $v_p(\mathscr{G}(2)) \le 2p^6$. This together with (2) gives $v_p(\mathscr{G}(1)) + v_p(\mathscr{G}(2)) \le 2v_p(\mathscr{G}(2)) \le 2p^6$.

Similarly using Lemma 8 we have

$$v_p(\mathscr{G}(3)) + v_p(\mathscr{G}(4)) + v_p(\mathscr{G}(5)) + v_p(\mathscr{G}((6)) \le p^5 + p^2(\alpha_{p,3}^3 + \alpha_{p,3}^2 + \alpha_{p,3}).$$

Since $\mathscr{G}(0) \subset 2^{[7,n]} - 2^{[10,n]}$ we have $v_p(\mathscr{G}(0)) \leq 1-q^3$. Consequently using (3) we have

$$\begin{aligned} w_p(\mathscr{F}) &= p^5 q \sum_{i=1}^{6} v_p(\mathscr{G}(i)) + p^6 v_p(\mathscr{G}(0)) \\ &\leq p^5 q\big(2p^6 + p^5 + p^2(\alpha_{p,3}^3 + \alpha_{p,3}^2 + \alpha_{p,3})\big) + p^6(1-q^3). \end{aligned}$$

For $p = \frac{1}{2}$ we have $w_p(\mathscr{F}) < 0.01948 < h(1/2)$, and we settle this subcase by Claim 1.

Subcase 1.2. *There exists $G_0 \in \mathscr{G}(0)$ such that $G_0 \cap [7,9] = \emptyset$ but $G \cap [7,10] \neq \emptyset$ holds for all $G \in \mathscr{G}(0)$.*

Since $\mathscr{F}$ is shifted, we have $E_7' = [n] - [7,9] \in \mathscr{F}$, and we also have $E_i' = [n] - [i,i+2] \in \mathscr{F}$ for $i \geq 7$. Then it follows that $E_7' \cap E_{10}' \cap [7,12] = \emptyset$.

Claim 5. *For $i = 4,5,6$, $\mathscr{G}(i)$ is 3-wise $(15-2i)$-intersecting.*

Proof. To prove the case $i = 4$, suppose, on the contrary, that $\mathscr{G}(4)$ is not 3-wise 7-intersecting. Then we can find $G_4, G_5, G_6 \in \mathscr{G}(4)$ such that $|G_4 \cap G_5 \cap G_6| \leq 6$. By the shiftedness we may assume that $G_4 \cap G_5 \cap G_6 = [7,12]$. For $4 \leq j \leq 6$ let $F_j = ([6] - \{j\}) \cup G_j \in \mathscr{F}(j)$, and choose $F_2 \in \mathscr{F}(2)$ and $F_3 \in \mathscr{F}(3)$ arbitrarily. Then we have $F_2 \cap \cdots \cap F_6 \cap E_7' \cap E_{10}' = \{1\}$, which contradicts that $\mathscr{F}$ is 7-wise 2-intersecting.

To prove the case $i = 5$, suppose that $\mathscr{G}(5)$ is not 3-wise 5-intersecting. Then we can find $G_5 \cap G_6 \cap G_7 \in \mathscr{G}(5)$ such that $G_5 \cap G_6 \cap G_7 = [7,10]$. For $5 \leq j \leq 7$ let $F_j = ([7] - \{j\}) \cup G_j \in \mathscr{F}$, and for $2 \leq j \leq 4$ choose $F_j \in \mathscr{F}(j)$ arbitrarily. Then we have $F_2 \cap \cdots \cap F_7 \cap E_8' = \{1\}$, which is a contradiction.

For the last case, suppose that $\mathscr{G}(6)$ is not 3-wise 3-intersecting. Then we can find $G_6 \cap G_7 \cap G_8 \in \mathscr{G}(6)$ such that $G_6 \cap G_7 \cap G_8 = [7,8]$. For $6 \leq j \leq 8$ let $F_j = ([8] - \{j\}) \cup G_j \in \mathscr{F}$, and for $2 \leq j \leq 5$ choose $F_j \in \mathscr{F}(j)$ arbitrarily. Then we have $F_2 \cap \cdots \cap F_8 = \{1\}$, which is a contradiction. ☐

We get the following table from Claim 5.

