THE RANDOM WALK METHOD FOR INTERSECTING FAMILIES

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ABSTRACT. Let $m(n,k,r,t)$ be the maximum size of $\mathscr{F} \subset \binom{[n]}{k}$ satisfying $|F_1 \cap \cdots \cap F_r| \geq t$ for all $F_1, \ldots, F_r \in \mathscr{F}$. We report some known results about $m(n,k,r,t)$. The random walk method introduced by Frankl is a strong tool to investigate $m(n,k,r,t)$. Using a concrete example, we explain the method and how to use it.

1. INTRODUCTION

Let n,k,r and t be positive integers, and let $[n] = \{1,2,\ldots,n\}$. A family $\mathscr{G} \subset 2^{[n]}$ is called r-wise t-intersecting if $|G_1 \cap \cdots \cap G_r| \geq t$ holds for all $G_1, \ldots, G_r \in \mathscr{G}$. Let us define a typical r-wise t-intersecting family $\mathscr{G}_i(n,r,t)$ and its k-uniform subfamily $\mathscr{F}_i(n,k,r,t)$, where $0 \leq i \leq \lfloor \frac{n-t}{r} \rfloor$, as follows:

$$\begin{aligned} \mathscr{G}_i(n,r,t) &= \{G \subset [n] : |G \cap [t+ri]| \geq t+(r-1)i\}, \\ \mathscr{F}_i(n,k,r,t) &= \mathscr{G}_i(n,r,t) \cap \binom{[n]}{k}. \end{aligned}$$

Two families $\mathscr{G}, \mathscr{G}' \subset 2^{[n]}$ are said to be isomorphic, and denoted by $\mathscr{G} \cong \mathscr{G}'$, if there exists a vertex permutation τ on $[n]$ such that $\mathscr{G}' = \{\{\tau(g) : g \in G\} : G \in \mathscr{G}\}$.

Let $m(n,k,r,t)$ be the maximum size of k-uniform r-wise t-intersecting families on n vertices. To determine $m(n,k,r,t)$ is one of the oldest problems in extremal set theory, which is still widely open. The case $r = 2$ was observed by Erdős–Ko–Rado [6], Frankl [10], Wilson [30], and then $m(n,k,2,t) = \max_i |\mathscr{F}_i(n,k,2,t)|$ was finally proved by Ahlswede and Khachatrian [2]. Frankl [8] showed $m(n,k,r,1) = |\mathscr{F}_0(n,k,r,1)|$ if $(r-1)n \geq rk$. Partial results for the cases $r \geq 3$ and $t \geq 2$ are found in [14, 16, 24, 26, 27, 23, 29]. All known results suggest

$$m(n,k,r,t) = \max_i |\mathscr{F}_i(n,k,r,t)|. \tag{1}$$

Now we introduce the p-weight version of the Erdős–Ko–Rado theorem. Throughout this paper, p and $q = 1-p$ denote positive real numbers. For $X \subset [n]$ and a family $\mathscr{G} \subset 2^X$ we define the p-weight of $\mathscr{G}$, denoted by $w_p(\mathscr{G} : X)$, as follows:

$$w_p(\mathscr{G} : X) = \sum_{G \in \mathscr{G}} p^{|G|} q^{|X|-|G|} = \sum_{i=0}^{|X|} \left| \mathscr{G} \cap \binom{X}{i} \right| p^i q^{|X|-i}.$$

We simply write $w_p(\mathscr{G})$ for the case $X = [n]$, for example, we have $w_p(\mathscr{G}_0(n,r,t)) = p^t$.

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Let $w(n,p,r,t)$ be the maximum p-weight of r-wise t-intersecting families on n vertices. It might be natural to expect

$$w(n,p,r,t) = \max_i w_p(\mathscr{G}_i(n,r,t)).$$

Ahlswede and Khachatrian proved that this is true for $r=2$ in [3] (cf. [5, 7, 23]). This includes the Katona theorem [19] about $w(n,1/2,2,t)$. It is shown in [15] that

$$w(n,p,r,1) = p \text{ for } p \leq (r-1)/r. \tag{2}$$

To state some more related results let us define some collections of families as follows.

$$\begin{aligned}
\mathbf{G}(n,r,t) &= \{\mathscr{G} \subset 2^{[n]} : \mathscr{G} \text{ is } r\text{-wise } t\text{-intersecting}\},\\
\mathbf{G}_j(n,r,t) &= \{\mathscr{G} \subset 2^{[n]} : \mathscr{G} \subset \mathscr{G}' \text{ for some } \mathscr{G}' \cong \mathscr{G}_j(n,r,t)\},\\
\mathbf{X}^i(n,r,t) &= \mathbf{G}(n,r,t) - \textstyle\bigcup_{0\leq j\leq i} \mathbf{G}_j(n,r,t),\\
\mathbf{Y}^i(n,k,r,t) &= \{\mathscr{F} \subset \binom{[n]}{k} : \mathscr{F} \in \mathbf{X}^i(n,r,t)\}.
\end{aligned}$$

Finally let us define

$$\begin{aligned}
m^i(n,k,r,t) &= \max\{|\mathscr{F}| : \mathscr{F} \in \mathbf{Y}^i(n,k,r,t)\},\\
w^i(n,p,r,t) &= \max\{w_p(\mathscr{G}) : \mathscr{G} \in \mathbf{X}^i(n,r,t)\}.
\end{aligned}$$

Ahlswede and Khachatrian [1] determined $m^0(n,k,2,t)$ completely, extending the earlier results by Hilton–Milner [18] and Frankl [11]. Brace and Daykin [4] determined $w^0(n,1/2,r,1)$ and Frankl determined $w^0(n,1/2,r,t)$ for $r \geq 5$ and $1 \leq t \leq 2^r - r - 1$; in both cases $\mathscr{G}_1(n,r,t)$ has the maximum p-weight. (But $\mathscr{G}_1$ is not always optimal for w^0, for example, we have $w^0(n,p,r,1) > w_p(\mathscr{G}_1(n,r,1))$ if $p > \frac{1}{2}$ and $r \leq 5$, see [28].) More results for $m^0(n,k,r,t)$ with $k/n \approx 1/2$, and $w^0(n,p,r,t)$ with $p \approx 1/2$ are found in [17, 28, 29].

In this article we will introduce the random walk method originated by Frankl, which is a strong tool to investigate $w(n,p,r,t)$. In the next section, we explain the key idea of the method. In Section 3 we prepare some tools to apply the method. Then in Section 4 we illustrate the method by determining $w(n,1/3,4,36)$, and a general setup to get $w(n,p,r,t)$ will be given in Section 5. In the last section we discuss how to derive $m(n,k,r,t)$ from $w(n,p,r,t)$ when $p \approx k/n$. As a consequence, we get the following result (see Theorem 10).

