MULTIPLY-INTERSECTING FAMILIES REVISITED

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ABSTRACT. Motivated by the Frankl's results in [11] ("Multiply-intersecting families," J. Combin. Theory (B) 1991), we consider some problems concerning the maximum size of multiply-intersecting families with additional conditions. Among other results, we show the following version of the Erdős–Ko–Rado theorem: for all $r \geq 8$ and $1 \leq t \leq 2^{r+1} - 3r - 1$ there exist positive constants ε and n_0 such that if $n > n_0$ and $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ then r-wise t-intersecting k-uniform families on n vertices have size at most $\max\{\binom{n-t}{k-t}, (t+r)\binom{n-t-r}{k-t-r+1} + \binom{n-t-r}{k-t-r}\}$.

1. INTRODUCTION

A family (or hypergraph) $\mathscr{G} \subset 2^{[n]}$ is called r-wise t-intersecting if $|G_1 \cap \cdots \cap G_r| \geq t$ holds for all $G_1, \ldots, G_r \in \mathscr{G}$. The aim of this paper is to find largest r-wise t-intersecting families with some additional conditions, which extends some of Frankl's results and his proof technique developed in [11]. Let us define a typical r-wise t-intersecting family $\mathscr{G}_i(n,r,t)$ and its k-uniform subfamily $\mathscr{F}_i(n,k,r,t)$ as follows:

$$\begin{aligned} \mathscr{G}_i(n,r,t) &= \{G \subset [n] : |G \cap [t+ri]| \geq t + (r-1)i\}, \\ \mathscr{F}_i(n,k,r,t) &= \mathscr{G}_i(n,r,t) \cap \binom{[n]}{k}. \end{aligned}$$

An r-wise t-intersecting family $\mathscr{G}$ is called *non-trivial* if $|\bigcap \mathscr{G}| < t$, where $\bigcap \mathscr{G} := \bigcap_{G \in \mathscr{G}} G$. Two families $\mathscr{G}, \mathscr{G}' \subset 2^{[n]}$ are said to be isomorphic and denoted by $\mathscr{G} \cong \mathscr{G}'$ if there exists a vertex permutation τ on $[n]$ such that $\mathscr{G}' = \{\{\tau(g) : g \in G\} : G \in \mathscr{G}\}$.

Let $m(n,k,r,t)$ be the maximal size of k-uniform r-wise t-intersecting families on n vertices. To determine $m(n,k,r,t)$ is one of the oldest problems in extremal set theory, which is still widely open. The case $r = 2$ was observed by Erdős–Ko–Rado[6], Frankl[9], Wilson[34], and then $m(n,k,2,t) = \max_i |\mathscr{F}_i(n,k,2,t)|$ was finally proved by Ahlswede and Khachatrian[2]. Frankl[8] showed $m(n,k,r,1) = |\mathscr{F}_0(n,k,r,1)|$ if $(r-1)n \geq rk$, see also [20, 27]. Partial results for the cases $r \geq 3$ and $t \geq 2$ are found in [12, 14, 29, 30, 31, 32]. All known results suggest

$$m(n,k,r,t) = \max_i |\mathscr{F}_i(n,k,r,t)|$$

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in general, and we will consider the case when the maximum is attained by $\mathscr{F}_0$ or $\mathscr{F}_1$. To state our result let us define a list A of acceptable parameters as follows.

$$\begin{aligned} A &= \{(r,t) : r \geq 5, 1 \leq t \leq 2^{r+1} - 3r - 1\} \\ &\quad -\{(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(7,1)\}. \end{aligned} \tag{1}$$

Theorem 1. *Let $(r,t) \in A$ be fixed. Then there exist positive constants ε, n_0 such that*

$$m(n,k,r,t) = \max\{|\mathscr{F}_0(n,k,r,t)|, |\mathscr{F}_1(n,k,r,t)|\}$$

holds for all $n > n_0$ and k with $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$. Moreover $\mathscr{F}_0(n,k,r,t)$ and $\mathscr{F}_1(n,k,r,t)$ are the only optimal configurations (up to isomorphism).

We note that $|\mathscr{F}_0(n,k,r,t)| = \binom{n-t}{k-t}$ and $|\mathscr{F}_1(n,k,r,t)| = (t+r)\binom{n-t-r}{k-t-r+1} + \binom{n-t-r}{k-t-r}$. Some computation shows that if $(r,t) \in A$ and $r \ll k$ then $\max\{|\mathscr{F}_0|, |\mathscr{F}_1|\}$ is attained by

$$\begin{cases} \mathscr{F}_0(n,k,r,t) & \text{if } 1 \leq t \leq 2^r - r - 2, \text{ or } t = 2^r - r - 1 \text{ and } n \geq 2k - 2^r + \lceil r/2 \rceil + 3, \\ \mathscr{F}_1(n,k,r,t) & \text{if } t \geq 2^r - r, \text{ or } t = 2^r - r - 1 \text{ and } n \leq 2k - 2^r + \lceil r/2 \rceil + 2. \end{cases}$$

Conjecture 1. *Theorem 1 is true for all $r \geq 3$ and $1 \leq t \leq 2^{r+1} - 3r - 1$.*

Let $m^*(n,k,r,t)$ be the maximal size of non-trivial k-uniform r-wise t-intersecting families on n vertices. Ahlswede and Khachatrian[1] determined $m^*(n,k,2,t)$ completely, which included earlier results of Hilton–Milner[21] and Frankl[10]. In [33] a k-uniform version of the Brace–Daykin theorem[4] is considered for $m^*(n,k,r \geq 7,2)$ and $k/n \approx 1/2$. To state our result let us define some families of k-uniform hypergraphs as follows.

$$\begin{aligned} \mathbf{F}(n,k,r,t) &= \{\mathscr{F} \subset \tbinom{[n]}{k} : \mathscr{F} \text{ is } r\text{-wise } t\text{-intersecting}\}, \\ \mathbf{F}_j(n,k,r,t) &= \{\mathscr{F} \subset \tbinom{[n]}{k} : \mathscr{F} \subset \mathscr{F}' \text{ for some } \mathscr{F}' \cong \mathscr{F}_j(n,k,r,t)\}, \\ \mathbf{Y}^i(n,k,r,t) &= \mathbf{F}(n,k,r,t) - \bigcup_{0 \leq j \leq i} \mathbf{F}_j(n,k,r,t). \end{aligned}$$

For fixed n,k,r,t, we clearly have $\mathbf{F}_j \subset \mathbf{F}$. We are interested in $m^* = \max\{|\mathscr{F}| : \mathscr{F} \in \mathbf{Y}^0\}$. It seems that hypergraphs in $\mathbf{F}$ with nearly largest size only come from some $\mathbf{F}_j$, moreover they are stable in a sense, namely, $\max\{|\mathscr{F}| : \mathscr{F} \in \mathbf{Y}^1\} < (1-\gamma)m^*$ for some fixed constant $\gamma > 0$. (See [16, 26] for more about stability type results.) We verify this phenomenon in the case $t \leq 2^{r+1} - 3r - 1$ and $k/n \approx 1/2$.

Theorem 2. *Let $(r,t) \in A$ be fixed, where A is defined by (1). Then there exist positive constants γ, ε, n_0 such that the following (i) and (ii) are true for all $n > n_0$ and k with $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$.*

(i) *$m^*(n,k,r,t) = |\mathscr{F}_1(n,k,r,t)|$.*

(ii) *If $\mathscr{F} \in \mathbf{Y}^1(n,k,r,t)$ then $|\mathscr{F}| < (1-\gamma)m^*(n,k,r,t)$.*

The above result immediately implies Theorem 1. We also apply this result to get a Sperner type inequality. A family $\mathscr{G} \subset 2^{[n]}$ is called a Sperner family if $G \not\subset G'$ holds for all distinct $G, G' \in \mathscr{G}$. Let $s(n,r,t)$ be the maximal size of r-wise t-intersecting Sperner families on n vertices. Milner[25] proved $s(n,r=2,t) = \binom{n}{\lceil (n+t)/2 \rceil}$. Frankl[8] and Gronau[17, 18, 19, 20] determined $s(n,r=3,t=1)$ for $n \geq 53$. Gronau[18] also proved $s(n,r \geq 4,t = 1) = \binom{n-1}{\lceil (n-1)/2 \rceil}$ for all n. For sufficiently large n, it was proved that $s(n,r \geq 4,t=2) = \binom{n-2}{\lceil (n-2)/2 \rceil}$ in [12], $s(n,r,t) = \binom{n-t}{\lceil (n-t)/2 \rceil}$ for $r \geq 5$ and $1 \leq t \leq 2^{r-2}\log 2 - 1$ in [29], and $s(n,r=3,t=2)$ was determined in [12, 14]. Using Theorem 2 we prove the following.

Theorem 3. *Let $r \geq 7$ and $1 \leq t \leq 2^{r+1} - 3r - 1$. Then there exists n_0 such that*

$$s(n,r,t) = \begin{cases} |\mathscr{F}_0(n,k_0,r,t)| & \text{if } 1 \leq t \leq 2^r - r - 2 \\ |\mathscr{F}_1(n,k_1,r,t)| & \text{if } 2^r - r - 1 \leq t \leq 2^{r+1} - 3r - 1 \end{cases}$$

for all $n > n_0$, where $k_0 \in \{t + \lceil \frac{n-t}{2} \rceil, t + \lfloor \frac{n-t}{2} \rfloor\}$ and $k_1 = t + r - 1 + \lceil \frac{n-t-r}{2} \rceil$. Moreover $\mathscr{F}_0(n,k_0,r,t)$ and $\mathscr{F}_1(n,k_1,r,t)$ are the only optimal configurations (up to isomorphism).

Conjecture 2. *Theorem 3 is true for $4 \leq r \leq 6$ as well.*

Due to the results mentioned above [18, 12], the conjecture is true for $t = 1, 2$. Our proof of Theorem 3 is valid for all $(r,t) \in A$, and the conjecture is open for $(r,t) \in \{(4,t) : 3 \leq t \leq 19\} \cup \{(5,3),(5,4),(6,3)\}$. The conjecture fails for $r = 3$. In fact it is known from [8, 17, 14] that $s(n = 2m, 3, 1) = \binom{n-1}{m} + 1$, $s(n = 2m+1, 3, 2) = \binom{n-2}{m} + 2$ (for n large enough). The exact value of $s(n,3,3)$ is not known, while $s(n = 2m, 3, 3) \geq \binom{n-3}{m-1} + 3$.

Finally we introduce a weighted version of Frankl's result in [11], which was a starting point of this research. Throughout this paper, p and $q = 1 - p$ denote positive real numbers. For a family $\mathscr{G} \subset 2^X$ we define the p-weight of $\mathscr{G}$, denoted by $w_p(\mathscr{G} : X)$, as follows:

$$w_p(\mathscr{G} : X) = \sum_{G \in \mathscr{G}} p^{|G|} q^{|X|-|G|} = \sum_{i=0}^{|X|} \left| \mathscr{G} \cap \binom{X}{i} \right| p^i q^{|X|-i}.$$

We simply write $w_p(\mathscr{G})$ for the case $X = [n]$.