$\mathscr{G}(i)$	$\mathscr{G}(4)$	$\mathscr{G}(5)$	$\mathscr{G}(6)$
ℓ-wise	3	3	3
t-int.	7	5	3

Since $\mathscr{G}(0) \subset 2^{[7,n]} - 2^{[11,n]}$ we have $v_p(\mathscr{G}(0)) \leq 1-q^4$. To bound $v_p(\mathscr{G}(i))$ for $1 \leq i \leq 6$ we use Lemma 8. Then we have

$$w_p(\mathscr{F}) \leq p^5 q(p^2(4\alpha_{p,3}^5 + \alpha_{p,3}^3 + \alpha_{p,3})) + p^6(1-q^4).$$

For $p = \frac{1}{2}$ we have $w_p(\mathscr{F}) < 0.0194$, and we are done.

Subcase 1.3. *There exists $G \in \mathscr{G}(0)$ such that $G \cap [7,10] = \emptyset$.*

In this case we use $E_i'' = [n] - [i,i+3] \in \mathscr{F}$ for $i \geq 7$, and we get the following table. (We omit the proof, which is similar to that of Claim 5.)

$\mathscr{G}(i)$	$\mathscr{G}(4)$	$\mathscr{G}(5)$	$\mathscr{G}(6)$
ℓ-wise	3	3	3
t-int.	9	6	3

To bound $v_p(\mathscr{G}(i))$ for $1 \le i \le 6$ we use Lemma 8. For $\mathscr{G}(0)$ we use a trivial bound $v_p(\mathscr{G}(0)) \le 1$. Then we have

$$w_p(\mathscr{F}) \le p^5 q(p^2(4\alpha_{p,3}^7 + \alpha_{p,3}^4 + \alpha_{p,3})) + p^6.$$

For $p = \frac{1}{2}$, we have $w_p(\mathscr{F}) < 0.0192$.

Case 2. $u = 6$.

Subcase 2.1. *$G \cap \{8,9\} \ne \emptyset$ holds for all $G \in \mathscr{G}(7)$.*

By Claim 3 and Claim 4, we get the following table.

$\mathscr{G}(i)$	$\mathscr{G}(3)$	$\mathscr{G}(4)$	$\mathscr{G}(5)$	$\mathscr{G}(6)$
ℓ-wise	4	3	3	3
t-int.	4	4	3	2

Since $\mathscr{G}(7) \subset 2^{[8,n]} - 2^{[10,n]}$, we have $v_p(\mathscr{G}(7)) \le 1 - q^2$. To bound $v_p(\mathscr{G}(i))$ we use Lemma 8 for $1 \le i \le 6$, and we use the trivial bound for $i = 0$. Then we have

$$w_p(\mathscr{F}) \le p^6 q(3p^4 + p^2(\alpha_{p,3}^2 + \alpha_{p,3} + 1) + (1 - q^2)) + p^7.$$

For $p = \frac{1}{2}$, we have $w_p(\mathscr{F}) < 0.0191$.

Subcase 2.2. *There exists $G \in \mathscr{G}(7)$ such that $G \cap \{8,9\} = \emptyset$.*

We use $E_i''' = [n] - [i, i+2] \in \mathscr{F}$ for $i \ge 7$ and we get the following table.

$\mathscr{G}(i)$	$\mathscr{G}(4)$	$\mathscr{G}(5)$	$\mathscr{G}(6)$
ℓ-wise	3	3	3
t-int.	6	4	2

To bound $w_p(\mathscr{G}(i))$ we use Lemma 8 for $1 \le i \le 6$. and we use trivial bounds for $i = 0, 7$. Then we have

$$w_p(\mathscr{F}) \le p^6 q(p^2(4\alpha_{p,3}^4 + \alpha_{p,3}^2 + 1) + 1) + p^7.$$

For $p = \frac{1}{2}$, we have $w_p(\mathscr{F}) < 0.01947$.

Case 3. $u = 7$.

By Claim 3 we find that $\mathscr{G}(5)$ is 3-wise 2-intersecting and $\mathscr{G}(6)$ is 3-wise 1-intersecting. To bound $v_p(\mathscr{G}(i))$ we use Lemma 8 for $1 \le i \le 6$, and we use trivial bounds for $i = 0, 7, 8$. Then we have

$$w_p(\mathscr{F}) \le p^7 q(5p^2 + p + 1 + 1) + p^8.$$

For $p = \frac{1}{2}$, we have $w_p(\mathscr{F}) < 0.0186$.