Theorem 1. *Let $p_0 \in (0,1)$ and $r,t,i \in \mathbb{N}$ be given. Suppose that $\max_j\{w_{p_0}(\mathscr{G}_j(n,r,t))\}$ is attained by $j = i-1$ or i. Then (W) implies (M).*

- (W) *There exist positive constants $\gamma_0, \varepsilon_0, n_0$ such that, for all p with $|p - p_0| < \varepsilon_0$ and all n with $n \geq n_0$, the following is true: If $\mathscr{G} \in \mathbf{X}^i(n,r,t)$ is shifted and $\bigcap \mathscr{G} = \emptyset$ then we have $w_p(\mathscr{G}) < (1-\gamma_0)\max\{w_p(\mathscr{G}_{i-1}(n,r,t)), w_p(\mathscr{G}_i(n,r,t))\}$.*
- (M) *There exist positive constants ε, n_1 such that, for all $n > n_1$ and k with $|\frac{k}{n} - p_0| < \varepsilon$, we have (1) with equality holding only if $\mathscr{F}_{i-1}(n,k,r,t)$ or $\mathscr{F}_i(n,k,r,t)$ (up to isomorphism).*

We can in fact show (W) in some particular choices of p_0, r, t, i by the random walk method. As an example we verify (1) for $r \geq 4$, $t \leq (3^r - 2r - 1)/2$, $k/n \leq 1/3$, and n large enough (Theorem 12). Although it is still beyond our reach to determine $m(n,k,r,t)$ and $w(n,p,r,t)$ completely, we hope that the strategy described in this article will provide a better understanding of multiply intersecting families.

2. THE RANDOM WALK METHOD

In [10] Frankl found a way to connect the number of walks of certain types with an upper bound for the size of intersecting families. He then extended the idea to bound the size of 3-wise 2-intersecting families in [9], where the random walk method was explicitly used for the first time. One of the highlights of the method is [13], where he got many interesting results on multiply intersecting families, and most of them have no alternative proofs so far. A survey [12] by himself is highly recommended.

In this section we explain the key idea of the method. Let p and q be positive reals with $p+q=1$, and let $\alpha_{r,p} \in (p,1)$ be the unique root of the equation $qx^r - x + p = 0$. The random walk method is basically to use the following inequality:

$$w(n,p,r,t) \le \alpha_{r,p}^t. \tag{3}$$

This inequality itself is not sharp, but we often get sharp upper bounds for the p-weight of intersecting families using (3) with some additional argument.

We outline how to get (3) here. (One can find the proof in [12] (for the case $p=1/2$) and we also include some more explanation about shifting technique etc. for convenience in the next section.) For $G \subset [n]$ we define the corresponding n-step walk on $\mathbb{Z}^2$, denoted by walk(G), as follows. The walk is from $(0,0)$ to $(|G|, n-|G|)$, and the i-th step is one unit up ($\uparrow$) if $i \in G$, or one unit to the right ($\rightarrow$) if $i \notin G$. Let $\mathscr{G} \in \mathbf{G}(n,r,t)$. We can find a shifted $\mathscr{G}^* \in \mathbf{G}(n,r,t)$ with $w_p(\mathscr{G}) = w_p(\mathscr{G}^*)$. Then, for each $G \in \mathscr{G}^*$, walk(G) touches the line $L : y = (r-1)x + t$ (see Lemma 4). Thus we have $\mathscr{G}^* \subset \mathscr{W}_n$, where $\mathscr{W}_n = \{W \subset [n] : \text{walk}(W) \text{ touches } L\}$. We note that $\mathscr{W}_n$ is not necessarily r-wise t-intersecting.

Now consider the infinite random walk in $\mathbb{Z}^2$ starting from $(0,0)$, taking $\uparrow$ with probability p and $\rightarrow$ with probability q at each step independently. Suppose that $\mathscr{G}$ has the maximum p-weight. Then it follows that

$$\begin{aligned} w(n,p,r,t) &= \sum_{G \in \mathscr{G}} p^{|G|} q^{n-|G|} \le \sum_{W \in \mathscr{W}_n} p^{|W|} q^{n-|W|} \le \lim_{n\to\infty} \sum_{W \in \mathscr{W}_n} p^{|W|} q^{n-|W|} \\ &= \mathbf{P}\,(\text{the infinite random walk touches } L) = \alpha_{r,p}^t. \end{aligned} \tag{4}$$

The last equality (4) can be shown as follows. Let X_s be the probability that the infinite random walk touches the line $y = (r-1)x + s$. After the first step, we are at $(1,0)$ with probability p, or at $(0,1)$ with probability q. Thus we have

$$X_t = pX_{t-1} + qX_{t+r-1}. \tag{5}$$

Let a_i be the number of walks from $(0,0)$ to $A_i = (i, (r-1)i + t)$ which touch L only at A_i. Then we have $X_t = \sum_{i \ge 0} a_i p^{(r-1)i+t} q^i$. To touch the line $L' : y = (r-1)x + t + 1$, we need to hit L somewhere, say, at A_i for the first time. Then the probability that we hit L' starting from A_i is equal to X_1. Thus we have

$$X_{t+1} = \sum_{i \ge 0} (a_i p^{(r-1)i+t} q^i) X_1 = X_t X_1 = X_1^{t+1}. \tag{6}$$

By (5) and (6) we have $X_1 = p + qX_1^r$. This equation has unique root $X_1 = \alpha_{r,p}$ in $(0,1)$, and then (6) gives $X_t = \alpha_{r,p}^t$, which proves (4). One can also show that $a_i = \frac{t}{ri+t}\binom{ri+t}{i}$ and $\sum_{i \ge 0} a_i p^{(r-1)i+t} q^i = \alpha_{r,p}^t$ in a different way, see e.g., [22].

To consider the k-uniform version problem, let us review the very original idea of the method from [10]. Let $\mathscr{F} \subset \binom{[n]}{k}$ be 2-wise t-intersecting. Then for every $F \in \mathscr{F}$, walk(F)

is from $(0,0)$ to $(n-k,k)$, which touches the line $y=x+t$. The total number of walks with this property is, by the reflection principle, equal to the total number of walks from $(-t,t)$ to $(n-k,k)$, which is $\binom{n}{k-t}$. This gives $m(n,k,2,t) \le \binom{n}{k-t} \le (\frac{k}{n-k})^t \binom{n}{k}$. On the other hand, by setting $p=\frac{k}{n}$, we have $\alpha_{2,p}=\frac{p}{q}=\frac{k}{n-k}$, and $m(n,k,2,t) \le \alpha_{2,p}^t \binom{n}{k}$. This suggests the following k-uniform version of (3):

$$m(n,k,r,t) \le \alpha_{r,p}^t \binom{n}{k},$$

where $p=\frac{k}{n}$. This is true if $p<\frac{r-1}{r+1}$ is fixed and n is large enough, see [25]. We will discuss how to get $m(n,k,r,t)$ from $w(n,p,r,t)$ in the last section.

3. Tools

Let us introduce the shifting operation. For integers $1 \le i < j \le n$ and a family $\mathscr{G} \subset 2^{[n]}$, we define the (i,j)-shift σ_{ij} as follows:

$$\sigma_{ij}(\mathscr{G}) = \{\sigma_{ij}(G) : G \in \mathscr{G}\},$$

where

$$\sigma_{ij}(G) = \begin{cases} (G-\{j\}) \cup \{i\} & \text{if } i \notin G,\ j \in G,\ (G-\{j\}) \cup \{i\} \notin \mathscr{G}, \\ G & \text{otherwise.} \end{cases}$$

This operation preserves r-wise t-intersecting property, namely, if $\mathscr{G}$ is r-wise t-intersecting, then so is $\sigma_{ij}(\mathscr{G})$. Note also that shifting does not change the p-weight, i.e., $w_p(\sigma_{ij}(\mathscr{G})) = w_p(\mathscr{G})$.