Let $w(n,p,r,t)$ be the maximal p-weight of r-wise t-intersecting families on n vertices, and let $w^*(n,p,r,t)$ be the maximal p-weight of non-trivial r-wise t-intersecting families on n vertices. It might be natural to expect

$$w(n,p,r,t) = \max_i w_p(\mathscr{G}_i(n,r,t)).$$

Ahlswede and Khachatrian proved that this is true for $r = 2$ in [3] (cf. [5, 7, 29]). This includes the Katona theorem[22] about $w(n, 1/2, 2, t)$. It is shown in [13] that

$$w(n,p,r,1) = w_p(\mathscr{G}_0(n,r,1)) = p \text{ for } p \leq (r-1)/r. \tag{2}$$

Partial results for $w^*(n,p,r,1)$ are found in [15, 33], which extend the result of Brace–Daykin[4]: $w^*(n,1/2,r,1) = w_{1/2}(\mathscr{G}_1(n,r,1))$. Let us define some families of hypergraphs as follows.

$$\begin{aligned}\mathbf{G}(n,r,t) &= \{\mathscr{G}\subset 2^{[n]} : \mathscr{G} \text{ is } r\text{-wise } t\text{-intersecting}\},\\ \mathbf{G}_j(n,r,t) &= \{\mathscr{G}\subset 2^{[n]} : \mathscr{G}\subset\mathscr{G}' \text{ for some } \mathscr{G}'\cong\mathscr{G}_j(n,r,t)\},\\ \mathbf{X}^i(n,r,t) &= \mathbf{G}(n,r,t) - \bigcup_{0\le j\le i}\mathbf{G}_j(n,r,t).\end{aligned}$$

Now we state the main result in this paper, which will imply Theorem 2.

Theorem 4. *Let $(r,t)\in A$ be fixed, where A is defined by (1). Then there exist positive constants γ,ε such that the following (i) and (ii) are true for all $n\ge r+t$ and p with $|p-\frac{1}{2}|<\varepsilon$.*
(i) $w^*(n,p,r,t)=w_p(\mathscr{G}_1(n,r,t))$.
(ii) *If $\mathscr{G}\in\mathbf{X}^1(n,r,t)$ then $w_p(\mathscr{G})<(1-\gamma)w^*(n,p,r,t)$.*

In [15] it is shown by construction that $w^*(n,p,5,1)>w_p(\mathscr{G}_1(n,5,1))$ for all $1/2<p<(1+\sqrt{21})/10$. Theorem 4 could be true for all $r\ge 5$ with only exception $r=5$ and $t=1$, and the same extension could be expected for Theorem 2. The upper bound for t set by (1) in Theorem 4 (and also Theorems 2 and 3) is best possible. In fact we have $w_p(\mathscr{G}_2(n,r,t))>w_p(\mathscr{G}_1(n,r,t))$ for $t\ge 2^{r+1}-3r$, see Lemma 2 in the next section. We emphasize that Frankl has already got a special case of (i) of Theorem 4 in [11] (Theorem 6.4), where he proved

$$w^*(n,1/2,r,t)=w_{1/2}(\mathscr{G}_1(n,r,t)) \text{ for } r\ge 5 \text{ and } 1\le t\le 2^r-r-1. \tag{3}$$

Our proof of (i) is based on his idea, but changing the weight from $1/2$ to p is not straightforward. As we mentioned above, (3) is no longer true if we replace $1/2$ with $1/2+\varepsilon$ for the case $r=5$ and $t=1$. One of the main reasons comes from the fact

$$w^*(n,1/2,3,2)<0.773\,(1/2)^2,$$

which Frankl used as a base case for his proof of (3), while in our case we only have

$$\lim_{n\to\infty} w^*(n,p,3,2)=p^2$$

for $p=1/2+\varepsilon$, see [12]. We will use results from [12, 29, 32] for our base case, which give $w(n,p,r,t)$ for $r=4,5$, see Lemma 5. Theorem 4 implies the following immediately.

Theorem 5. *Let $(r,t)\in A$ be fixed. Then there exists positive constant ε such that*

$$w(n,p,r,t)=\max\{w_p(\mathscr{G}_0(n,r,t)),\,w_p(\mathscr{G}_1(n,r,t))\}$$

holds for all $n\ge r+t$ and p with $|p-\frac{1}{2}|<\varepsilon$. Moreover $\mathscr{G}_0(n,r,t)$ and $\mathscr{G}_1(n,r,t)$ are the only optimal configurations (up to isomorphism).

Comparing $w_p(\mathscr{G}_1)$ and $w_p(\mathscr{G}_2)$ (see Lemma 1 in the next section), we find that if $(r,t)\in A$ then $\max\{w_p(\mathscr{G}_1),\,w_p(\mathscr{G}_2)\}$ is attained by

$$\begin{cases}\mathscr{G}_0(n,r,t) & \text{if } 1\le t\le 2^r-r-2, \text{ or } t=2^r-r-1 \text{ and } p\le 1/2,\\ \mathscr{G}_1(n,r,t) & \text{if } t\ge 2^r-r, \text{ or } t=2^r-r-1 \text{ and } p>1/2.\end{cases}$$

In Theorems 1 and 5, we focused on the case when the range for k/n or p is around $1/2$. We can extend this range for the case $t \le 2^r - r - 1$ as follows.

Theorem 6. *Let $(r,t) \in A$ and $t \le 2^r - r - 1$. Then for all $\varepsilon > 0$ there exist positive constants γ, n_0 such that $m^*(n,k,r,t) < (1-\gamma)\binom{n-t}{k-t}$ holds for all $n > n_0$ and k with $\frac{k}{n} < \frac{1}{2} - \varepsilon$. In particular, we have $m(n,k,r,t) = \binom{n-t}{k-t}$, and $\mathscr{F}_0(n,k,r,t)$ is the only optimal family (up to isomorphism).*

Theorem 7. *Let $(r,t) \in A$ and $t \le 2^r - r - 1$. Then for all $\varepsilon > 0$ there exists positive constant γ such that $w^*(n,p,r,t) < (1-\gamma)p^t$ holds for all $n \ge t$ and p with $p < \frac{1}{2} - \varepsilon$. In particular, we have $w(n,p,r,t) = p^t$, and $\mathscr{G}_0(n,r,t)$ is the only optimal family (up to isomorphism).*

As the reader might expect, $m(n,k,r,t)/\binom{n}{k}$ and $w(n,p,r,t)$ are closely related when $p \approx k/n$. This was observed by Dinur and Safra in [7] for the case $r = 2$. See also [29] for more general setting. We will fully use this relation to prove our results.

In Section 2, we prepare some tools for the proofs. We prove Theorem 4 in Section 3. In the last section, we prove the other theorems in the following implication.

$$\text{Theorem 3} \Leftarrow \text{Theorem 2} \Leftarrow \text{Theorem 4} \Rightarrow \text{Theorem 6} \Rightarrow \text{Theorem 7}$$

2. Tools

2.1. **Some inequalities.** To find $w(n,p,r,t)$ we need to know $\max_i w_p(\mathscr{G}_i(n,r,t))$. So let us start with comparing $w_p(\mathscr{G}_0(n,r,t)) = p^t$ and $w_p(\mathscr{G}_1(n,r,t)) = (t+r)p^{t+r-1}q + p^{t+r}$. Then we have $w_p(\mathscr{G}_0) \ge w_p(\mathscr{G}_1)$ iff $t \le (p^{1-r} - p)/q - r =: f(p)$. We note that $f(1/2) = 2^r - r - 1$, and $f(p)$ is decreasing iff $1 - qr - p^r < 0$ (and this is so for $p = 1/2$ and $r \ge 2$). Thus we have the following.

Lemma 1. *For every $r \ge 2$ there exists $\varepsilon > 0$ such that $w_p(\mathscr{G}_0(n,r,t)) \ge w_p(\mathscr{G}_1(n,r,t))$ holds for $p \in (1/2 - \varepsilon, 1/2]$ iff $1 \le t \le 2^r - r - 1$, and $w_p(\mathscr{G}_0(n,r,t)) > w_p(\mathscr{G}_1(n,r,t))$ holds for $p \in (1/2, 1/2 + \varepsilon)$ iff $1 \le t \le 2^r - r - 2$.*

Lemma 2. *For every $r \ge 3$ there exists $\varepsilon > 0$ such that $w_p(\mathscr{G}_1(n,r,t)) > w_p(\mathscr{G}_2(n,r,t))$ holds for all p with $|p - 1/2| < \varepsilon$ iff $1 \le t \le 2^{r+1} - 3r - 1$.*

Proof. Since $w_p(\mathscr{G})$ is a continuous function of p (for fixed $\mathscr{G}$), it is sufficient to show the case $p = 1/2$. So set $p = 1/2$ and let $\mathscr{G}_1 = \mathscr{G}_1(n,r,t)$ and $\mathscr{G}_2 = \mathscr{G}_2(n,r,t)$. First we note that $w_p(\mathscr{G}_1) > w_p(\mathscr{G}_2)$ iff $w_p(\mathscr{G}_1 \setminus \mathscr{G}_2) > w_p(\mathscr{G}_2 \setminus \mathscr{G}_1)$, and

$$\begin{aligned}
\mathscr{G}_1 \setminus \mathscr{G}_2 &= \{G \subset [n] : [t+r] \subset G,\ |G \cap [t+r+1, t+2r]| < r-2\} \\
&\quad \cup \{G \subset [n] : |G \cap [t+r]| = t+r-1,\ |G \cap [t+r+1, t+2r]| < r-1\}, \\
\mathscr{G}_2 \setminus \mathscr{G}_1 &= \{G \subset [n] : |G \cap [t+r]| = t+r-2,\ [t+r+1, t+2r] \subset G\}.
\end{aligned}$$

Then we have

$$\begin{aligned} w_p(\mathscr{G}_1 \setminus \mathscr{G}_2) &= p^{t+2r}\big(\textstyle\sum_{j=0}^{r-3}\binom{r}{j} + (t+r)\sum_{j=0}^{r-2}\binom{r}{j}\big) \\ &= p^{t+2r}\big((t+r+1)(2^r-1-r) - \binom{r}{2}\big), \\ w_p(\mathscr{G}_2 \setminus \mathscr{G}_1) &= p^{t+2r}\binom{t+r}{2}. \end{aligned}$$

Thus we have $w_p(\mathscr{G}_1) = w_p(\mathscr{G}_2)$ iff $f(t) := (t+r+1)(2^r-1-r) - \binom{r}{2} - \binom{t+r}{2} = 0$, and this quadratic equation of t has only one positive root. We have $f(2^{r+1}-3r-1) = 2^r - r^2/2 - r/2 - 1 > 0$ and $f(2^{r+1}-3r) = -(r^2-r+2)/2 < 0$, which completes the proof. □

Similarly one can prove the following.