Case 4. $u = 4$.

Claim 6. *$\mathscr{G}(0)$ is 3-wise 2-intersecting.*

Proof. Suppose that $\mathscr{G}(0)$ is not 3-wise 2-intersecting. Then by the shiftedness we can find $G_6, G_7, G_8 \in \mathscr{G}(0)$ such that $G_6 \cap G_7 \cap G_8 = \{6\}$. For $j = 2,3,4$ choose $F_j \in \mathscr{F}(j)$ arbitrarily, for $j = 6,7,8$ let $F_j = [5] \cup G_j \in \mathscr{F}$, and recall that $E_5 = [k+2] - \{5,6\} \in \mathscr{F}$ by (4). Then we have $F_2 \cap F_3 \cap F_4 \cap E_5 \cap F_6 \cap F_7 \cap F_8 = \{1\}$, which is a contradiction. □

By Claim 3 and Claim 6, we find that $\mathscr{G}(5)$ is 3-wise 5-intersecting and $\mathscr{G}(0)$ is 3-wise 2-intersecting. To bound $w_p(\mathscr{G}(i))$ for $0 \leq i \leq 5$ we use Lemma 8. Then we have

$$w_p(\mathscr{F}) \leq p^4 q(5p^2 \alpha_{p,3}^3) + p^5 p^2.$$

For $p = \frac{1}{2}$ and sufficiently large n, we have $w_p(\mathscr{F}) < 0.0171$. This completes the proof of the initial step $r = 7$ of Lemma 12.

Next we show the induction step. Let $r > 7$ and let $\mathscr{F} \subset 2^{[n]}$ be a tame r-wise 2-intersecting family with $\mathscr{F} \not\subset \mathscr{E}(n,r,2)$. Let us define

$$\mathscr{F}_1 = \{F - \{1\} : 1 \in F \in \mathscr{F}\} \subset 2^{[2,n]}, \quad \mathscr{F}_1^- = \{F \in \mathscr{F} : 1 \notin F\} \subset 2^{[2,n]},$$

and we consider the p-weights of these families in $2^{[2,n]}$.

We may assume that $\mathscr{F}$ is p-weight maximal. Since $\mathscr{F}$ is tame, we have $[n] - \{i\} \in \mathscr{F}$ for $1 \leq i \leq n$. Thus $\mathscr{F}_1$ is also tame and $(r-1)$-wise 2-intersecting. Since $\mathscr{F} \not\subset \mathscr{E}(n,r,2)$ we have $[n] - \{r+1, r+2\} \in \mathscr{F}$ and so $\mathscr{F}_1 \not\subset \mathscr{E}(n-1, r-1, 2)$. Then using the induction hypothesis we have some $\gamma > 0$ and

$$w_p(\mathscr{F}_1 : [2,n]) < (1-\gamma) w_p(\mathscr{E}(n-1, r-1, 2)) = (1-\gamma)\big((r+1)p^r q + p^{r+1}\big).$$

On the other hand, $\mathscr{F}_1^-$ is r-wise $(r+1)$-intersecting. To see this fact, suppose on the contrary, that there exist $F_1, \ldots, F_r \in \mathscr{F}_1^-$ such that $|F_1 \cap \cdots \cap F_r| < r+1$. Since $\mathscr{F}$ is shifted, we may assume that $F_1 \cap \cdots \cap F_r = [2, r+1]$. Then we have $F_i' = (F_i - \{i\}) \cup \{1\} \in \mathscr{F}$ for $2 \leq i \leq r$, and $F_1 \cap F_2' \cap \cdots \cap F_r' = \{r+1\}$, a contradiction. Therefore $\mathscr{F}_1^-$ is r-wise $(r+1)$-intersecting and using Lemma 11 we have $w_p(\mathscr{F}_1^- : [2,n]) \leq p^{r+1}$. Consequently it follows that

$$\begin{aligned} w_p(\mathscr{F}) &= p w_p(\mathscr{F}_1 : [2,n]) + q w_p(\mathscr{F}_1^- : [2,n]) \\ &< p(1-\gamma)\big((r+1)p^r q + p^{r+1}\big) + q p^{r+1} = (1-\gamma')\big((r+2)p^{r+1} q + p^{r+2}\big), \end{aligned}$$