A family $\mathscr{G} \subset 2^{[n]}$ is called *shifted* if $\sigma_{ij}(\mathscr{G}) = \mathscr{G}$ for all $1 \le i < j \le n$, and $\mathscr{G}$ is called *tame* if it is shifted and $\bigcap \mathscr{G} = \emptyset$. Starting from a given $\mathscr{G}$ we can always get a shifted $\mathscr{G}'$ by a finite sequence of shifting operations. To see this fact, let $s(\mathscr{G}) = \sum\{\sum\{g : g \in G\} : G \in \mathscr{G}\} \in \mathbb{N}$ and observe $s(\sigma_{ij}(\mathscr{G})) < s(\mathscr{G})$ if $\sigma_{ij}(\mathscr{G}) \ne \mathscr{G}$.

Lemma 2. $\mathbf{X}^0(n,r,t) \subset \mathbf{X}^0(n,r-1,t+1)$ *and* $w^0(n,p,r,t) \le w^0(n,p,r-1,t+1)$.

Proof. Let $\mathscr{G} \in \mathbf{X}^0(n,r,t)$. Then clearly we have $\mathscr{G} \notin \mathbf{G}_0(n,r-1,t+1)$. Thus it suffices to show that $\mathscr{G} \in \mathbf{G}(n,r-1,t+1)$. If it is not, then we can find $G_1,\dots,G_{r-1} \in \mathscr{G}$ such that $|G_1 \cap \cdots \cap G_{r-1}| = t$. But $\mathscr{G}$ is r-wise t-intersecting and so every $G \in \mathscr{G}$ must contain $G_1 \cap \cdots \cap G_{r-1}$. This means $\mathscr{G} \notin \mathbf{X}^0(n,r,t)$, a contradiction. □

Lemma 3. *If* $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ *has maximum p-weight then we can find a tame* $\mathscr{G}' \in \mathbf{X}^0(n,r,t)$ *with* $w_p(\mathscr{G}') = w_p(\mathscr{G})$.

Proof. If $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ then $\mathscr{G} \in \mathbf{X}^0(n,r-1,t+1)$ by Lemma 2. We apply shifting operations to $\mathscr{G}$ to get a shifted family $\mathscr{G}' \in \mathbf{G}(n,r,t) \subset \mathbf{G}(n,r-1,t+1)$.

We have to show that $\bigcap \mathscr{G}' = \emptyset$. Otherwise we may assume that $1 \in \bigcap \mathscr{G}'$ and $H = [2,n] \notin \mathscr{G}'$. Since $\mathscr{G}'$ is p-weight maximum we can find $G_1,\dots,G_{r-1} \in \mathscr{G}'$ such that $|G_1 \cap \cdots \cap G_{r-1} \cap H| < t$. Then we have $|G_1 \cap \cdots \cap G_{r-1}| < t+1$, which is a contradiction. □

Lemma 4. *Let* $\mathscr{G} \in \mathbf{G}(n,r,t)$ *be shifted. Then* walk(G) *touches the line* $L : y = (r-1)x + t$ *for all* $G \in \mathscr{G}$.

Proof. Let $H = [n] - \{t, t+r, t+2r, t+3r, \dots\}$. Then walk$(H)$ does not touch L. Moreover this walk is the maximal one with this property. Namely, if walk(F) does not touch L, then we can find $F' \supset F$ such that H is obtained from F' by a sequence of shifting operations.

Let $\mathcal{G} \in \mathbf{G}(n,r,t)$. Suppose that we have some $G \in \mathcal{G}$ such that walk(G) does not touch L. We may assume that $\mathcal{G}$ is size maximal, and so $G = H$. For $1 \le i < r$, let $H_i = [n] - \{t+i, t+r+i, t+2r+i, t+3r+i, \dots\}$. We get H_i from H by shifting. Since $\mathcal{G}$ is shifted we have $H, H_1, \dots, H_{r-1} \in \mathcal{G}$ and $H \cap H_1 \cap \cdots \cap H_{r-1} = [t-1]$, which is a contradiction. □

Lemma 5 ([28]). *Let p, r, t_0, c be fixed constants, and let $\alpha \in (p,1)$ be the root of the equation $qx^r - x + p = 0$. Suppose that $w(n,p,r,t_0) \le c$ holds for all $n \ge t_0$. Then we have $w(n,p,r,t) \le c\alpha^{t-t_0}$ for all $t \ge t_0$ and $n \ge t$.*

Proof. If $\mathcal{G} \subset 2^{[n]}$ is trivial r-wise t_0-intersecting, i.e., $|\bigcap \mathcal{G}| \ge t_0$, then we have $\mathcal{G} \subset \{G \subset [n] : [t_0] \subset G\}$ and $w_p(\mathcal{G}) \le p^{t_0}$. Thus we may assume that $c \ge p^{t_0}$. Note also that $p < \alpha$.

We prove the result by double induction on $s = n - t$ and t. One of the initial steps for $t = t_0$ follows from our assumption. For the other initial step for s, we prove the result for the cases $0 \le s \le r-1$, or equivalently, $t \le n \le t+r-1$. Suppose that $\mathcal{G} \subset 2^{[n]}$ satisfies $w_p(\mathcal{G}) = w(n,p,r,t)$. We may assume that $\mathcal{G}$ is shifted and size maximal. If $\mathcal{G}$ is trivial, i.e., $|\bigcap \mathcal{G}| \ge t$, then we have $w_p(\mathcal{G}) \le p^t = p^{t_0} p^{t-t_0} < c\alpha^{t-t_0}$ and we are done. Otherwise we have $G \in \mathcal{G}$ such that $[t] \not\subset G$, and we may assume that $G_t = [n] - \{t\} \in \mathcal{G}$ because $\mathcal{G}$ is shifted and maximal. Then again by the shiftedness we have $G_i = [n] - \{i\} \in \mathcal{G}$ for all $t \le i \le n$. This implies $|\bigcap_{i=t}^{n} G_i| = t-1$. But this is impossible because $\mathcal{G}$ is r-wise t-intersecting and $n - t + 1 \le r$.

Next we show the induction step. Let $s \ge r$ and $t > t_0$. We show the case (s,t). We assume that the result holds for $\{(s,b) : b < t\} \cup \{(a,b) : a < s, b \ge t_0\}$. In particular, we can apply induction hypothesis to the case $(s, t-1)$ and $(s-r, t+r-1)$.

Let $\mathcal{G} \subset 2^{[n]}$ be r-wise t-intersecting. Define $\mathcal{G}_1, \mathcal{G}_{\bar{1}} \subset 2^{[2,n]}$ as follows:

$$\mathcal{G}_1 = \{G - \{1\} : 1 \in G \in \mathcal{G}\}, \quad \mathcal{G}_{\bar{1}} = \{G : 1 \notin G \in \mathcal{G}\}.$$

Then $\mathcal{G}_1$ is clearly r-wise $(t-1)$-intersecting. On the other hand, $\mathcal{G}_{\bar{1}}$ is r-wise $(t+r-1)$-intersecting. To see this fact suppose, on the contrary, that there exist $G_2 \dots G_{r+1} \in \mathcal{G}_{\bar{1}}$ such that $\bigcap_{i=2}^{r+1} G_i = [2, t+r-1]$. By the shiftedness we have $G_i' = \{1\} \cup (G_i - \{i\}) \in \mathcal{G}$ for all $2 \le i \le r+1$. But then we have $\bigcap_{i=2}^{r+1} G_i' = [t+r-1] - [2, r+1]$, which contradicts r-wise t-intersecting property of $\mathcal{G}$.