Lemma 3. *Let $j = 3,4$. For every $r \geq j+2$ there exists $\varepsilon > 0$ such that $w_p(\mathscr{G}_{j-1}(n,r,t)) > w_p(\mathscr{G}_j(n,r,t))$ holds for all p with $|p-1/2| < \varepsilon$ iff $1 \leq t \leq j(2^r-2r+1)+r-3$.*

Throughout this paper, let $\alpha_{r,p} \in (p,1)$ be the root of the equation $X = p + qX^r$. We write α_r omitting p for the case $p = 1/2$. For later use, we record the numerical data: $\alpha_3 = (\sqrt{5}-1)/2 \approx 0.618$, $\alpha_4 \approx 0.543689$, $\alpha_5 \approx 0.51879$, $\alpha_6 \approx 0.50866$, $\alpha_7 \approx 0.504138$. We list inequalities about $w(n,p,r,t)$ below, which will be used to prove Theorem 4. Lemma 6 follows from Lemma 4 and Lemma 5.

Lemma 4 ([33])**.** *Let p,r,t_0,c be fixed constants. Suppose that $w(n,p,r,t_0) = c$ holds for all $n \geq t_0$. Then we have $w(n,p,r,t) \leq c\alpha_{r,p}^{t-t_0}$ for all $t \geq t_0$ and $n \geq t$.*

Lemma 5 ([12, 29, 32])**.** *Let $r = 3$ and $1 \leq t \leq 2$, or $r = 4$ and $1 \leq t \leq 7$, or $r = 5$ and $1 \leq t \leq 18$. Then there exists $\varepsilon > 0$ such that $w(n,p,r,t) = p^t$ holds for all $n \geq t$ and p with $|p-\frac{1}{2}| < \varepsilon$.*

Lemma 6. *Let $s \geq 2$ and $t \geq 7$. Then there exists $\varepsilon > 0$ such that*

$$w(n,p,3,s) \leq p^2\alpha_{3,p}^{s-2} \ \text{ and } \ w(n,p,4,t) \leq p^7\alpha_{4,p}^{t-7}$$

hold for all $n \geq s$ (resp. $n \geq t$) and p with $|p-\frac{1}{2}| < \varepsilon$.

We will use Lemma 8 in our main reduction step to prove Theorem 4, see Claim 9. To prove Lemma 8 we need the following lemma, which is essentially proved in [11], cf. Proposition 2.8 and 7.7 of [11].

Lemma 7. *We have (i) $(2\alpha_r)^{2^{r+1}} < 8$ for $r \geq 8$, and (ii) $1/(2\alpha_r) < 1-(1/2)^r$.*

Proof. Recall that α_r is the unique root of $f(x) = 0$ in $(1/2,1)$, where $f(x) = x^r - 2x + 1$. We note that $f(1/2) > 0$ and $f(1) = 0$.

(i) is equivalent to $2\alpha_r < 8^b$, where $b = 1/2^{r+1}$. It is sufficient to show $f(8^b/2) < 0$. We use $br = r/2^{r+1} \leq 8/2^9 = 1/64$, $2 \times 8^{1/64} < 2.07 < \log 8$, and $8^b = e^{b\log 8} > 1 + b\log 8$. Then we have $(8^b/2)^r = 8^{br}/2^r \leq 8^{1/64}/2^r < (\log 8)/2^{r+1} = b\log 8 < 8^b - 1$, as desired.

(ii) is equivalent to $\alpha_r > \beta := 2^{r-1}/(2^r-1)$. It is sufficient to show $f(\beta) > 0$, and this follows from $\beta^r = \big(\frac{1}{2}\big(\frac{2^r}{2^r-1}\big)\big)^r = \frac{1}{2^r}\big(\frac{2^r}{2^r-1}\big)^r > \frac{1}{2^r}\big(\frac{2^r}{2^r-1}\big) = \frac{1}{2^r-1} = 2\beta - 1$. □

Lemma 8. *Let $r \geq 9$, $t_r = 2^{r+1} - 3r - 1$ and $p = 1/2$. Then we have*

$$w_p(\mathscr{G}_1(n, r-1, t_{r-1}))\alpha_{r-1}^{(t+3)-t_{r-1}} < w_p(\mathscr{G}_1(n, r, t)) \tag{4}$$

for $t_{r-1} \leq t \leq t_r$, where $w_p(\mathscr{G}_1(n, a, b)) = (a+b+1)p^{a+b}$.

Proof. Set $\alpha = \alpha_{r-1}$, $t = t_r - i$ and we prove (4) by induction on i, $0 \leq i \leq t_r - t_{r-1} = 2^r - 3$. First we show the case $i = 0$, i.e., $t = t_r$. In this case (4) is

$$(2^r - 2r + 2)p^{2^r - 2r + 1}\alpha^{2^r} < (2^{r+1} - 2r)p^{2^{r+1} - 2r - 1},$$

or equivalently,

$$\alpha^{2^r} < \frac{2^{r+1} - 2r}{2^r - 2r + 2}\, p^{2^r - 2}.$$

The RHS is more than $2p^{2^r-2} = 8p^{2^r}$, and so it is sufficient to show $\alpha^{2^r} < 8p^{2^r}$, i.e., $(2\alpha_{r-1})^{2^r} < 8$, which is true for $r \geq 9$ by Lemma 7 (i).

To show the induction step, we assume that (4) is true for i, that is,

$$R\,(2\alpha)^{2^r - i} < 2^{r+1} - 2r - i,$$

where $R = (2^r - 2r + 2)/4$. Then, for the case $i + 1$, we have

$$R(2\alpha)^{2^r - (i+1)} = R(2\alpha)^{2^r - i}/(2\alpha) < (2^{r+1} - 2r - i)/(2\alpha).$$

We have to show that the RHS is less than $2^{r+1} - 2r - (i+1)$, that is,

$$\frac{1}{2\alpha} < 1 - \frac{1}{2^{r+1} - 2r - i}.$$

By Lemma 7 (ii) and $i \leq 2^r - 3$ we have

$$\frac{1}{2\alpha_{r-1}} < 1 - \frac{1}{2^{r-1}} < 1 - \frac{1}{2^{r+1} - 2r - (2^r - 3)} \leq 1 - \frac{1}{2^{r+1} - 2r - i}$$

as desired. □

We use Lemmas 9 and 10 to prove Theorems 4 and 7 respectively.

Lemma 9. *$w^*(n, p, r, t) \leq w^*(n, p, r-1, t+1)$.*

Proof. If $\mathscr{G} \in \mathbf{X}^0(n, r, t)$ then $\mathscr{G} \in \mathbf{X}^0(n, r-1, t+1)$. In fact, if $\mathscr{G}$ is not $(r-1)$-wise $(t+1)$-intersecting, then we can find $G_1, \ldots, G_{r-1} \in \mathscr{G}$ such that $|G_1 \cap \cdots \cap G_{r-1}| = t$. But $\mathscr{G}$ is r-wise t-intersecting and so every $G \in \mathscr{G}$ must contain $G_1 \cap \cdots \cap G_{r-1}$, which contradicts the fact that $\mathscr{G}$ is non-trivial. □

Lemma 10. *$w^*(n+1, p, r, t) \geq w^*(n, p, r, t)$.*

Proof. Choose $\mathscr{G} \in \mathbf{X}^0(n, r, t)$ with $w_p(\mathscr{G}) = w^*(n, p, r, t)$. Then we have $\mathscr{G}' := \mathscr{G} \cup \{G \cup \{n+1\} : G \in \mathscr{G}\} \in \mathbf{X}^0(n+1, r, t)$ and $w_p(\mathscr{G}' : [n+1]) = w_p(\mathscr{G} : [n])(q+p) = w^*(n, p, r, t)$, which means $w^*(n+1, p, r, t) \geq w^*(n, p, r, t)$. □

2.2. **Shifting and shadow.** For integers $1 \le i < j \le n$ and a family $\mathscr{G} \subset 2^{[n]}$, we define the (i,j)-shift σ_{ij} as follows:

$$\sigma_{ij}(\mathscr{G}) = \{\sigma_{ij}(G) : G \in \mathscr{G}\},$$

where

$$\sigma_{ij}(G) = \begin{cases} (G - \{j\}) \cup \{i\} & \text{if } i \notin G,\ j \in G,\ (G-\{j\}) \cup \{i\} \notin \mathscr{G}, \\ G & \text{otherwise.} \end{cases}$$

A family $\mathscr{G} \subset 2^{[n]}$ is called *shifted* if $\sigma_{ij}(\mathscr{G}) = \mathscr{G}$ for all $1 \le i < j \le n$, and $\mathscr{G}$ is called *tame* if it is shifted and $\bigcap \mathscr{G} = \emptyset$. If $\mathscr{G}$ is r-wise t-intersecting, then so is $\sigma_{ij}(\mathscr{G})$. Note also that $w_p(\mathscr{G}) = w_p(\sigma_{ij}(\mathscr{G}))$, namely, shifting operations keep the p-weight.

Lemma 11. *Let $\mathscr{G} \subset 2^{[n]}$ be a non-trivial r-wise t-intersecting family with maximal p-weight. Then we can find a tame r-wise t-intersecting family $\mathscr{G}' \subset 2^{[n]}$ with $w_p(\mathscr{G}') = w_p(\mathscr{G})$.*

Proof. If $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ then $\mathscr{G} \in \mathbf{X}^0(n,r-1,t+1)$ (see Lemma 9). We apply all possible shifting operations to $\mathscr{G}$ to get a shifted family $\mathscr{G}' \in \mathbf{X}^0(n,r-1,t+1)$ with the same p-weight.

We have to show that $\bigcap \mathscr{G}' = \emptyset$. Otherwise we may assume that $1 \in \bigcap \mathscr{G}'$ and $H = [2,n] \notin \mathscr{G}'$. Since $\mathscr{G}'$ is p-weight maximal we can find $G_1, \dots, G_{r-1} \in \mathscr{G}'$ such that $|G_1 \cap \cdots \cap G_{r-1} \cap H| < t$. Then we have $|G_1 \cap \cdots \cap G_{r-1}| < t+1$, which is a contradiction. □

To prove Theorems 2, 3 and 6, we will use some basic facts about shadow. For a family $\mathscr{G} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓ-th lower shadow of $\mathscr{G}$, denoted by $\Delta_\ell(\mathscr{G})$, as follows:

$$\Delta_\ell(\mathscr{G}) = \{F \in \tbinom{[n]}{\ell} : F \subset \exists G \in \mathscr{G}\}.$$

Similarly, the ℓ-th upper shadow of $\mathscr{G}$ is defined by $\nabla_\ell(\mathscr{G}) = \{H \in \binom{[n]}{\ell} : H \supset \exists G \in \mathscr{G}\}$. We define the complement family of $\mathscr{G} \subset 2^{[n]}$ by $\mathscr{G}^c := \{[n] - G : G \in \mathscr{G}\}$. We note that $\nabla_\ell(\mathscr{G}) = (\Delta_{n-\ell}(\mathscr{G}^c))^c$ and so $|\nabla_\ell(\mathscr{G})| = |\Delta_{n-\ell}(\mathscr{G}^c)|$.