which completes the proof of Lemma 12, and also (i) of Theorem 5. □

3.2. **Proof of (ii).** Set $\mathscr{E}_1 = \mathscr{E}(n,r,2)$. Let $\mathscr{G} \subset 2^{[n]}$ be a (not necessarily shifted) non-trivial r-wise 2-intersecting family, and suppose that $\mathscr{G} \in \mathbf{X}(n,r,2)$. By Lemma 6 we can find a tame r-wise 2-intersecting family $\mathscr{G}^*$ with $w_p(\mathscr{G}^*) = w_p(\mathscr{G})$. If $\mathscr{G}^* \not\subset \mathscr{E}_1$ then we have already shown that $w_p(\mathscr{G}^*) < (1-\gamma) w_p(\mathscr{E}_1)$. Thus we may assume that $\mathscr{G}^* \subset \mathscr{E}_1$, and in particular (by renaming the starting family if necessary) we may assume that $\mathscr{G}^* = \sigma_{xy}(\mathscr{G}) \subset \mathscr{E}_1$, where $x = r+2, y = r+3$. We note that $|[x] \cap G| \geq r$ for all $G \in \mathscr{G}$. Moreover if $|[x] \cap G| = r$ then $G \cap \{x,y\} = \{y\}$ and $(G - \{y\}) \cup \{x\} \notin \mathscr{G}$.

For $i \in [x]$ set $\mathscr{G}(i) = \{G \in \mathscr{G} : [y] \setminus G = \{i\}\}$, and for $j \in [x-1]$ and $z \in \{x,y\}$ let $\mathscr{G}_z(j) = \{G \in \mathscr{G} : [y] \setminus G = \{j,z\}\}$. Since $\sigma_{xy}(\mathscr{G}) \subset \mathscr{E}_1$ we have $\mathscr{G}_x(j) \cap \mathscr{G}_y(j) = \emptyset$ and so

$w_p(\mathscr{G}_x(j))+w_p(\mathscr{G}_y(j)) \le p^{x-1}q^2$. Set $\mathscr{G}(\emptyset)=\{G\in\mathscr{G} : [x]\subset G\}$, $\mathscr{G}_{xy}=\{G\in\mathscr{G} : G\cap[y]=[x-1]\}$ and let $e=\min_{i\in[x]} w_p(\mathscr{G}(i))$. Then we have

$$\begin{aligned} w_p(\mathscr{G}) &= \sum_{i\in[x]} w_p(\mathscr{G}(i))+\sum_{j\in[x-1]}\big(w_p(\mathscr{G}_x(j))+w_p(\mathscr{G}_y(j))\big)+w_p(\mathscr{G}(\emptyset))+w_p(\mathscr{G}_{xy}) \quad (5)\\ &\le e+(x-1)p^xq+(x-1)p^{x-1}q^2+p^x+p^{x-1}q^2 = e+(\eta-1)p^xq, \quad (6)\end{aligned}$$

where $\eta=\frac{x}{p}+\frac{1}{q}$. Note that $e\le p^xq$, and (6) coincides $w_p(\mathscr{E}_1)=\eta p^xq$ iff $e=p^xq$. If there is some $j\in[x-1]$ such that $\mathscr{G}_x(j)\cup\mathscr{G}_y(j)=\emptyset$, then by (5) we get $w_p(\mathscr{G})\le w_p(\mathscr{E}_1)-p^{x-1}q^2=\big(1-q/(\eta p)\big)w_p(\mathscr{E}_1)$, and we are done. Thus we may assume that

$$\mathscr{G}_x(j)\cup\mathscr{G}_y(j)\ne\emptyset \text{ for all } j\in[x-1]. \quad (7)$$

To prove $w_p(\mathscr{G})<(1-\gamma)w_p(\mathscr{E}_1)$ by contradiction, let us assume that for any $\gamma>0$ and any n_0 there is some $n>n_0$ such that

$$w_p(\mathscr{G})>(1-\gamma)w_p(\mathscr{E}_1)=(1-\gamma)\eta p^xq. \quad (8)$$

By (6) and (8) we have $e>(1-\gamma\eta)p^xq$. This means, letting $\mathscr{H}(i)=\{G\setminus[y] : G\in\mathscr{G}(i)\}$ and $Y=[y+1,n]$,

$$w_p(\mathscr{H}(i):Y) \text{ only misses at most } \gamma\eta\ p\text{-weight for all } i\in[x]. \quad (9)$$