Note that s for $\mathcal{G}_1$ is $(n-1) - (t-1) = s$ and s for $\mathcal{G}_{\bar{1}}$ is $(n-1) - (t+r-1) = s - r$. Therefore using the induction hypothesis, we have

$$\begin{aligned} w_p(\mathcal{G}) &= p w_p(\mathcal{G}_1 : [2,n]) + q w_p(\mathcal{G}_{\bar{1}} : [2,n]) \le pc\alpha^{t-t_0-1} + qc\alpha^{t+r-t_0-1} \\ &= c\alpha^{t-t_0-1}(p + q\alpha^r) = c\alpha^{t-t_0}. \quad \square \end{aligned}$$

Lemma 6. *For any $i \ge 0$ we have $w^i(n+1,p,r,t) \ge w^i(n,p,r,t)$.*

Proof. Choose $\mathcal{G} \in \mathbf{X}^i(n,r,t)$ with $w_p(\mathcal{G}) = w^i(n,p,r,t)$. Then $\mathcal{G}' := \mathcal{G} \cup \{G \cup \{n+1\} : G \in \mathcal{G}\} \in \mathbf{X}^i(n+1,r,t)$ and $w_p(\mathcal{G}' : [n+1]) = w_p(\mathcal{G} : [n])(q+p) = w^i(n,p,r,t)$, which means $w^i(n+1,p,r,t) \ge w^i(n,p,r,t)$. □

4. An example

As a toy example, we consider the case $r = 4$ and $t = 36$. Let $p \in (0,1)$ and $q = 1 - p$, and set $\mathcal{G}_j = \mathcal{G}_j(n,4,36)$. Simple computation shows that $w_p(\mathcal{G}_0) \ge w_p(\mathcal{G}_1)$ iff $p \le 1/3$.

To give a feel of the random walk method, we will show that

$$w(n,p,4,36) = w_p(\mathscr{G}_0) = p^{36} \tag{7}$$

for all $n \geq 40$ and $p \leq 1/3$.

Clearly we have $w(n,p,4,36) \leq w(n,p,2,36)$, and the Ahlswede–Khachatrian result [3] already shows (7) for $p \leq 1/(t+1) = 1/37$. We can easily improve this upper bound for p using (3). Suppose that $\mathscr{G} \in \mathbf{G}(n,4,36)$. If $\mathscr{G} \in \mathbf{G}_0(n,4,36)$ then we have $w_p(\mathscr{G}) \leq p^{36}$. Otherwise we have $\mathscr{G} \in \mathbf{X}^0(n,4,36) \subset \mathbf{X}^0(n,3,37)$ by Lemma 2. Now by (3) we have $w_p(\mathscr{G}) \leq \alpha_{3,p}^{37}$. Then we find that $\alpha_{3,p}^{37} < p^{36}$ if $p \leq 1/5$. In this way we get (7) for $p \leq 1/5$.

To get (7) for $p \leq 1/3$, we will prove the following slightly stronger inequality, that is,

$$w^1(n,p,4,36) < 0.9999 \max\{w_p(\mathscr{G}_0), w_p(\mathscr{G}_1)\} \tag{8}$$

for all $n \geq 40$ and $p \leq 0.34$. This gives $w(n,p,4,36) = \max\{w_p(\mathscr{G}_0), w_p(\mathscr{G}_1)\}$ for $p \leq 0.34$, and in particular this implies (7) for $p \leq 1/3$.

Choose $\mathscr{G} \in \mathbf{X}^1(n,4,36)$ with the maximum p-weight, and choose a tame $\mathscr{G}^* \in \mathbf{X}^0(n,4,36)$ with $w_p(\mathscr{G}) = w_p(\mathscr{G}^*)$ by Lemma 3. We will show the following.

(i) If $\mathscr{G}^* \not\subset \mathscr{G}_1$ then $w_p(\mathscr{G}^*) < 0.99\, w_p(\mathscr{G}_0)$ for $p \leq 0.34$.
(ii) If $\mathscr{G}^* \subset \mathscr{G}_1$ then $w_p(\mathscr{G}^*) < 0.9999\, w_p(\mathscr{G}_1)$ for $p \leq 0.34$.

We can show (ii) in a more general setting as we will see in the next section. Here we show (i). So we assume that $\mathscr{G}^* \not\subset \mathscr{G}_1$ and rename it $\mathscr{G}$.

Let $s = \max\{j : \mathscr{G} \in \mathbf{G}(n,3,j)\}$. By Lemma 2 we have $s \geq 37$. If $s \geq 40$ then by (3) we have

$$w_p(\mathscr{G}) \leq w(n,p,3,40) \leq \alpha_{3,p}^{40} < 0.99 p^{36} \tag{9}$$

for $p \leq 0.34$. Thus we may assume that $37 \leq s \leq 39$. After [13] let

$$h = \min\{j : |G \cap [36+j]| \geq 36 \text{ for all } G \in \mathscr{G}\}.$$

This is the maximum size of “holes” in $[36+h]$.

Claim 1. $1 \leq h \leq s - 36 \leq 3$.

Proof. Since $\mathscr{G} \in \mathbf{X}^0(n,4,36)$, we have $h \geq 1$. By the definition of s and the shiftedness of $\mathscr{G}$, we have $G_1, G_2, G_3 \in \mathscr{G}$ such that $G_1 \cap G_2 \cap G_3 = [s]$. Since $\mathscr{G} \in \mathbf{G}(n,4,36)$ it follows that $|G \cap [s]| \geq 36$ for all $G \in \mathscr{G}$, namely, $36 + h \leq s$. □

Let $b = 36 + (h-1) = 35 + h$ and let $T_i = [b+1-i, b]$ be the right-most i-set in $[b]$ ($T_0 = \emptyset$). For $A \subset [b]$ let

$$\mathscr{G}(A) = \{G \cap [b+1,n] : G \in \mathscr{G}, [b] \setminus G = A\}.$$

Since $\mathscr{G}$ is shifted, we have $\mathscr{G}(A) \subset \mathscr{G}(T_i)$ for all $A \in \binom{[b]}{i}$, and thus we have

$$w_p(\mathscr{G}) \leq \sum_{i=0}^{h} \binom{b}{i} p^{b-i} q^i\, w_p(\mathscr{G}(T_i) : [b+1,n]). \tag{10}$$

To bound $w_p(\mathscr{G}(T_i) : [b+1,n])$ we use the fact that $\mathscr{G}(T_i)$ is highly-intersecting as we see below.

Claim 2. *For* $0 \leq i < h$ *we have* $\mathscr{G}(T_i) \in \mathbf{G}(n,3,3i+1)$.

Proof. Suppose that $\mathscr{G}(T_i) \notin \mathbf{G}(n,3,3i+1)$. Then we can find $G_1, G_2, G_3 \in \mathscr{G}(T_i)$ such that $|G_1 \cap G_2 \cap G_3| \leq 3i$. Since $\mathscr{G}$ is shifted, we may assume that $G_1 \cap G_2 \cap G_3 \subset [b+1, b+3i]$. For $1 \leq \ell \leq 3$, by shifting $(G_\ell \cup [b]) - T_i \in \mathscr{G}$, we get $G'_\ell := (G_\ell \cup [b]) - [b+1+(\ell-1)i, b+\ell i] \in \mathscr{G}$. By the definition of h we have some $H \in \mathscr{G}$ such that $|H \cap [b]| < 36$ and due to the shiftedness of $\mathscr{G}$ we may assume that $H = [n] - [36, b]$. Then we have $G'_1 \cap G'_2 \cap G'_3 \cap H = [35]$, which contradicts the fact $\mathscr{G} \in \mathbf{G}(n,4,36)$. □