Lemma 12. *Let $0 < a < b$ and $\emptyset \neq \mathscr{G}_a \subset \binom{[n]}{a}$. Then we have*

$$\frac{|\nabla_b(\mathscr{G}_a)|}{|\mathscr{G}_a|} \ge \frac{\binom{n}{b}}{\binom{n}{a}}.$$

Moreover if $a + b < n$ then we have $|\nabla_b(\mathscr{G}_a)| > |\mathscr{G}_a|$.

Proof. Choose a real $x \le n$ so that $|\mathscr{G}_a| = \binom{x}{n-a}$. By the Kruskal–Katona Theorem[24, 23], we have $|\nabla_b(\mathscr{G}_a)| = |\Delta_{n-b}(\mathscr{G}_a^c)| \ge \binom{x}{n-b}$, and $|\nabla_b(\mathscr{G}_a)|/|\mathscr{G}_a| \ge \binom{x}{n-b}/\binom{x}{n-a} \ge \binom{n}{b}/\binom{n}{a}$, where we used $x \le n$ in the last inequality. If $a+b<n$ then $\binom{n}{b}/\binom{n}{a} > 1$ and the result follows. □

Lemma 13. *Let $\mathscr{A}, \mathscr{B} \subset 2^{[n]}$ be Sperner families, and let $c > 1$ be a real. Suppose that*

$$\mathscr{A} \cap \Delta(\mathscr{B}) = \emptyset, \tag{5}$$

where $\Delta(\mathscr{B}) = \{C : C \subset \exists B \in \mathscr{B}\}$*. Then we have*

$$c|\mathscr{A}| + |\mathscr{B}| \le c\binom{n}{\lceil n/2 \rceil} + \binom{n}{\lceil n/2 \rceil - 1},$$

with equality holding iff $\mathscr{A} = \binom{[n]}{\lceil n/2 \rceil}$ *and* $\mathscr{B} = \binom{[n]}{\lceil n/2 \rceil - 1}$*.*

Proof. First suppose that n is odd and let $n = 2m+1$. Then by the Sperner theorem[28], $\mathscr{A}$ and $\mathscr{B}$ have size at most $\binom{n}{m+1} = \binom{n}{m}$, which gives the desired upper bound. Possible optimal configurations for $\mathscr{A}, \mathscr{B}$ are $\binom{[n]}{m+1}$ and $\binom{[n]}{m}$. Only the case $\mathscr{A} = \binom{[n]}{m+1}$ and $\mathscr{B} = \binom{[n]}{m}$ satisfies (5).

Next suppose that n is even and let $n = 2m$. Set $a_i = |\mathscr{A} \cap \binom{[n]}{i}|$, $b_i = |\mathscr{B} \cap \binom{[n]}{i}|$ and $x_i = ca_i + b_i$. Using the Yamamoto[35] (or LYM) inequality, we have

$$\sum_i \frac{x_i}{\binom{n}{i}} = c\sum_i \frac{a_i}{\binom{n}{i}} + \sum_i \frac{b_i}{\binom{n}{i}} \le c+1,$$

and

$$\sum_{i \ne m} \frac{x_i}{\binom{n}{i}} \le c + 1 - \frac{x_m}{\binom{n}{m}}. \tag{6}$$

By (5) we have $a_m + b_m \le \binom{n}{m}$, and

$$x_m = ca_m + b_m \le c(a_m + b_m) \le c\binom{n}{m}. \tag{7}$$

Consequently we have

$$\begin{aligned} \sum_i x_i &= x_m + \sum_{i \ne m} x_i \le x_m + \binom{n}{m-1} \sum_{i \ne m} \frac{x_i}{\binom{n}{i}} \\ &\le x_m + \binom{n}{m-1}\Big(c + 1 - \frac{x_m}{\binom{n}{m}}\Big) = (c+1)\binom{n}{m-1} + \frac{x_m}{m+1} \\ &\le (c+1)\binom{n}{m-1} + \frac{c}{m+1}\binom{n}{m} = c\binom{n}{m} + \binom{n}{m-1}, \end{aligned}$$

which is the desired inequality. For the equality, we need $ca_m + b_m = c(a_m + b_m) = c\binom{n}{m}$ in (7), which implies $b_m = 0$ and $a_m = \binom{n}{m}$. Since $\sum_i a_i / \binom{n}{i} \le 1$, we have $a_i = 0$ if $i \ne m$, i.e., $\mathscr{A} = \binom{[n]}{m}$. By (5) we have $b_i = 0$ if $i > m$, and $c|\mathscr{A}| + |\mathscr{B}| = c\binom{n}{m} + \binom{n}{m-1}$ implies $|\mathscr{B}| = \sum_{i<m} b_i = \binom{n}{m-1}$. We also need equality in (6), which gives $\sum_{i<m} b_i / \binom{n}{i} = 1$. Consequently we have $\binom{n}{m-1} = \sum_{i<m} b_i \le \binom{n}{m-1} \sum_{i<m} b_i / \binom{n}{i} = \binom{n}{m-1}$, and so $b_{m-1} = \binom{n}{m-1}$, namely $\mathscr{B} = \binom{[n]}{m-1}$. □

3. PROOF OF THEOREM 4

First we show (i). Let $(r,t) \in A$ and let $\mathscr{G} \subset 2^{[n]}$ be a non-trivial r-wise t-intersecting family with maximal p-weight. By Lemma 11 we may assume that $\mathscr{G}$ is tame, namely, it is shifted and $\bigcap \mathscr{G} = \emptyset$. If $\mathscr{G} \in \mathbf{G}_1(n,r,t)$ then there is nothing to prove. Thus we assume that $\mathscr{G} \in \mathbf{X}^1(n,r,t)$ and we will show that there exist $\gamma, \varepsilon > 0$ such that

$$w_p(\mathscr{G}) < (1-\gamma) w_p(\mathscr{G}_1(n,r,t)) \tag{8}$$

holds for all $n \geq r+t$ and p with $|p-1/2| < \varepsilon$. If $\mathscr{G} \in \mathbf{X}^1 - \mathbf{X}^4 = \mathbf{G}_2 \cup \mathbf{G}_3 \cup \mathbf{G}_4$ then (8) follows from Lemmas 2 and 3. Thus we may assume that $\mathscr{G} \in \mathbf{X}^4(n,r,t)$. Let $\tilde{w}^*(n,p,r,t)$ be the maximal p-weight of tame families in $\mathbf{X}^4(n,r,t)$. Then it suffices to show

$$w_p(\mathscr{G}) = \tilde{w}^*(n,p,r,t) < (1-\gamma) w_p(\mathscr{G}_1(n,r,t)). \tag{9}$$

Recall that $w_p(\mathscr{G}_1(n,r,t)) = (t+r)p^{t+r-1}q + p^{t+r}$ and let $\omega := w_{1/2}(\mathscr{G}_1(n,r,t)) = (t+r+1)(1/2)^{t+r}$. The following simple observation is useful.

Claim 1. *Suppose that $w_p(\mathscr{G}) \leq f(p)$ holds for some continuous function $f(p)$, and suppose that $f(1/2) < \omega$. Then there exist $\gamma, \varepsilon > 0$ such that $w_p(\mathscr{G}) < (1-\gamma) w_p(\mathscr{G}_1(n,r,t))$ for all $|p-1/2| < \varepsilon$.*

Let $t^{(i)} = \max\{j : \mathscr{G}$ is i-wise j-intersecting$\}$, and let $s = t^{(r-1)}$. Since $\mathscr{G}$ is p-weight maximal we have $t^{(r)} = t$. Due to $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ we have $t < s$ and

$$w_p(\mathscr{G}) \leq w^*(n,p,r-1,s) \leq w(n,p,r-1,s). \tag{10}$$

After [11] let $h := \min\{i : |G \cap [t+i]| \geq t \text{ for all } G \in \mathscr{G}\}$. This is the maximum size of "holes" in $[t+h]$.

Claim 2. $1 \leq h \leq s-t$.

Proof. Since $\mathscr{G}$ is non-trivial, we have $h \geq 1$. By the definition of s and the shiftedness of $\mathscr{G}$, we have $G_1, \ldots, G_{r-1} \in \mathscr{G}$ such that $G_1 \cap \cdots \cap G_{r-1} = [s]$. Then it follows from $t^{(r)} = t$ that $|[s] \cap G| \geq t$ for all $G \in \mathscr{G}$, namely, $t+h \leq s$. □

Let $b = t+h-1$ and let $T_i = [b+1-i, b]$ be the right-most i-set in $[b]$. For $A \subset [b]$ let

$$\mathscr{G}(A) = \{G \cap [b+1,n] : G \in \mathscr{G},\, [b] \setminus G = A\}.$$

Since $\mathscr{G}$ is shifted, we have $\mathscr{G}(A) \subset \mathscr{G}(T_i)$ for all $A \in \binom{[b]}{i}$, and thus we have

$$w_p(\mathscr{G}) \leq \sum_{i=0}^{h} \binom{b}{i} p^{b-i} q^i \, w_p(\mathscr{G}(T_i) : [b+1,n]). \tag{11}$$

Claim 3. *For $0 \leq i < h$ and $2 \leq j < r$, $\mathscr{G}(T_i)$ is j-wise $(ij + (r-1-j)h + 1)$-intersecting.*

Proof. Suppose that $\mathscr{G}(T_i)$ is not j-wise v-intersecting, where $v = ij + (r-1-j)h + 1$. Then we can find $G_1, \ldots, G_j \in \mathscr{G}(T_i)$ such that $|G_1 \cap \cdots \cap G_j| < v$. Since $\mathscr{G}$ is shifted, we may assume that $G_1 \cap \cdots \cap G_j \subset [b+1, b+v-1]$. By shifting $(G_\ell \cup [b]) - T_i \in \mathscr{G}$, we get $G'_\ell := (G_\ell \cup [b]) - [b+1+(\ell-1)i, b+\ell i] \in \mathscr{G}$ for $1 \leq \ell \leq j$.

