Two-colorings with many monochromatic cliques in both colors

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Abstract

Color the edges of the n-vertex complete graph in red and blue, and suppose that red k-cliques are fewer than blue k-cliques. We show that the number of red k-cliques is always less than $c_k n^k$, where $c_k \in (0,1)$ is the unique root of the equation $z^k = (1-z)^k + kz(1-z)^{k-1}$. On the other hand, we construct a coloring in which there are at least $c_k n^k - O(n^{k-1})$ red k-cliques and at least the same number of blue k-cliques.

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1. Introduction

Let n and k be positive integers with $n > k \geq 3$, and let K_n denote the complete graph of order n. Let $\chi : E(K_n) \to \{\text{red}, \text{blue}\}$ be a two-coloring, and let $r_k(n,\chi)$ (resp. $b_k(n,\chi)$) be the number of red (resp. blue) K_k's (k-cliques). We will find a coloring which has many monochromatic k-cliques both red and blue. Namely, we want to know the asymptotic behavior of the following function:

$$f_k(n) := \max\{ \min\{r_k(n,\chi), b_k(n,\chi)\} : \chi \text{ is an edge two-coloring of } K_n\}.$$

Let μ_k be the unique real root in $(0,1)$-interval of the following equation

$$z^k = (1-z)^k + kz(1-z)^{k-1}. \tag{1}$$

Now we can state our main result.

Theorem 1. *For all $n > k \geq 3$ we have $f_k(n) < \frac{\mu_k^k}{k!} n^k$.*

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The above upper bound is asymptotically sharp as shown by the example below.

Example 1. *There is a coloring χ which gives $f_k(n) \geq \frac{\mu_k^k}{k!} n^k - O(n^{k-1})$.*

Construction. Let $V(K_n) = A \cup B$ be a partition with $|A| = (1-c)n$ and $|B| = cn$. Let χ be a coloring such that all edges in B are blue, and all the other edges are red. Then we can count the number of monochromatic k-cliques as follows:

$$\begin{aligned} b_k(n,\chi) &= \binom{|B|}{k} = \binom{cn}{k}, \\ r_k(n,\chi) &= \binom{|A|}{k} + \binom{|A|}{k-1}|B| = \binom{(1-c)n}{k} + \binom{(1-c)n}{k-1} cn. \end{aligned}$$

Suppose that the coloring is balanced. Namely, $b_k(n,\chi) = r_k(n,\chi) + O(n^{k-1})$, that is,

$$\frac{(cn)^k}{k!} = \frac{\big((1-c)n\big)^k}{k!} + \frac{\big((1-c)n\big)^{k-1}(cn)}{(k-1)!} + O(n^{k-1}),$$

or equivalently, $c^k = (1-c)^k + kc(1-c)^{k-1} + O(n^{-1})$. We are only interested in terms of order n^k, so c actually satisfies (1). Thus, by setting $c = \mu_k$, we have

$$f_k(n) \geq \min\{r_k(n,\chi), b_k(n,\chi)\} = \binom{\mu_k n}{k} + O(n^{k-1}) = \frac{\mu_k^k}{k!} n^k - O(n^{k-1})$$

as desired. □

In this paper we deal with $f_k(n)$. One can consider the opposite problem, namely, the problem to capture the following function:

$$\tilde{f}_k(n) := \min\{\max\{r_k(n,\chi), b_k(n,\chi)\} : \chi \text{ is an edge two-coloring of } K_n\}.$$

In this problem we want to find a balanced coloring with fewest possible monochromatic k-cliques. Erdős conjectured that the optimal one comes from a random coloring, that is,

$$\lim_{n\to\infty} \frac{\tilde{f}_k(n)}{\binom{n}{k}} = 2^{-\binom{k}{2}}.$$

This was known to be true for $k = 3$ by Goodman [3], but it turned out to be false for all $k \geq 4$ by Thomason, Franěk and Rödl [6, 1, 2].

One can also consider these problems in uniform hypergraphs or with more colors, which should be interesting and more difficult.

2. Proof

We outline the proof of Theorem 1. In §2.1 we show that the optimal coloring that gives $f_k(n)$ comes from a coloring defined on a unimodal sequence (Theorem 2). This observation enables us to translate the problem into a problem concerning a degree sequence of a graph. For this translation we use a result of Gale and Ryser (Theorem 3), which we will explain in §2.2. Then in §2.3 we restate the problem as a max-min problem of a function on Young diagrams (Theorem 4). We solve its continuous version (Theorem 5) in §2.4 and §2.5, then this implies our main result Theorem 1.