Claim 3. *If $\mathscr{G} \not\subset \mathscr{G}_h$ then $\mathscr{G}(T_h) \in \mathbf{G}(n,3,3h+2)$.*

Proof. Suppose that $\mathscr{G}(T_h) \notin \mathbf{G}(n,3,w)$, where $w = 3h+2$. Then we can find $G_1, G_2, G_3 \in \mathscr{G}(T_h)$ such that $G_1 \cap G_2 \cap G_3 \subset [b+1, b+w-1]$. By shifting $(G_\ell \cup [b]) - T_h \in \mathscr{G}$ we get $G'_\ell := (G_\ell \cup [b]) - [36+(\ell-1)h, 35+\ell h] \in \mathscr{G}$ for $1 \leq \ell \leq 3$. Since $\mathscr{G} \not\subset \mathscr{G}_h$ we have $G'_4 := [n] - [36+3h, 36+4h] \in \mathscr{G}$. Then we have $|G'_1 \cap \cdots \cap G'_4| < 36$, a contradiction. □

We may assume that $\mathscr{G} \not\subset \mathscr{G}_h$ for $1 \leq h \leq 3$. In fact, we have already assumed $\mathscr{G} \not\subset \mathscr{G}_1$, and we have $w_p(\mathscr{G}_i) < 0.99 \max\{w_p(\mathscr{G}_0), w_p(\mathscr{G}_1)\}$ for $i = 2,3$ and $p \leq 0.34$.

First we consider the case $h = 1$. In this case, by Claim 2 we have $\mathscr{G}(T_0) \in \mathbf{G}(n,3,1)$. Since $\mathscr{G} \not\subset \mathscr{G}_1$ it follows from Claim 3 that $\mathscr{G}(T_1) \in \mathbf{G}(n,3,5)$. Thus (3) gives $w_p(\mathscr{G}(T_0) : [b+1,n]) \leq \alpha_{3,p}$ and $w_p(\mathscr{G}(T_1) : [b+1,n]) \leq \alpha_{3,p}^5$. Finally by (10) we have

$$w_p(\mathscr{G}) \leq p^{36}\alpha_{3,p} + 36p^{35}q\alpha_{3,p}^5 < 0.99p^{36}$$

for $p \leq 0.34$.

Next we consider the case $h = 2$. In this case, Claim 2 gives $\mathscr{G}(T_0) \in \mathbf{G}(n,3,1)$ and $\mathscr{G}(T_1) \in \mathbf{G}(n,3,4)$, and Claim 3 gives $\mathscr{G}(T_2) \in \mathbf{G}(n,3,8)$. Thus (3) and (10) imply

$$w_p(\mathscr{G}) \leq p^{37}\alpha_{3,p} + 37p^{36}q\alpha_{3,p}^4 + \binom{37}{2}p^{35}q^2\alpha_{3,p}^8 < 0.99p^{36}.$$

Similarly, in the case $h = 3$, we have

$$w_p(\mathscr{G}) \leq p^{38}\alpha_{3,p} + 38p^{37}q\alpha_{3,p}^4 + \binom{38}{2}p^{36}q^2\alpha_{3,p}^7 + \binom{38}{3}p^{35}q^3\alpha_{3,p}^{11} < 0.99p^{36}. \quad (11)$$

This completes the proof of (i). □

If we have more information about $w(n,p,3,*)$ then we get simpler proof. For example, using a result in [27] we have $w(n,p,3,8) \leq p^8$ for $p \leq 0.34$. This together with Lemma 5 gives

$$w(n,p,3,39) \leq p^8\alpha_{3,p}^{31} < 0.99p^{36}.$$

By replacing (9) with the above estimation, we can conclude that $37 \leq s \leq 38$ and so $1 \leq h \leq 2$. This means we do not have to deal with (11).

5. A general setup

Let n,p,r,t be fixed and let $\mathscr{G}_i = \mathscr{G}_i(n,r,t)$. Suppose that $\max\{w_p(\mathscr{G}_{i-1}), w_p(\mathscr{G}_i)\} > w_p(\mathscr{G}_j)$ for all $j \notin \{i-1, i\}$, and consider the situation that we are trying to show

$$w(n,p,r,t) = \max\{w_p(\mathscr{G}_{i-1}), w_p(\mathscr{G}_i)\}, \quad (12)$$

with equality holding only if $\mathscr{G} \cong \mathscr{G}_{i-1}$ or $\mathscr{G}_i$. If $\mathscr{G} \notin \mathbf{X}^i(n,r,t)$ then there is nothing to show. So suppose that $\mathscr{G} \in \mathbf{X}^i(n,r,t)$ and we want to show that $w_p(\mathscr{G})$ is much less than $\max\{w_p(\mathscr{G}_{i-1}), w_p(\mathscr{G}_i)\}$. Let $\mathscr{G}^*$ be a tame family obtained from $\mathscr{G}$ by shifting. Then we have two cases:

(a) $\mathscr{G}^* \not\subset \mathscr{G}_j(n,r,t)$ for all $0 \le j \le i$. This is the essential case we need to estimate $w_p(\mathscr{G}^*)$ by the random walk method. To use the method, it is important that $\mathscr{G}^*$ is shifted. We saw an example in this case in the previous section.

(b) $\mathscr{G}^* \subset \mathscr{G}_j(n,r,t)$ for some $0 \le j \le i$. In this case, we will see that $w_p(\mathscr{G}^*)$ cannot be large by Theorem 7 below.

Consequently, to get (12) with the uniqueness of the optimal configuration, it is enough to consider a tame $\mathscr{G} \in \mathbf{X}^i(n,r,t)$ from the beginning.

Theorem 7. *Let r,t and i be positive integers with $r \ge 4$, and let $p \in (0, \frac{r-3}{r-2}]$. Then there exists $\gamma > 0$ such that for all $n \ge t+r$ the following is true.*

Let $\mathscr{G} \in \mathbf{X}^i(n,r,t)$, and let $\mathscr{G}^* \in \mathbf{X}^0(n,r,t)$ be a tame family obtained from $\mathscr{G}$ by shifting. If $\mathscr{G}^* \subset \mathscr{G}_i(n,r,t)$ then

$$w_p(\mathscr{G}) < (1-\gamma) w_p(\mathscr{G}_i(n,r,t)).$$

Proof. Set $\mathscr{G}_i = \mathscr{G}_i(n,r,t)$. Note that $\mathscr{G}$ is not necessarily shifted. Since $\mathscr{G}^* \subset \mathscr{G}_i$, we may assume (by renaming the starting family if necessary) that $\mathscr{G}^* = \sigma_{xy}(\mathscr{G}) \subset \mathscr{G}_i$, where $x = t+ri$, $y = x+1$. We note that $|[x] \setminus G| \le i+1$ for all $G \in \mathscr{G}$. Moreover if $|[x] \setminus G| = i+1$ then $G \cap \{x,y\} = \{y\}$ and $(G - \{y\}) \cup \{x\} \notin \mathscr{G}$.