By the definition of h we have some $H \in \mathscr{G}$ such that $|H \cap [b]| < t$ and due to the shiftedness of $\mathscr{G}$ we may assume that $H = [n] - [t,b]$. By shifting H, we get $G'_\ell := [n] - [b+ij+1+(\ell-1-j)h, b+ij+(\ell-j)h] \in \mathscr{G}$ for $j < \ell < r$. Then we have $G'_1 \cap \cdots \cap G'_{r-1} \cap H = [t-1]$, which contradicts the r-wise t-intersecting property of $\mathscr{G}$. □

Claim 4. *$\mathscr{G}(T_h)$ is r-wise $((r-1)h+1)$-intersecting, and if $\mathscr{G} \not\subset \mathscr{G}_h(n,r,t)$ then $\mathscr{G}(T_h)$ is $(r-1)$-wise $((r-1)h+2)$-intersecting.*

Proof. First suppose that $\mathscr{G}(T_h)$ is not r-wise v-intersecting, where $v=(r-1)h+1$. Then we can find $G_1,\dots,G_r \in \mathscr{G}(T_h)$ such that $G_1\cap\cdots\cap G_r \subset [b+1,b+v-1]$. By shifting $(G_\ell \cup [b]) - T_h \in \mathscr{G}$ we get $G'_\ell := (G_\ell\cup[b]) - [t+(\ell-1)h, t+\ell h-1] \in \mathscr{G}$ for $1 \le \ell \le r$. Then we have $|G'_1\cap\cdots\cap G'_r| < t$, a contradiction.

Next suppose that $\mathscr{G}(T_h)$ is not $(r-1)$-wise w-intersecting, where $w=(r-1)h+2$. Then we can find $G_1,\dots,G_{r-1} \in \mathscr{G}(T_h)$ such that $G_1\cap\cdots\cap G_{r-1} \subset [b+1,b+w-1]$. By shifting $(G_\ell\cup[b]) - T_h \in \mathscr{G}$ we get $G'_\ell := (G_\ell\cup[b]) - [t+(\ell-1)h, t+\ell h-1] \in \mathscr{G}$ for $1\le \ell < r$. Since $\mathscr{G}\not\subset\mathscr{G}_h(n,r,t)$ we have $G'_r := [n]-[t+(r-1)h, t+rh] \in \mathscr{G}$. Then we have $|G'_1\cap\cdots\cap G'_r| < t$. $\square$

Now we explain the outline of our proof for (9) (cf. Claims 5–9). If s is large then (9) follows from (10). Thus we may assume s is small, actually we will find that we may assume $s \le t+4$. Then we have $1\le h\le 4$ by Claim 2 and we can apply Claim 4 since $\mathscr{G}\in \mathbf{X}^4(n,r,t)$. Using Claims 3 and 4 we define an upper bound function $g^{(i)}(p)$ for $w_p(\mathscr{G}(T_i):[b+1,n])$ by

$$g^{(i)}(p) = \begin{cases} \min\{w(n',p,r-1,t'), w(n',p,r-2,t'')\} & \text{if } 0\le i<h \\ \min\{w(n',p,r,(r-1)h+1), w(n',p,r-1,(r-1)h+2)\} & \text{if } i=h, \end{cases}$$

where $n'=n-b$, $t'=(r-1)i+1$ and $t''=(r-2)i+h+1$. We will find continuous functions $f^{(i)}$ such that $g^{(i)}(p)\le f^{(i)}(p)$ and $\sum_{i=0}^{h}\binom{b}{i}p^{b-i}q^i f^{(i)}(1/2) < \omega$. Then this together with (11) and Claim 1 will give (9). We will apply Claim 1 several times with different $f^{(i)}$, and our $\varepsilon>0$ will be chosen sufficiently small to get through all the cases.

Let $t_r := 2^{r+1}-3r-1$.

Claim 5. *Let $r=5$ and $5\le t\le t_5=48$. Then we have (9).*

Proof. We show that (9) holds if $s\ge t+5$, and then we proceed the casewise analysis for the cases $s\le t+4$, i.e., $1\le h\le 4$.

First suppose that $s=t^{(4)}\le 7$. Since $s>t$ we have $t\le 6$. By (10) and Lemma 5 it follows $w_p(\mathscr{G})\le w(n,p,4,s)=p^s$. To apply Claim 1 as $f(p)=p^s$, we note that $(1/2)^s<\omega$ holds iff $2^{t-s+5}<t+6$. This is true if $t\le 6$ and $s\ge t+3$, and we are done in this case. Thus for the case $t\le 6$ we may assume that $s\le t+2$, i.e., $1\le h\le 2$ by Claim 2.

Next suppose that $s\ge 8$. By (10) and Lemma 6 we have $w_p(\mathscr{G})\le w(n,p,4,s)\le p^7\alpha_{4,p}^{s-7}$. If $s\ge t+5$ then the RHS is less than ω at $p=1/2$ for $1\le t\le 50$. Thus we may assume that $s\le t+4$ and so $1\le h\le 4$ by Claim 2.

Case 5-1. $h=1$. We find that $\mathscr{G}(T_0)$ is $(r-2)$-wise 2-intersecting by Claim 3, and $\mathscr{G}(T_1)$ is $(r-1)$-wise $(r+1)$-intersecting by Claim 4. Then $w_p(\mathscr{G}(T_0):[b+1,n])\le p^2$ and $w_p(\mathscr{G}(T_1):[b+1,n])\le p^{r+1}$ follow from Lemma 5. Thus using (11) we have

$$w_p(\mathscr{G}) \le p^t\cdot p^2 + t p^{t-1} q\cdot p^{r+1}, \tag{12}$$

and the RHS is less than ω at $p=1/2$ for $t>2^{r-1}-2r-2$. Then Claim 1 gives (9).

Case 5-2. $h=2$. Since $\mathscr{G}(T_0)$ is 3-wise 3-intersecting, $\mathscr{G}(T_1)$ is 4-wise 5-intersecting, and $\mathscr{G}(T_2)$ is 4-wise 10-intersecting, we have

$$w_p(\mathscr{G}) \leq p^{t+1}\cdot p^2\alpha_{3,p} + (t+1)p^t q\cdot p^5 + \binom{t+1}{2}p^{t-1}q^2\cdot p^7\alpha_{4,p}^3,$$

and the RHS is less than ω at $p=1/2$ for $1\leq t\leq 54$.

Case 5-3. $h=3$. Since $\mathscr{G}(T_0)$ is 3-wise 4-intersecting, $\mathscr{G}(T_1)$ is 3-wise 7-intersecting, $\mathscr{G}(T_2)$ is 4-wise 9-intersecting, and $\mathscr{G}(T_3)$ is 5-wise 13-intersecting, we have

$$w_p(\mathscr{G}) \leq p^{t+2}\cdot p^2\alpha_{3,p}^2 + (t+2)p^{t+1}q\cdot p^2\alpha_{3,p}^5 + \binom{t+2}{2}p^t q^2\cdot p^7\alpha_{4,p}^2 + \binom{t+2}{3}p^{t-1}q^3\cdot p^{13},$$

and the RHS is less than ω at $p=1/2$ for $1\leq t\leq 49$.

Case 5-4. $h=4$. Since $\mathscr{G}(T_0)$ is 3-wise 5-intersecting, $\mathscr{G}(T_1)$ is 3-wise 8-intersecting, $\mathscr{G}(T_2)$ is 4-wise 9-intersecting, $\mathscr{G}(T_3)$ is 4-wise 13-intersecting, and $\mathscr{G}(T_4)$ is 5-wise 17-intersecting, we have

$$\begin{aligned} w_p(\mathscr{G}) \;\leq\; & p^{t+3}\cdot p^2\alpha_{3,p}^3 + (t+3)p^{t+2}q\cdot p^2\alpha_{3,p}^6 + \binom{t+3}{2}p^{t+1}q^2\cdot p^7\alpha_{4,p}^2 \\ & + \binom{t+3}{3}p^t q^3\cdot p^7\alpha_{4,p}^6 + \binom{t+3}{4}p^{t-1}q^4\cdot p^{17}, \end{aligned}$$

and the RHS is less than ω at $p=1/2$ for $1\leq t\leq 57$. □

We note that similarly to Lemma 9 we have

$$\tilde{w}^*(n,p,r,t) \leq \tilde{w}^*(n,p,r-1,t+1). \tag{13}$$

Claim 6. *Let $r=6$ and $4\leq t\leq t_6=109$. Then we have (9).*

Proof. If $5\leq t+1\leq t_5=48$ then using (13) with Claim 5 we have

$$\tilde{w}^*(n,p,6,t) \leq \tilde{w}^*(n,p,5,t+1) < (1-\gamma)w_p(\mathscr{G}_1(n,5,t+1)) = (1-\gamma)w_p(\mathscr{G}_1(n,6,t)).$$

Thus we may assume that $s\geq t+1\geq 49$. By (10) and Lemma 4 with Claim 5 we have

$$w_p(\mathscr{G}) \leq w(n,p,5,s) \leq w_p(\mathscr{G}_1(n,5,48))\alpha_{5,p}^{s-48}.$$

If $s\geq t+4$ then the RHS is less than ω at $p=1/2$ for $t\leq 124$. Thus we may assume that $s\leq t+3$ and $1\leq h\leq 3$.

Case 6-1. $h=1$. Same as Case 5-1. (We need (12) for $t\geq t_5$. This is true in general for $r\geq 6$. In fact we have (12) for $t>2^{r-1}-2r-2$ and $t_{r-1}>2^{r-1}-2r-2$.)

Case 6-2. $h=2$. Since $\mathscr{G}(T_0)$ is $(r-2)$-wise 3-intersecting, $\mathscr{G}(T_1)$ is $(r-2)$-wise $(r+1)$-intersecting, and $\mathscr{G}(T_2)$ is $(r-1)$-wise $(2r)$-intersecting, we have

$$w_p(\mathscr{G}) \leq p^{t+1}\cdot p^3 + (t+1)p^t q\cdot p^{r+1} + \binom{t+1}{2}p^{t-1}q^2\cdot p^{2r}, \tag{14}$$

and the RHS is less than ω at $p=1/2$ for $t_{r-1}\leq t\leq 2^{r+1}$.