Since $\mathscr{G}\in\mathbf{X}(n,r,2)$ both $\bigcup_{j\in[x-1]}\mathscr{G}_x(j)$ and $\bigcup_{j\in[x-1]}\mathscr{G}_y(j)$ are non-empty. Using this with (7), we can choose $G\in\mathscr{G}_x(j)$ and $G'\in\mathscr{G}_y(j')$ with $j\ne j'$, say, $j=x-1, j'=x-2$. Let $L=[r-2]$ and $\mathscr{H}^*=\bigcap_{\ell\in L}\mathscr{H}(\ell)$. Then by (9) we have

$$w_p(\mathscr{H}^*:Y)>1-(r-2)\gamma\eta. \quad (10)$$

If $\mathscr{H}^*\subset 2^Y$ is not $(r-2)$-wise 1-intersecting, then we can find $H_\ell\in\mathscr{H}^*$ for $\ell\in L$ so that $H_1\cap\cdots\cap H_{r-2}=\emptyset$. Setting $G_\ell:=([y]-\{\ell\})\cup H_\ell\in\mathscr{G}$ we have $|G_1\cap\cdots\cap G_{r-2}\cap G\cap G'|=1$, which contradicts the r-wise 2-intersecting property of $\mathscr{G}$. Thus $\mathscr{H}^*$ is $(r-2)$-wise 1-intersecting and $w_p(\mathscr{H}^*:Y)\le p$ by Lemma 2. But this contradicts (10) because we can choose γ so small that $p\ll 1-(r-2)\gamma\eta$. □

4. Proof of Theorem 4

We deduce (ii) from Theorem 5, then (i) follows from (ii). Assuming the negation of Theorem 4, we will construct a counterexample to Theorem 5.

For reals $0<b<a$ we write $a\pm b$ to mean the open interval $(a-b,a+b)$ and $n(a\pm b)$ means $((a-b)n,(a+b)n)\cap\mathbb{N}$. Fix $\gamma_0:=\gamma_{\mathrm{Thm5}}$ and $\varepsilon_0:=\varepsilon_{\mathrm{Thm5}}$ from Theorem 5. For fixed r we note that $f(p):=w^*(n,p,r,2)=(r+2)p^{r+1}q+p^{r+2}$ is a uniformly continuous function of p on $\frac12\pm\varepsilon_0$. Let $\varepsilon=\frac{\varepsilon_0}{2}$, $\gamma=\frac{\gamma_0}{4}$, and $I=\frac12\pm\varepsilon$.

Choose $\varepsilon_1\ll\varepsilon$ so that

$$(1-3\gamma)f(p)>(1-4\gamma)f(p+\delta) \quad (11)$$

holds for all $p \in I$ and all $0 < \delta \le \varepsilon_1$. Choose n_1 so that

$$\sum_{i \in J} \binom{n}{i} p_0^i (1-p_0)^{n-i} > (1-3\gamma)/(1-2\gamma) \tag{12}$$

holds for all $n > n_1$ and all $p_0 \in I_0 := \frac{1}{2} \pm \frac{3\varepsilon}{2}$, where $J = n(p_0 \pm \varepsilon_1)$. Choose n_2 so that

$$(1-\gamma)|\mathscr{F}(n,k,r,2)| > (1-2\gamma) f(k/n) \binom{n}{k} \tag{13}$$

holds for all $n > n_2$ and k with $k/n \in I$. Finally set $n_0 = \max\{n_1, n_2\}$.