For $A \in \binom{[x]}{i}$ set $\mathscr{G}(A) = \{G \in \mathscr{G} : [y] \setminus G = A\}$, and for $B \in \binom{[x-1]}{i}$ and $z \in \{x,y\}$ let $\mathscr{G}_z(B) = \{G \in \mathscr{G} : [y] \setminus G = B \cup \{z\}\}$. Since $\sigma_{xy}(\mathscr{G}) \subset \mathscr{G}_i$ we have $\mathscr{G}_x(B) \cap \mathscr{G}_y(B) = \emptyset$ and so $w_p(\mathscr{G}_x(B)) + w_p(\mathscr{G}_y(B)) \le p^{x-i} q^{i+1}$. Set $\mathscr{G}' = \{G \in \mathscr{G} : |[x] \setminus G| < i\}$, $\mathscr{G}'' = \{G \in \mathscr{G} : |[x-1] \setminus G| = i-1,\ G \cap \{x,y\} = \emptyset\}$ and let $e = \min\{w_p(\mathscr{G}(A)) : A \in \binom{[x]}{i}\}$. Then we have

$$\begin{aligned} w_p(\mathscr{G}) &= \sum_{A \in \binom{[x]}{i}} w_p(\mathscr{G}(A)) + \sum_{B \in \binom{[x-1]}{i}} \big(w_p(\mathscr{G}_x(B)) + w_p(\mathscr{G}_y(B))\big) + w_p(\mathscr{G}') + w_p(\mathscr{G}'') \quad (13)\\ &\le e + (\tbinom{x}{i} - 1) p^{x-i+1} q^i + \tbinom{x-1}{i} p^{x-i} q^{i+1} + \sum_{j=0}^{i-1} \tbinom{x}{j} p^{x-j} q^j + \tbinom{x-1}{i-1} p^{x-i} q^{i+1}\\ &= e + (\eta - 1) p^{x-i+1} q^i, \quad (14) \end{aligned}$$

where $\eta = \sum_{j=0}^{i} \binom{x}{j} p^{i-j-1} q^{-i+j}$. Note that $e \le p^{x-i+1} q^i$, and (14) coincides $w_p(\mathscr{G}_i) = \eta p^{x-i+1} q^i$ iff $e = p^{x-i+1} q^i$. If there is some $B \in \binom{[x-1]}{i}$ such that $\mathscr{G}_x(B) \cup \mathscr{G}_y(B) = \emptyset$, then by (13) we get $w_p(\mathscr{G}) \le w_p(\mathscr{G}_i) - p^{x-i} q^{i+1} = \big(1 - q/(\eta p)\big) w_p(\mathscr{G}_i)$, and we are done. Thus we may assume that

$$\mathscr{G}_x(B) \cup \mathscr{G}_y(B) \ne \emptyset \text{ for all } B \in \tbinom{[x-1]}{i}. \quad (15)$$

To prove $w_p(\mathscr{G}) < (1-\gamma) w_p(\mathscr{G}_i)$ by contradiction, let us assume that for any $\gamma > 0$ and any n_0 there is some $n > n_0$ such that

$$w_p(\mathscr{G}) > (1-\gamma) w_p(\mathscr{G}_i) = (1-\gamma) \eta p^{x-i+1} q^i. \quad (16)$$

By (14) and (16) we have $e > (1 - \gamma\eta) p^{x-i+1} q^i$. This means, letting $\mathscr{H}(A) = \{G \setminus [y] : G \in \mathscr{G}(A)\}$ and $Y = [y+1, n]$, we have $w_p(\mathscr{H}(A) : Y) > 1 - \gamma\eta$, namely,

$$w_p(2^Y - \mathscr{H}(A) : Y) > \gamma\eta \text{ for all } A \in \tbinom{[x]}{i}. \quad (17)$$

Since $\mathscr{G} \in \mathbf{X}^i(n,r,t)$ both $\bigcup_{B \in \binom{[x-1]}{i}} \mathscr{G}_x(B)$ and $\bigcup_{B \in \binom{[x-1]}{i}} \mathscr{G}_y(B)$ are non-empty. Using this with (15), we can choose $G \in \mathscr{G}_x(B)$ and $G' \in \mathscr{G}_y(B')$ with $B, B' \in \binom{[x-1]}{i}$ and $B \cap B' = \emptyset$.

Let $L=[x-1]-(B\cup B')$ and $\mathscr{H}^*=\bigcap_{A\in\binom{L}{i}}\mathscr{H}(A)$. Then by (17) we have

$$\begin{aligned} w_p(\mathscr{H}^*:Y) &= 1-w_p(2^Y-\mathscr{H}^*:Y)=1-w_p(\textstyle\bigcup_{A\in\binom{L}{i}}(2^Y-\mathscr{H}(A)):Y) \\ &\geq 1-\textstyle\sum_{A\in\binom{L}{i}}w_p(2^Y-\mathscr{H}(A):Y)>1-\binom{|L|}{i}\gamma\eta. \end{aligned} \tag{18}$$

If $\mathscr{H}^*\subset 2^Y$ is not $(r-2)$-wise 1-intersecting, then we can find $H_1,\dots H_{r-2}\in\mathscr{H}^*$ such that $H_1\cap\cdots\cap H_{r-2}=\emptyset$. Choose disjoint i-sets $B_\ell\subset L$, $1\leq\ell\leq r-2$, and set $G_\ell:=([y]-B_\ell)\cup H_\ell\in\mathscr{G}$. Then we have $|G_1\cap\cdots\cap G_{r-2}\cap G\cap G'|=t-1$, which contradicts the r-wise t-intersecting property of $\mathscr{G}$. Thus $\mathscr{H}^*$ is $(r-2)$-wise 1-intersecting and $w_p(\mathscr{H}^*:Y)\leq p$ by (2). (We need $r\geq 4$ and $p\leq\frac{r-3}{r-2}$ here.) But this contradicts (18) because we can choose γ so small that $p\ll 1-\binom{|L|}{i}\gamma\eta$. □

This theorem implies (ii) of the previous section by taking $\gamma=0.0001$. In fact we have $q/(\eta p)>\gamma$ and $p\leq 1-\binom{|L|}{i}\gamma\eta=1-37\gamma(\frac{1}{q}+\frac{40}{p})$ for $p\leq 0.34$. Consequently we have proved (8). It is an easy exercise to get

$$w^1(n,p,4,t)<(1-\gamma)\max\{w_p(\mathscr{G}_0(n,4,t)),w_p(\mathscr{G}_1(n,4,t))\}$$

for all $n\geq 40$, $1\leq t\leq 36$ and $p\leq 0.34$, where $\gamma>0$ is an absolute constant. Then using induction on r with more careful analysis (but very much in the same way we did for the case $r=4$ and $t=36$) one can show the following.

Theorem 8. *For all $r\geq 4$ there exist positive constants ε,γ such that*

$$w^1(n,p,r,t)<(1-\gamma)\max\{w_p(\mathscr{G}_0(n,r,t)),w_p(\mathscr{G}_1(n,r,t))\}$$

holds for all $n\geq t+r$, $1\leq t\leq(3^r-2r-1)/2$ and $p\leq\frac{1}{3}+\varepsilon$.

We note that $w_p(\mathscr{G}_0(n,r,t))=w_p(\mathscr{G}_1(n,r,t))$ if $p=1/3$ and $t=(3^r-2r-1)/2$. As a corollary we get the following.

Corollary 9. *For all $r\geq 4$, $n\geq t+r$, $1\leq t\leq(3^r-2r-1)/2$ and $p\leq 1/3$ we have*

$$w(n,p,r,t)=w_p(\mathscr{G}_0(n,r,t))=p^t.$$

Moreover if $t=(3^r-2r-1)/2$ and $p=1/3$ then $\mathscr{G}_0(n,r,t)$ and $\mathscr{G}_1(n,r,t)$ are the only optimal configurations (up to isomorphism). Otherwise $\mathscr{G}_0(n,r,t)$ is the only optimal configuration (up to isomorphism).