Case 6-3. $h=3$. Since $\mathscr{G}(T_0)$ is $(r-2)$-wise 4-intersecting, $\mathscr{G}(T_1)$ is $(r-2)$-wise $(r+2)$-intersecting, $\mathscr{G}(T_2)$ is $(r-2)$-wise $(2r)$-intersecting, and $\mathscr{G}(T_3)$ is $(r-1)$-wise $(3r-1)$-intersecting, we have

$$w_p(\mathscr{G}) \leq p^{t+2}\cdot p^4 + (t+2)p^{t+1}q\cdot p^7\alpha_{4,p} + \binom{t+2}{2}p^t q^2\cdot p^7\alpha_{4,p}^5 + \binom{t+2}{3}p^{t-1}q^3\cdot p^{17}, \tag{15}$$

and the RHS is less than ω at $p=1/2$ for $t_{r-1}\leq t\leq 2^{r+1}$. □

Claim 7. *Let $r=7$ and $2\le t\le t_7=234$. Then we have (9).*

Proof. The case $t=2$ was proved in [33]. Using (13) with Claim 6 we have (9) for $4\le t+1\le 109$. Thus we may assume that $s\ge t+1\ge 110$, and we have

$$w_p(\mathscr{G})\le w(n,p,6,s)\le w_p(\mathscr{G}_1(n,6,109))\alpha_{6,p}^{s-109}.$$

If $s\ge t+4$ then the RHS is less than ω at $p=1/2$ for $t\le 278$. Thus we may assume that $s\le t+3$ and $1\le h\le 3$. Then we repeat the casewise check as in Claim 6. In this case we can replace (15) with the following:

$$w_p(\mathscr{G})\le p^{t+2}\cdot p^4+(t+2)p^{t+1}q\cdot p^{r+2}+\binom{t+2}{2}p^tq^2\cdot p^{2r}+\binom{t+2}{3}p^{t-1}q^3\cdot p^{3r-1}.$$

□

Similarly we can prove the following.

Claim 8. *Let $r=8$ and $1\le t\le t_8=487$. Then we have (9).*

Finally we are ready to prove the general case $r\ge 9$.

Claim 9. *Let $r\ge 9$ and $1\le t\le t_r$. Then we have (9).*

Proof. We prove the result by induction on r. We have (9) for $1\le t+1\le t_{r-1}$ using (13) with our induction hypothesis for $r-1$. Thus we may assume that $s\ge t+1>t_{r-1}$, and we have

$$w_p(\mathscr{G})\le w(n,p,r-1,s)\le w_p(\mathscr{G}_1(n,r-1,t_{r-1}))\alpha_{r-1,p}^{s-t_{r-1}}.$$

If $s\ge t+3$ then the RHS is less than ω at $p=1/2$ for $t_{r-1}\le t\le t_r$ by Lemma 8. Thus we may assume that $s\le t+2$ and $1\le h\le 2$.

Case 9-1. $h=1$. Same as Case 5-1.

Case 9-2. $h=2$. We use the same estimation as in Case 6-2. Then the RHS of (14) is less than ω at $p=1/2$ iff

$$(a-b)/2\le t\le (a+b)/2, \tag{16}$$

where $a=3\cdot 2^r-1$, $b=\sqrt{1+2^{2r+3}+(8r+3)2^{r+1}}$. Since $t_{r-1}\le t\le t_r$, we have (16). □

This completes the proof of (i) of the theorem. Moreover we have proved the inequality (8) if $\mathscr{G}$ is tame and $\mathscr{G}\in\mathbf{X}^1(n,r,t)$.

Next we show (ii). We include the proof of this part from [33] for self-completeness. Set $\mathscr{G}_1=\mathscr{G}_1(n,r,t)$. Let $\mathscr{G}\subset 2^{[n]}$ be a (not necessarily shifted) non-trivial r-wise t-intersecting family, and suppose that $\mathscr{G}\in\mathbf{X}^1(n,r,t)$. By Lemma 11 we can find a tame r-wise t-intersecting family $\mathscr{G}^*$ with $w_p(\mathscr{G}^*)=w_p(\mathscr{G})$. If $\mathscr{G}^*\not\subset\mathscr{G}_1$ then we have already shown that $w_p(\mathscr{G}^*)<(1-\gamma)w_p(\mathscr{G}_1)$. Thus we may assume that $\mathscr{G}^*\subset\mathscr{G}_1$, and in particular (by renaming the starting family if necessary) we may assume that $\mathscr{G}^*=\sigma_{xy}(\mathscr{G})\subset\mathscr{G}_1$, where $x=t+r$, $y=x+1$. We note that $|[x]\cap G|\ge x-2$ for all $G\in\mathscr{G}$. Moreover if $|[x]\cap G|=x-2$ then $G\cap\{x,y\}=\{y\}$ and $(G-\{y\})\cup\{x\}\notin\mathscr{G}$.

For $i\in[x]$ set $\mathscr{G}(i)=\{G\in\mathscr{G}:[y]\setminus G=\{i\}\}$, and for $j\in[x-1]$ and $z\in\{x,y\}$ let $\mathscr{G}_z(j)=\{G\in\mathscr{G}:[y]\setminus G=\{j,z\}\}$. Since $\sigma_{xy}(\mathscr{G})\subset\mathscr{G}_1$ we have $\mathscr{G}_x(j)\cap\mathscr{G}_y(j)=\emptyset$ and so

$w_p(\mathscr{G}_x(j))+w_p(\mathscr{G}_y(j)) \le p^{x-1}q^2$. Set $\mathscr{G}(\emptyset)=\{G\in\mathscr{G} : [x]\subset G\}$, $\mathscr{G}_{xy}=\{G\in\mathscr{G} : G\cap[y]=[x-1]\}$ and let $e=\min_{i\in[x]} w_p(\mathscr{G}(i))$. Then we have

$$w_p(\mathscr{G}) = \sum_{i\in[x]} w_p(\mathscr{G}(i))+\sum_{j\in[x-1]}\big(w_p(\mathscr{G}_x(j))+w_p(\mathscr{G}_y(j))\big)+w_p(\mathscr{G}(\emptyset))+w_p(\mathscr{G}_{xy}) \quad (17)$$

$$\le e+(x-1)p^xq+(x-1)p^{x-1}q^2+p^x+p^{x-1}q^2 = e+(\eta-1)p^xq, \quad (18)$$

where $\eta=\frac{x}{p}+\frac{1}{q}$. Note that $e\le p^xq$, and (18) coincides $w_p(\mathscr{G}_1)=\eta p^xq$ iff $e=p^xq$. If there is some $j\in[x-1]$ such that $\mathscr{G}_x(j)\cup\mathscr{G}_y(j)=\emptyset$, then by (17) we get $w_p(\mathscr{G})\le w_p(\mathscr{G}_1)-p^{x-1}q^2=\big(1-q/(\eta p)\big)w_p(\mathscr{G}_1)$, and we are done. Thus we may assume that

$$\mathscr{G}_x(j)\cup\mathscr{G}_y(j)\ne\emptyset \text{ for all } j\in[x-1]. \quad (19)$$

To prove $w_p(\mathscr{G})<(1-\gamma)w_p(\mathscr{G}_1)$ by contradiction, let us assume that for any $\gamma>0$ and any n_0 there is some $n>n_0$ such that

$$w_p(\mathscr{G})>(1-\gamma)w_p(\mathscr{G}_1)=(1-\gamma)\eta p^xq. \quad (20)$$

By (18) and (20) we have $e>(1-\gamma\eta)p^xq$. This means, letting $\mathscr{H}(i)=\{G\setminus[y] : G\in\mathscr{G}(i)\}$ and $Y=[y+1,n]$,

$$w_p(\mathscr{H}(i) : Y) \text{ only misses at most } \gamma\eta\ p\text{-weight for all } i\in[x]. \quad (21)$$

Since $\mathscr{G}\in\mathbf{X}^1(n,r,t)$ both $\bigcup_{j\in[x-1]}\mathscr{G}_x(j)$ and $\bigcup_{j\in[x-1]}\mathscr{G}_y(j)$ are non-empty. Using this with (19), we can choose $G\in\mathscr{G}_x(j)$ and $G'\in\mathscr{G}_y(j')$ with $j\ne j'$, say, $j=x-1$, $j'=x-2$. Let $L=[r-2]$ and $\mathscr{H}^*=\bigcap_{\ell\in L}\mathscr{H}(\ell)$. Then by (21) we have

$$w_p(\mathscr{H}^* : Y)>1-(r-2)\gamma\eta. \quad (22)$$

If $\mathscr{H}^*\subset 2^Y$ is not $(r-2)$-wise 1-intersecting, then we can find $H_\ell\in\mathscr{H}^*$ for $\ell\in L$ so that $H_1\cap\cdots\cap H_{r-2}=\emptyset$. Setting $G_\ell:=([y]-\{\ell\})\cup H_\ell\in\mathscr{G}$ we have $|G_1\cap\cdots\cap G_{r-2}\cap G\cap G'|=t-1$, which contradicts the r-wise t-intersecting property of $\mathscr{G}$. Thus $\mathscr{H}^*$ is $(r-2)$-wise 1-intersecting and $w_p(\mathscr{H}^* : Y)\le p$ by (2). But this contradicts (22) because we can choose γ so small that $p\ll 1-(r-2)\gamma\eta$. □

4. APPLICATION

4.1. **Proof of Theorem 2.** We deduce (ii) from Theorem 4, then (i) follows from (ii). We include the proof of this part from [33] for self-completeness. Assuming the negation of Theorem 2 for some fixed $(r,t)\in A$, we will construct a counterexample to Theorem 4 (ii).

For reals $0<b<a$ we write $a\pm b$ to mean the open interval $(a-b,a+b)$ and $n(a\pm b)$ means $((a-b)n,(a+b)n)\cap\mathbb{N}$. Fix $\gamma_0:=\gamma_{\text{Thm4}}$ and $\varepsilon_0:=\varepsilon_{\text{Thm4}}$ from Theorem 4. For fixed r and t we note that $f(p):=w^*(n,p,r,t)=(t+r)p^{t+r-1}q+p^{t+r}$ is a uniformly continuous function of p on $\frac{1}{2}\pm\varepsilon_0$. Let $\gamma=\frac{\gamma_0}{4}$, $\varepsilon=\frac{\varepsilon_0}{2}$, and $I=\frac{1}{2}\pm\varepsilon$.

Choose $\varepsilon_1\ll\varepsilon$ so that

$$(1-3\gamma)f(p)>(1-4\gamma)f(p+\delta) \quad (23)$$

holds for all $p \in I$ and all $0 < \delta \le \varepsilon_1$. Choose n_1 so that

$$\sum_{i \in J} \binom{n}{i} p_0^i (1-p_0)^{n-i} > (1-3\gamma)/(1-2\gamma) \tag{24}$$

holds for all $n > n_1$ and all $p_0 \in I_0 := \frac{1}{2} \pm \frac{3\varepsilon}{2}$, where $J = n(p_0 \pm \varepsilon_1)$. Choose n_2 so that

$$(1-\gamma)|\mathscr{F}_1(n,k,r,t)| > (1-2\gamma) f(k/n) \binom{n}{k} \tag{25}$$

holds for all $n > n_2$ and k with $k/n \in I$. Finally set $n_0 = \max\{n_1, n_2\}$.

Suppose that Theorem 2 fails. Then for our choice of γ, ε and n_0, we can find some n, k and $\mathscr{F} \in \mathbf{Y}^1(n,k,r,t)$ with $|\mathscr{F}| \ge (1-\gamma)|\mathscr{F}_1(n,k,r,t)|$, where $n > n_0$ and $\frac{k}{n} \in I$. We fix n, k and $\mathscr{F}$, and let $p = \frac{k}{n}$. By (25) we have $|\mathscr{F}| > c\binom{n}{k}$, where $c = (1-2\gamma) f(p)$. Let $\mathscr{G} = \bigcup_{k \le i \le n} (\nabla_i(\mathscr{F}))$ be the collection of all upper shadows of $\mathscr{F}$, which belongs to $\mathbf{X}^1(n,r,t)$. Let $p_0 = p + \varepsilon_1 \in I_0$.