Suppose that Theorem 4 fails. Then for our choice of ε, γ and n_0, we can find some n, k and $\mathscr{F} \in \mathbf{Y}(n,k,r,2)$ with $|\mathscr{F}| \ge (1-\gamma)|\mathscr{F}(n,k,r,2)|$, where $n > n_0$ and $\frac{k}{n} \in I$. We fix n, k and $\mathscr{F}$, and let $p = \frac{k}{n}$. By (13) we have $|\mathscr{F}| > c\binom{n}{k}$, where $c = (1-2\gamma) f(p)$. Let $\mathscr{G} = \bigcup_{k \le i \le n} (\nabla_i(\mathscr{F}))$ be the collection of all upper shadows of $\mathscr{F}$, where $\nabla_i(\mathscr{F}) = \{H \in \binom{[n]}{i} : H \supset \exists F \in \mathscr{F}\}$. Then we have $\mathscr{G} \in \mathbf{X}(n,r,2)$. Let $p_0 = p + \varepsilon_1 \in I_0$.

Claim 7. $|\nabla_i(\mathscr{F})| \ge c\binom{n}{i}$ *for* $i \in J$.

Proof. Choose a real $x \le n$ so that $c\binom{n}{k} = \binom{x}{n-k}$. Since $|\mathscr{F}| > c\binom{n}{k} = \binom{x}{n-k}$ the Kruskal–Katona Theorem[10, 9] implies that $|\nabla_i(\mathscr{F})| \ge \binom{x}{n-i}$. Thus it suffices to show that $\binom{x}{n-i} \ge c\binom{n}{i}$, or equivalently,

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}} \ge \frac{c\binom{n}{i}}{c\binom{n}{k}}.$$

Using $i \ge k$ this is equivalent to $i \cdots (k+1) \ge (x-n+i) \cdots (x-n+k+1)$, which follows from $x \le n$. □

By the claim we have

$$w_{p_0}(\mathscr{G}) \ge \sum_{i \in J} |\nabla_i(\mathscr{F})|\, p_0^i (1-p_0)^{n-i} \ge c \sum_{i \in J} \binom{n}{i} p_0^i (1-p_0)^{n-i}. \tag{14}$$

Using (12) and (11), the RHS of (14) is more than

$$c\,(1-3\gamma)/(1-2\gamma) = (1-3\gamma) f(p) > (1-4\gamma) f(p+\varepsilon_1) = (1-\gamma_0) f(p_0).$$

This means $w_{p_0}(\mathscr{G}) > (1-\gamma_0)\, w^*(n, p_0, r, 2)$, which contradicts Theorem 5 (ii). □

Acknowledgment. The author would like to thank the referee for careful reading and valuable comments.

References

[1] R. Ahlswede, L.H. Khachatrian. The complete nontrivial-intersection theorem for systems of finite sets. *J. Combin. Theory (A)*, 76:121-138, 1996.

[2] A. Brace and D. E. Daykin. A finite set covering theorem. *Bull. Austral. Math. Soc.*, 5:197–202, 1971.

[3] P. Frankl, Z. Füredi. Non-trivial intersecting families. *J. Combin. Theory (A)*, 20:1–11, 1976.

[4] P. Frankl, N. Tokushige. Weighted multiply intersecting families. *Studia Sci. Math. Hungarica* 40:287–291, 2003.

[5] P. Frankl, N. Tokushige. Weighted 3-wise 2-intersecting families. *J. Combin. Theory (A)*, 100:94–115, 2002.

[6] P. Frankl, N. Tokushige. Weighted non-trivial multiply intersecting families. *Combinatorica*, 26:37–46, 2006.

[7] A. J. W. Hilton, E. C. Milner, Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser.* 18:369–384, 1967.

[8] S. Janson, T. Łuczak, A. Rucinski. Random Graphs. *Wiley-Interscience Series in Discrete Mathematics and Optimization*. Wiley-Interscience, New York, 2000.

[9] G.O.H. Katona. A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, 1966 (Akademiai Kiadó, 1968) 187–207.

[10] J.B. Kruskal. The number of simplices in a complex, in: Math. Opt. Techniques (Univ. of Calif. Press, 1963) 251–278.

[11] N. Tokushige. Extending the Erdős–Ko–Rado theorem. *J. Combin. Designs*, 14:52–55 (2006).

[12] N. Tokushige. EKR type inequalities for 4-wise intersecting families. *J. Combin. Theory (A)*, to appear.

[13] N. Tokushige. Intersecting families — uniform versus weighted, *Ryukyu Math. J.*, 18:89–103 (2005).

COLLEGE OF EDUCATION, RYUKYU UNIVERSITY, NISHIHARA, OKINAWA, 903-0213 JAPAN
E-mail address: hide@edu.u-ryukyu.ac.jp