6. From p-weight version to k-uniform version

In this section, we show that a k-uniform version problem for $m(n,k,r,t)$ can be reduced to a p-weight version problem for $w(n,p,r,t)$ when $k/n\approx p$ (Theorems 10 and 11). Using these results, we will get a k-uniform version (Theorem 12) corresponding to Theorem 8. Theorem 1 in the introduction is an immediate consequence of the following result.

Theorem 10. *Let $p_0\in(0,1)$ and $r,t,i\in\mathbb{N}$ be given. Then (W) implies (M).*

(W) *There exist positive constants $\gamma_0,\varepsilon_0,n_0$ such that*

$$w^i(n,p,r,t)<(1-\gamma_0)\max\{w_p(\mathscr{G}_{i-1}(n,r,t)),w_p(\mathscr{G}_i(n,r,t))\}$$

holds for all p with $|p-p_0|<\varepsilon_0$ and all n with $n\geq n_0$.

(M) *There exist positive constants* γ, ε, n_1 *such that*

$$m^i(n,k,r,t) < (1-\gamma)\max\{|\mathscr{F}_{i-1}(n,k,r,t)|, |\mathscr{F}_i(n,k,r,t)|\}$$

holds for all $n > n_1$ *and* k *with* $|\frac{k}{n} - p_0| < \varepsilon$. *(We can choose* $\varepsilon = \frac{\varepsilon_0}{2}$, $\gamma = \frac{\gamma_0}{4}$.*)*

For reals $0 < b < a$ we write $a \pm b$ to mean the open interval $(a-b, a+b)$, and for $n \in \mathbb{N}$, $n(a \pm b)$ means $((a-b)n, (a+b)n) \cap \mathbb{N}$.

Proof. Assuming the negation of (M), we will construct a counterexample to (W).

For fixed r and t we note that

$$f(p) := \max\{w_p(\mathscr{G}_{i-1}(n,r,t)), w_p(\mathscr{G}_i(n,r,t))\}$$

is a uniformly continuous function of p on $p_0 \pm \varepsilon_0$. Let $\varepsilon = \frac{\varepsilon_0}{2}$, $\gamma = \frac{\gamma_0}{4}$, and $I = p_0 \pm \varepsilon$.

Choose $\varepsilon_1 \ll \varepsilon$ so that

$$(1-3\gamma) f(p) > (1-4\gamma) f(p+\delta) \tag{19}$$

holds for all $p \in I$ and all $0 < \delta \le \varepsilon_1$. Choose n_2 so that

$$\sum_{j \in J} \binom{n}{j} p_1^j (1-p_1)^{n-j} > (1-3\gamma)/(1-2\gamma) \tag{20}$$

holds for all $n > n_2$ and all $p_1 \in I_0 := p_0 \pm \frac{3\varepsilon}{2}$, where $J = n(p_1 \pm \varepsilon_1)$. Choose n_3 so that

$$(1-\gamma)\max\{|\mathscr{F}_{i-1}(n,k,r,t)|, |\mathscr{F}_i(n,k,r,t)|\} > (1-2\gamma) f(k/n) \binom{n}{k} \tag{21}$$

holds for all $n > n_3$ and k with $k/n \in I$. Finally set $n_1 = \max\{n_0, n_2, n_3\}$.

Suppose that (M) fails. Then for our choice of ε, γ and n_1, we can find some n, k and $\mathscr{F} \in \mathbf{Y}^i(n,k,r,t)$ with $|\mathscr{F}| \ge (1-\gamma)\max\{|\mathscr{F}_{i-1}(n,k,r,t)|, |\mathscr{F}_i(n,k,r,t)|\}$, where $n > n_1$ and $\frac{k}{n} \in I$. We fix n, k and $\mathscr{F}$, and let $p = \frac{k}{n}$. By (21) we have $|\mathscr{F}| > c\binom{n}{k}$, where $c = (1-2\gamma) f(p)$. Let $\mathscr{G} = \bigcup_{k \le j \le n} (\nabla_j(\mathscr{F})) \in \mathbf{X}^i(n,r,t)$ be the collection of all upper shadows of $\mathscr{F}$, where $\nabla_j(\mathscr{F}) = \{H \in \binom{[n]}{j} : H \supset \exists F \in \mathscr{F}\}$. Let $p_1 = p + \varepsilon_1 \in I_0$, and $J = n(p_1 \pm \varepsilon_1) = (k, k+2\varepsilon_1 n) \cap \mathbb{N}$.

Claim 4. $|\nabla_j(\mathscr{F})| \ge c\binom{n}{j}$ *for* $j \in J$.

Proof. Choose a real $x \le n$ so that $c\binom{n}{k} = \binom{x}{n-k}$. Since $|\mathscr{F}| > c\binom{n}{k} = \binom{x}{n-k}$ the Kruskal–Katona Theorem [21, 20] implies that $|\nabla_j(\mathscr{F})| \ge \binom{x}{n-j}$. Thus it suffices to show that $\binom{x}{n-j} \ge c\binom{n}{j}$, or equivalently,

$$\frac{\binom{x}{n-j}}{\binom{x}{n-k}} \ge \frac{c\binom{n}{j}}{c\binom{n}{k}}.$$

Using $j \ge k$ this is equivalent to $j \cdots (k+1) \ge (x-n+j) \cdots (x-n+k+1)$, which follows from $x \le n$. □

By the claim we have

$$w_{p_1}(\mathscr{G}) \ge \sum_{j \in J} |\nabla_j(\mathscr{F})|\, p_1^j (1-p_1)^{n-j} \ge c \sum_{j \in J} \binom{n}{j} p_1^j (1-p_1)^{n-j}. \tag{22}$$

Using (20) and (19), the RHS of (22) is more than

$$c(1-3\gamma)/(1-2\gamma) = (1-3\gamma) f(p) > (1-4\gamma) f(p+\varepsilon_1) = (1-\gamma_0) f(p_1).$$

This means $w_{p_1}(\mathscr{G}) > (1-\gamma_0)\max\{w_{p_1}(\mathscr{G}_{i-1}(n,r,t)), w_{p_1}(\mathscr{G}_i(n,r,t))\}$, which contradicts (W) because $p_1 \in I_0 \subset p_0 \pm \varepsilon_0$. $\square$

Theorem 11. *Let $r,t \in \mathbb{N}$ with $r \geq 4$, and let $p_0 \in (0, \frac{r-3}{r-2}]$. Suppose that*

$$p_0^t > (t+r)p_0^{t+r-1}(1-p_0) + p_0^{t+r},$$

i.e., $w_{p_0}(\mathscr{G}_0(n,r,t)) > w_{p_0}(\mathscr{G}_1(n,r,t))$ for all $n \geq t+r$. Then (W0) implies (M1) and (W1).

(W0) *There exist positive constants $\gamma_0, \varepsilon_0, n_0$ such that $w^0(n,p,r,t) < (1-\gamma_0)p^t$ holds for all p with $|p-p_0| < \varepsilon_0$ and all n with $n \geq n_0$.*

(M1) *There exist positive constants $\gamma_1, \varepsilon_1, n_1$ such that*

$$m^0(n,k,r,t) < (1-\gamma_1)\binom{n-t}{k-t} \tag{23}$$

holds for all $n > n_1$ and k with $\frac{k}{n} < p_0 + \varepsilon_1$.

(W1) *There exist positive constants γ_2, ε_2 such that $w^0(n,p,r,t) < (1-\gamma_2)p^t$ holds for all p with $p < p_0 + \varepsilon_2$ and all n with $n \geq t$.*

Proof. For simplicity, we write $\mathscr{G}_j$ for $\mathscr{G}_j(n,r,t)$ and $\mathscr{F}_j$ for $\mathscr{F}_j(n,k,r,t)$.