Claim 10. $|\nabla_i(\mathscr{F})| \ge c\binom{n}{i}$ *for* $i \in J$.

Proof. Choose a real $x \le n$ so that $c\binom{n}{k} = \binom{x}{n-k}$. Since $|\mathscr{F}| > c\binom{n}{k} = \binom{x}{n-k}$ the Kruskal–Katona Theorem implies that $|\nabla_i(\mathscr{F})| \ge \binom{x}{n-i}$. Thus it suffices to show that $\binom{x}{n-i} \ge c\binom{n}{i}$, or equivalently,

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}} \ge \frac{c\binom{n}{i}}{c\binom{n}{k}}. \tag{26}$$

Since $i \in J$ we have $i > n(p_0 - \varepsilon_1) = np = k$, and (26) is equivalent to $i \cdots (k+1) \ge (x-n+i) \cdots (x-n+k+1)$, which follows from $x \le n$. □

By the claim we have

$$w_{p_0}(\mathscr{G}) \ge \sum_{i \in J} |\nabla_i(\mathscr{F})|\, p_0^i (1-p_0)^{n-i} \ge c \sum_{i \in J} \binom{n}{i} p_0^i (1-p_0)^{n-i}. \tag{27}$$

Using (24) and (23), the RHS of (27) is more than

$$c\,(1-3\gamma)/(1-2\gamma) = (1-3\gamma) f(p) > (1-4\gamma) f(p+\varepsilon_1) = (1-\gamma_0) f(p_0).$$

This means $w_{p_0}(\mathscr{G}) > (1-\gamma_0)\, w^*(n, p_0, r, t)$, which contradicts Theorem 4 (ii) because $p_0 \in I_0 = \frac{1}{2} \pm \frac{3\varepsilon}{2} = \frac{1}{2} \pm \frac{3\varepsilon_0}{4} \subset \frac{1}{2} \pm \varepsilon_0$. □

4.2. **Proof of Theorem 3.** For the cases $t = 1, 2$, it follows from [18, 12] that $s(n,r,t) \le s(n,4,t) \le |\mathscr{F}_0(n,k_0,r,t)|$ with the only optimal family $\mathscr{F}_0(n,k_0,r,t)$. So we may assume that $t \ge 3$, though our proof will be valid for all $(r,t) \in A$. We are going to prove

$$s(n,r,t) = \max\{|\mathscr{F}_0(n,k_0,r,t)|, |\mathscr{F}_1(n,k_1,r,t)|\}.$$

Let $\mathscr{G} \subset 2^{[n]}$ be an r-wise t-intersecting Sperner family with maximal size. If $|\bigcap \mathscr{G}| \ge t$, say $[t] \subset \bigcap \mathscr{G}$, then $\mathscr{G}' = \{G - [t] : [t] \subset G \in \mathscr{G}\}$ is Sperner, and by the Sperner theorem we have $|\mathscr{G}| = |\mathscr{G}'| \le \binom{n-t}{\lceil (n-t)/2 \rceil} = |\mathscr{F}_0|$ with equality holding iff $\mathscr{G}' \cong \binom{[t+1,n]}{\lceil (n-t)/2 \rceil}$ or $\binom{[t+1,n]}{\lfloor (n-t)/2 \rfloor}$, that is, $\mathscr{G} \cong \mathscr{F}_0(n,k_0,r,t)$.

So we assume that $|\bigcap \mathscr{G}| < t$. Let

$$u(\mathscr{G}) = \max\{i : |G \cap [i+1]| \ge i \text{ for all } G \in \mathscr{G}\}.$$

For a permutation τ on $[n]$ let $\tau(\mathscr{G}) = \{\tau(G) : G \in \mathscr{G}\}$, and define $\tilde{u}(\mathscr{G}) = \max_\tau u(\tau(\mathscr{G}))$, where the max is taken over all possible vertex permutations. We further assume that this max is attained when τ is the identity, that is, $\tilde{u}(\mathscr{G}) = u(\mathscr{G})$. Set $x = t + r$.

First suppose that $\tilde{u}(\mathscr{G}) \geq x-1$, i.e., $|G \cap [x]| \geq x-1$ for all $G \in \mathscr{G}$. For $i \in [x]$ let $\mathscr{G}(i) = \{G \cap [x+1,n] : i \notin G \in \mathscr{G}\}$, and let $\mathscr{G}(\emptyset) = \{G \cap [x+1,n] : [x] \subset G \in \mathscr{G}\}$. Choose i_0 such that $|\mathscr{G}(i_0)| = \max_i |\mathscr{G}(i)|$. Then we have $|\mathscr{G}| \leq x|\mathscr{G}(i_0)| + |\mathscr{G}(\emptyset)|$. Set $\mathscr{A} = \mathscr{G}(i_0)$, $\mathscr{B} = \mathscr{G}(\emptyset)$, where both $\mathscr{A}$ and $\mathscr{B}$ are Sperner in $2^{[x+1,n]}$. Moreover we have $\mathscr{A} \cap \Delta(\mathscr{B}) = \emptyset$. Thus by Lemma 13 we have

$$|\mathscr{G}| \leq x|\mathscr{A}| + |\mathscr{B}| \leq x\binom{n-x}{\lceil \frac{n-x}{2} \rceil} + \binom{n-x}{\lceil \frac{n-x}{2} \rceil - 1} = |\mathscr{F}_1(n,k_1,r,t)|,$$

with equality holding iff $\mathscr{G} \cong \mathscr{F}_1(n,k_1,r,t)$. This completes the proof for the case $\tilde{u}(\mathscr{G}) \geq x-1$.

From now on we assume that $\tilde{u}(\mathscr{G}) < x-1$. We will show that

$$|\mathscr{G}| < (1 - \tfrac{\xi}{2}) \max\{|\mathscr{F}_0|, |\mathscr{F}_1|\}$$

for some $\xi > 0$. Let $\mathscr{G}_\ell = \mathscr{G} \cap \binom{[n]}{\ell}$ and $L = \{\ell : \mathscr{G}_\ell \neq \emptyset\}$.

Claim 11. $L \subset [\lfloor \frac{n}{2} \rfloor, n]$.

Proof. Let a and b be the least and second least element of L respectively, and let $\mathscr{H} = (\mathscr{G} - \mathscr{G}_a) \cup \nabla_b(\mathscr{G}_a)$. Then $\mathscr{H}$ is r-wise t-intersecting Sperner. If $a+b < n$ then we have $|\nabla_b(\mathscr{G}_a)| > |\mathscr{G}_a|$ by Lemma 12, which means $|\mathscr{H}| > |\mathscr{G}|$. Thus we may assume $|L \cap [0, \lfloor \frac{n}{2} \rfloor - 1]| \leq 1$. If this number is one, then we repeat the same exchange operation for $a = \min L$ and $b = \lfloor \frac{n}{2} \rfloor$. Consequently $L \subset [\lfloor \frac{n}{2} \rfloor, n]$ follows from the maximality of $\mathscr{G}$. □

Choose $\varepsilon > 0$ from Theorem 2 and set $a = \min\big(L \cap [\lfloor \frac{n}{2} \rfloor, (\frac{1}{2} + \varepsilon)n)\big)$. We choose a vertex permutation ρ so that $\tilde{u}(\mathscr{G}_a) = u(\rho(\mathscr{G}_a))$. Since $\tilde{u}(\mathscr{G}) < x-1$ we still have $u(\rho(\mathscr{G})) < x-1$. We rearrange the vertex set so that ρ is the identity. For a real $p \in (0,1)$, let $f_1(p) = p^t$, $f_2(p) = xp^{x-1}(1-p) + p^x$ and $f(p) = \max\{f_1(p), f_2(p)\}$. We note that

$$\max\{|\mathscr{F}_0(n,k_0,r,t)|, |\mathscr{F}_1(n,k_1,r,t)|\} = (f(\tfrac{1}{2}) + o(1))\binom{n}{\lfloor n/2 \rfloor}. \tag{28}$$

Claim 12. *There exists $\xi > 0$ such that $|\mathscr{G}_a| < (1-2\xi) f(\frac{a}{n})\binom{n}{a}$.*

Proof. First suppose that $\mathscr{G}_a$ is trivial and $[t] \subset G$ for all $G \in \mathscr{G}_a$. Since $\mathscr{G}$ is non-trivial we can find $H \in \mathscr{G}$ such that $|[t] \cap H| < t$. Thus $\mathscr{G}'_a := \{G - [t] : G \in \mathscr{G}_a\}$ is $(r-1)$-wise 1-intersecting and

$$\begin{aligned} |\mathscr{G}_a| &= |\mathscr{G}'_a| \leq m(n-t, a-t, r-1, 1) = \tbinom{n-t-1}{a-t-1} \\ &= ((a/n)^{t+1} + o(1))\tbinom{n}{a} < (1-\gamma_1) f_1(a/n)\tbinom{n}{a}. \end{aligned}$$

Next suppose that $\mathscr{G}_a$ is non-trivial, i.e., $|\bigcap \mathscr{G}_a| < t$. If $\tilde{u}(\mathscr{G}_a) < x-1$, namely, if $\mathscr{G}_a \in \mathbf{Y}^1(n,a,r,t)$, then $|\mathscr{G}_a| < (1-\gamma_2) f_2(a/n)\binom{n}{a}$ follows from Theorem 2. Thus we may assume that $\tilde{u}(\mathscr{G}_a) = u(\mathscr{G}_a) \geq x-1$.