Assume (W0). First we show (M1). Choose ε_0 from (W0). Since $w_{p_0}(\mathscr{G}_0) > w_{p_0}(\mathscr{G}_1)$ we may assume that $w_p(\mathscr{G}_0) > w_p(\mathscr{G}_1)$ for all p with $|p-p_0| < \varepsilon_0$ (if necessary we replace ε_0 so that this property holds). We can choose n_1 so that $|\mathscr{F}_0| > |\mathscr{F}_1|$ holds for all $n > n_1$ and k with $|\frac{k}{n} - p_0| < \varepsilon_0$. Then for the parameters chosen as above, we have $w^0(n,p,r,t) = w^1(n,p,r,t)$ and $m^0(n,k,r,t) = m^1(n,k,r,t)$. Thus (23) for the case $|\frac{k}{n} - p_0| < \varepsilon_1 := \frac{\varepsilon_0}{2}$ follows from Theorem 10 by setting $i = 1$. We will show (23) for $\frac{k}{n} \leq p_0 - \varepsilon_1$. Let $p = p_0 - \frac{\varepsilon_1}{2}$. Since $p < p_0$ and $w_p(\mathscr{G}_0) = p^t > w_p(\mathscr{G}_1)$ we can choose $\gamma_1 > 0$ so that

$$(1-2\gamma_1)p^t > w_p(\mathscr{G}_1(n,r,t)). \tag{24}$$

Then choose n_0 so that

$$\sum_{i \in J} \binom{n-t}{i-t} p^i (1-p)^{n-i} > p^t\,(1-2\gamma_1)/(1-\gamma_1) \tag{25}$$

holds for all $n > n_0$, where $J = n(p \pm \frac{\varepsilon_1}{2}) = ((p_0-\varepsilon_1)n, p_0 n) \cap \mathbb{N}$.

To show (23), suppose, on the contrary, that we can find some n,k and $\mathscr{F} \in \mathbf{Y}^0(n,k,r,t)$ with $|\mathscr{F}| \geq (1-\gamma_1)\binom{n-t}{k-t}$, where $n > n_1$ and $\frac{k}{n} \leq p_0 - \varepsilon_1$. We fix n,k and $\mathscr{F}$. Let $\mathscr{G} = \bigcup_{k \leq i \leq n}(\nabla_i(\mathscr{F})) \in \mathbf{X}^0(n,r,t)$ be the collection of all upper shadows of $\mathscr{F}$.

Claim 5. $|\nabla_i(\mathscr{F})| \geq (1-\gamma_1)\binom{n-t}{i-t}$ *for $i \in J$.*

Proof. Choose a real $x \leq n-t$ so that $(1-\gamma_1)\binom{n-t}{k-t} = \binom{x}{n-k}$. Since $|\mathscr{F}| \geq \binom{x}{n-k}$ the Kruskal–Katona Theorem implies that $|\nabla_i(\mathscr{F})| \geq \binom{x}{n-i}$. Thus it suffices to show that $\binom{x}{n-i} \geq (1-\gamma_1)\binom{n-t}{i-t}$, or equivalently,

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}} \geq \frac{(1-\gamma_1)\binom{n-t}{i-t}}{(1-\gamma_1)\binom{n-t}{k-t}}.$$

Using $i > (p_0-\varepsilon_1)n \geq k$ this is equivalent to $(i-t)\cdots(k-t+1) \geq (x-n+i)\cdots(x-n+k+1)$, which follows from $x \leq n-t$. $\square$

By the claim we have

$$w_p(\mathscr{G}) \geq \sum_{i\in J} |\nabla_i(\mathscr{F})|\, p^i(1-p)^{n-i} \geq (1-\gamma_1)\sum_{i\in J}\binom{n-t}{i-t} p^i(1-p)^{n-i}. \qquad (26)$$

By (25) and (24), the RHS of (26) is more than $(1-\gamma_1)\cdot p^t(1-2\gamma_1)/(1-\gamma_1) = p^t(1-2\gamma_1) > w_p(\mathscr{G}_1(n,r,t))$, which contradicts (W0). This completes the proof of (M1).

Next we show (W1). Let $\varepsilon_1 = \frac{\varepsilon_0}{2}$ and let $p \leq p_0 - \varepsilon_1$ be given. By (M1) we can find $\gamma_1 > 0$ and n_1 such that $m^0(n,k,r,t) < (1-\gamma_1)\binom{n-t}{k-t}$ holds for all $n > n_1$ and k with $\frac{k}{n} < p_0$. Choose $0 < \delta \ll \varepsilon_1$ so that $p \pm \delta \subset (0, p_0)$. Choose n_2 so that

$$(1-\gamma_1)\sum_{k\in J}\binom{n-t}{k-t} p^k q^{n-k} + \sum_{k\notin J}\binom{n}{k} p^k q^{n-k} < (1-\tfrac{\gamma_1}{2})p^t \qquad (27)$$

holds for all $n > n_2$, where $J = n(p\pm\delta)$. Let $n > \max\{n_1, n_2\}$ and choose $\mathscr{G} \in \mathbf{X}^0(n,r,t)$ with $w_p(\mathscr{G}) = w^0(n,p,r,t)$. Let $\mathscr{G}^{(k)} = \mathscr{G} \cap \binom{[n]}{k}$ for $k \in J$.

If $\mathscr{G}^{(k)} \in \mathbf{Y}^0(n,k,r,t)$ then we have $|\mathscr{G}^{(k)}| \leq m^0(n,k,r,t) < (1-\gamma_1)\binom{n-t}{k-t}$. If $\mathscr{G}^{(k)}$ fixes t vertices, say $[t]$, then $\tilde{\mathscr{G}}^{(k)} := \{G-[t] : G \in \mathscr{G}^{(k)}\}$ is $(r-1)$-wise 1-intersecting. (Otherwise $\mathscr{G}$ fixes $[t]$.) Thus we have $|\mathscr{G}^{(k)}| = |\tilde{\mathscr{G}}^{(k)}| \leq \binom{n-t-1}{k-t-1} = \frac{k-t}{n-t}\binom{n-t}{k-t} < p_0\binom{n-t}{k-t}$ by (2). Consequently, in both cases, we have

$$|\mathscr{G}^{(k)}| < (1-\gamma_1)\binom{n-t}{k-t}. \qquad (28)$$

Using (28) and (27), we have

$$w_p(\mathscr{G}) \leq \sum_{k\in J} |\mathscr{G}^{(k)}| p^k q^{n-k} + \sum_{k\notin J}\binom{n}{k} p^k q^{n-k} < (1-\tfrac{\gamma_1}{2})p^t,$$

and this is true for all $n \geq t$ by Lemma 6. This completes the proof of (W1). □

By Theorems 8, 10 and 11, we have the following.

Theorem 12. *Let $r \geq 4$. There exists n_1 such that*

$$m(n,k,r,t) = \max\{|\mathscr{F}_0(n,k,r,t)|, |\mathscr{F}_1(n,k,r,t)|\}$$

holds for all t with $1 \leq t \leq (3^r - 2r - 1)/2$, and for all $n > n_1$ and k with $\frac{k}{n} < \frac{1}{3} + \varepsilon$. Moreover $\mathscr{F}_0(n,k,r,t)$ and $\mathscr{F}_1(n,k,r,t)$ are the only possible optimal configurations (up to isomorphism).

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