Let $\mathscr{G}_a(i) = \{G \cap [x+1,n] : i \notin G \in \mathscr{G}_a\}$ and $\mathscr{G}_a(\emptyset) = \{G - [x] : [x] \subset G \in \mathscr{G}_a\}$. Set $e = \min_{i\in[x]} |\mathscr{G}_a(i)|$. Since $|\mathscr{G}_a| = \sum_{i=1}^{x} |\mathscr{G}_a(i)| + |\mathscr{G}_a(\emptyset)|$ we have

$$|\mathscr{G}_a| \le e + (x-1)\binom{n-x}{a-x+1} + \binom{n-x}{a-x}. \tag{29}$$

Suppose that $|\mathscr{G}_a| > (1-\gamma_3) f_2(a/n)\binom{n}{a} = (1-\gamma_3)(1+o(1))(x\binom{n-x}{a-x+1} + \binom{n-x}{a-x})$ holds for any $\gamma_3 > 0$. Then by (29) we have $e > (1-\gamma_3(x+2))\binom{n-x}{a-x+1}$. This means $\mathscr{G}_a(i)$ only misses at most $\gamma_3(x+2)$ portion of $\binom{[x+1,n]}{a-x+1}$ for all $i \in [x]$. Since $u(\mathscr{G}) < x-1$ we can find some $G \in \mathscr{G} - \mathscr{G}_a$ such that $|G \cap [x]| \le x-2$, say, $G \not\ni x-1, x$. Let $\mathscr{G}_a^* = \bigcap_{i=1}^{r-1} \mathscr{G}_a(i)$. Then we have

$$|\mathscr{G}_a^*| > \bigl(1-(r-1)\gamma_3(x+2)\bigr)\binom{n-x}{a-x+1}. \tag{30}$$

If $\mathscr{G}_a^* \subset \binom{[x+1,n]}{a-x+1}$ is not $(r-1)$-wise 1-intersecting, then we can find $G_i^* \in \mathscr{G}_a^*$ for $i \in [r-1]$ so that $G_1^* \cap \cdots \cap G_{r-1}^* = \emptyset$. Setting $G_i := ([x] - \{i\}) \cup G_i^* \in \mathscr{G}$ we have $|G_1 \cap \cdots \cap G_{r-1} \cap G| = t-1$, which contradicts the r-wise t-intersecting property of $\mathscr{G}$. Thus $\mathscr{G}_a^*$ is $(r-1)$-wise 1-intersecting and $|\mathscr{G}_a^*| \le \binom{n-x-1}{a-x}$, which contradicts (30) because we can choose $\gamma_3 > 0$ arbitrarily small. Therefore there is some $\gamma_3 > 0$ such that $|\mathscr{G}_a| < (1-\gamma_3) f_2(a/n)\binom{n}{a}$.

Finally we get the claim by setting $\xi = (1/2)\max\{\gamma_1, \gamma_2, \gamma_3\}$. □

Since f is continuous, we can chose a constant μ, $0 < \mu \ll \varepsilon$, so that

$$(1-2\xi) f(\tfrac{1}{2}+\mu) < (1-\xi) f(\tfrac{1}{2}).$$

Set $M = M(\mathscr{G}) = \{k \in [\lfloor \frac{n}{2} \rfloor, (\frac{1}{2}+\mu)n) : \mathscr{G}_k \neq \emptyset\}$.

Claim 13. $\sum_{k\in M} |\mathscr{G}_k|/\binom{n}{k} < (1-\xi) f(\frac{1}{2})$.

Proof. It will be shown by induction on $m = |M|$. The case $M = \{k\}$ follows from Claim 12; in fact noting that f is increasing on $[\frac{1}{2}, \frac{1}{2}+\mu]$ we have

$$|\mathscr{G}_k|/\binom{n}{k} < (1-2\xi) f(\tfrac{k}{n}) < (1-2\xi) f(\tfrac{1}{2}+\mu) < (1-\xi) f(\tfrac{1}{2}).$$

Next we assume that our claim holds for $m-1$. Let a and b be the least and second least element of M, and let $\mathscr{H} = (\mathscr{G} - \mathscr{G}_a) \cup \nabla_b(\mathscr{G}_a)$. Then $\mathscr{H}$ is r-wise t-intersecting Sperner and $M(\mathscr{H}) = M(\mathscr{G}) - \{a\}$. By Lemma 12, we have $|\mathscr{G}_a|/\binom{n}{a} \le |\nabla_b(\mathscr{G}_a)|/\binom{n}{b}$, which means

$$\sum_{k\in M(\mathscr{G})} \frac{|\mathscr{G}_k|}{\binom{n}{k}} \le \sum_{k\in M(\mathscr{H})} \frac{|\mathscr{H}_k|}{\binom{n}{k}},$$

and the RHS is less than $(1-\xi) f(\frac{1}{2})$ by the induction hypothesis. □

By Claim 13 we have

$$(1-\xi) f(\tfrac{1}{2}) > \sum_{k\in M} \frac{|\mathscr{G}_k|}{\binom{n}{k}} \ge \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k\in M} |\mathscr{G}_k|.$$

On the other hand, by the Yamamoto inequality, we have

$$1 \ge \sum_{k\in L-M} \frac{|\mathscr{G}_k|}{\binom{n}{k}} \ge \frac{1}{\binom{n}{(\frac{1}{2}+\mu)n}} \sum_{k\in L-M} |\mathscr{G}_k|,$$

where we used $L-M\subset[(\frac{1}{2}+\mu)n,n]$ by Claim 11. Consequently we have

$$|\mathscr{G}|=\textstyle\sum_{k\in L}|\mathscr{G}_k|<(1-\xi)f(\frac{1}{2})\binom{n}{\lfloor n/2\rfloor}+\binom{n}{(\frac{1}{2}+\mu)n}<(1-\frac{\xi}{2})f(\frac{1}{2})\binom{n}{\lfloor n/2\rfloor},$$

and the RHS is less than $\max\{|\mathscr{F}_0|,|\mathscr{F}_1|\}$ by (28). □

4.3. **Proof of Theorem 6.** Let r and t be fixed. Assuming the negation of Theorem 6, we will construct a counterexample to (i) of Theorem 4. Fix $\varepsilon:=\varepsilon_{\rm Thm4}$ from Theorem 4 and let $p_0=\frac{1}{2}-\frac{\varepsilon}{2}$. Since $p_0<\frac{1}{2}$ and $t\le 2^r-r-1$ we have $w_{p_0}(\mathscr{G}_0(n,r,t))=p_0^t>w_{p_0}(\mathscr{G}_1(n,r,t))$ by Lemma 1. Thus we can choose $\gamma>0$ so that

$$(1-2\gamma)p_0^t>w_{p_0}(\mathscr{G}_1(n,r,t)). \tag{31}$$

Then choose n_0 so that

$$\sum_{i\in J}\binom{n-t}{i-t}p_0^i(1-p_0)^{n-i}>p_0^t(1-2\gamma)/(1-\gamma) \tag{32}$$

holds for all $n>n_0$, where $J=((p_0-\frac{\varepsilon}{2})n,(p_0+\frac{\varepsilon}{2})n)\cap\mathbb{N}$.

Suppose that Theorem 6 fails. Then for our choice of ε,γ and n_0, we can find some n,k and $\mathscr{F}\in\mathbf{Y}^0(n,k,r,t)$ with $|\mathscr{F}|\ge(1-\gamma)\binom{n-t}{k-t}$, where $n>n_0$ and $\frac{k}{n}<\frac{1}{2}-\varepsilon=p_0-\frac{\varepsilon}{2}$. We fix n,k and $\mathscr{F}$. Let $\mathscr{G}=\bigcup_{k\le i\le n}(\nabla_i(\mathscr{F}))$ be the collection of all upper shadows of $\mathscr{F}$, which is non-trivial r-wise t-intersecting, i.e., $\mathscr{G}\in\mathbf{X}^0(n,r,t)$.

Claim 14. *$|\nabla_i(\mathscr{F})|\ge(1-\gamma)\binom{n-t}{i-t}$ for $i\in J$.*

Proof. Choose a real $x\le n-t$ so that $(1-\gamma)\binom{n-t}{k-t}=\binom{x}{n-k}$. Since $|\mathscr{F}|\ge\binom{x}{n-k}$ the Kruskal–Katona Theorem implies that $|\nabla_i(\mathscr{F})|\ge\binom{x}{n-i}$. Thus it suffices to show that $\binom{x}{n-i}\ge(1-\gamma)\binom{n-t}{i-t}$, or equivalently,

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}}\ge\frac{(1-\gamma)\binom{n-t}{i-t}}{(1-\gamma)\binom{n-t}{k-t}}. \tag{33}$$

Since $i\in J$ we have $i>(p_0-\frac{\varepsilon}{2})n>k$, and (33) is equivalent to $(i-t)\cdots(k-t+1)\ge(x-n+i)\cdots(x-n+k+1)$, which follows from $x\le n-t$. □

By the claim we have

$$w_{p_0}(\mathscr{G})\ge\sum_{i\in J}|\nabla_i(\mathscr{F})|\,p_0^i(1-p_0)^{n-i}\ge(1-\gamma)\sum_{i\in J}\binom{n-t}{i-t}p_0^i(1-p_0)^{n-i}. \tag{34}$$

By (32) and (31), the RHS of (34) is more than $(1-\gamma)\cdot p_0^t(1-2\gamma)/(1-\gamma)=p_0^t(1-2\gamma)>w_{p_0}(\mathscr{G}_1(n,r,t))$, which contradicts Theorem 4 (i). □

4.4. **Proof of Theorem 7.** Let $\varepsilon>0$ and $p<\frac{1}{2}-\varepsilon$ be given. By Theorem 6 we can find $0<\gamma\ll 1/4$ and n_0 so that $m^*(n,k,r,t)<(1-2\gamma)\binom{n-t}{k-t}$ for all $n>n_0$ and k with $\frac{k}{n}<\frac{1}{2}-\frac{\varepsilon}{2}$. Choose $0<\delta\ll\varepsilon$ so that $(p-\delta,p+\delta)\subset(0,\frac{1}{2}-\delta)$. Choose n_1 so that

$$(1-2\gamma)\sum_{k\in J}\binom{n-t}{k-t}p^kq^{n-k}+\sum_{k\notin J}\binom{n}{k}p^kq^{n-k}<(1-\gamma)p^t \tag{35}$$

holds for all $n > n_1$, where $J = ((p-\delta)n, (p+\delta)n) \cap \mathbb{N}$. Let $n > \max\{n_0, n_1\}$ and choose $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ with $w_p(\mathscr{G}) = w^*(n,p,r,t)$. Let $\mathscr{G}_k = \mathscr{G} \cap \binom{[n]}{k}$ for $k \in J$.

If $\mathscr{G}_k \in \mathbf{Y}^0(n,k,r,t)$ then by Theorem 6 we have $|\mathscr{G}_k| < (1-2\gamma)\binom{n-t}{k-t}$. If $\mathscr{G}_k$ fixes t vertices, say $[t]$, then $\mathscr{G}'_k := \{G - [t] : G \in \mathscr{G}\}$ is $(r-1)$-wise 1-intersecting. (Otherwise $\mathscr{G}$ fixes $[t]$.) Thus we have $|\mathscr{G}_k| = |\mathscr{G}'_k| \le \binom{n-t-1}{k-t-1}$. Consequently, in both cases, we have

$$|\mathscr{G}_k| < (1-2\gamma)\binom{n-t}{k-t}. \tag{36}$$

Using (36) and (35), we have

$$w_p(\mathscr{G}) \le \sum_{k \in J} |\mathscr{G}_k| p^k q^{n-k} + \sum_{k \notin J} \binom{n}{k} p^k q^{n-k} < (1-\gamma)p^t,$$

and this is true for all $n \ge t$ by Lemma 10. □

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