2.1. Unimodal sequences

Let $z = (z_1, \ldots, z_n)$ be a permutation of $V(K_n) = [n]$, where $[n] = \{1, 2, \ldots, n\}$. Define an edge two-coloring χ of K_n corresponding to z by

$$\chi(z_i z_j) := \begin{cases} \text{red} & \text{if } i < j \text{ and } z_i < z_j \\ \text{blue} & \text{if } i < j \text{ and } z_i > z_j. \end{cases} \tag{2}$$

A sequence $z = (z_1, \ldots, z_n)$ is called unimodal if there is some m such that

$$z_1 < z_2 < \cdots < z_m < z_{m+1} > z_{m+2} > \cdots > z_n. \tag{3}$$

The corresponding coloring to a unimodal sequence z satisfying (3) is as follows: Let $V(K_n) = V_r \cup V_b$ where $V_r = \{z_1, \ldots, z_m\}$ and $V_b = \{z_{m+1}, \ldots, z_n\}$. Color all pairs in V_r red and all pairs in V_b blue. The remaining pairs are the edges of a complete bipartite graph $K_{m,n-m}$ with partition $V_r \cup V_b$. These pairs are colored according to the rule (2). For example, the unimodal sequence corresponding to Example 1 is $z = (1, 2, 3, \ldots, m, n, n-1, \ldots, m+2, m+1)$, where $m = \mu n$.

Theorem 2. *$f_k(n)$ is given by a unimodal sequence of permutation of $[n]$.*

To prove Theorem 2, it is convenient to consider k-uniform hypergraphs. For a coloring χ of K_n, we assign a family of red k-cliques $\mathcal{F}_1 \subset \binom{[n]}{k}$ by

$$\mathcal{F}_1 = \{F \subset V(K_n) : |F| = k \text{ and } \chi(ij) = \text{red for all } i, j \in F\},$$

and a family of blue k-cliques $\mathcal{F}_2 \subset \binom{[n]}{k}$ similarly. Then we have

$$|F_1 \cap F_2| \le 1 \text{ for all } F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2. \tag{4}$$

Now we forget about the coloring χ for a while, and we will look at (not necessarily k-uniform) families $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ satisfying (4).

For a family $\mathcal{F} \subset 2^{[n]}$ and $a, b \in [n]$ the $S_{a,b}$ shift $S_{a,b}(\mathcal{F})$ is defined as follows. First for $F \in \mathcal{F}$ define

$$S_{a,b}(F) = \begin{cases} (F - \{b\}) \cup \{a\} & \text{if } b \in F, a \notin F \text{ and } ((F - \{b\}) \cup \{a\}) \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Then define $S_{a,b}(\mathcal{F}) = \{S_{a,b}(F) : F \in \mathcal{F}\}$. Obviously $|S_{a,b}(F)| = |F|$ and $|S_{a,b}(\mathcal{F})| = |\mathcal{F}|$ hold.

Lemma 1. *If $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ satisfy (4), then the same holds for $S_{a,b}(\mathcal{F}_1)$ and $S_{b,a}(\mathcal{F}_2)$.*

Proof. Suppose instead that $|S_{a,b}(F_1) \cap S_{b,a}(F_2)| \ge 2$ for some pair $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$. This cannot happen if $F_1 = S_{a,b}(F_1)$ and $F_2 = S_{b,a}(F_2)$, or if $F_1 \ne S_{a,b}(F_1)$ and $F_2 \ne S_{b,a}(F_2)$. So we may assume by symmetry that $F_1 = S_{a,b}(F_1)$ and $F_2 \ne S_{b,a}(F_2)$.

Since $|F_1 \cap F_2| \le 1$ and $|S_{a,b}(F_1) \cap S_{b,a}(F_2)| \ge 2$, we need $F_1 \cap \{a, b\} = \{b\}$ and $F_2 \cap \{a, b\} = \{a\}$. Since $F_1 = S_{a,b}(F_1)$, the element b is not removed by the shift $S_{a,b}$. By definition it means that $F_1' := (F_1 - \{b\}) \cup \{a\}$ was already in $\mathcal{F}_1$. Then $|F_1' \cap F_2| = |S_{a,b}(F_1) \cap S_{b,a}(F_2)| \ge 2$, which contradicts (4). □

Starting with two families $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$, and repeated simultaneous shifting in opposite directions we eventually come to a halt, that is, families $\mathcal{F}_1, \mathcal{F}_2$ with $S_{a,b}(\mathcal{F}_1) = \mathcal{F}_1$ and $S_{b,a}(\mathcal{F}_2) = \mathcal{F}_2$ holding for all $1 \le a < b \le n$. From now on, let $\mathcal{F}_1, \mathcal{F}_2 \subset \binom{[n]}{k}$, satisfying (4) and shifted (in opposite directions).

Lemma 2. *Let $\{x_1, \dots, x_k\} \in \mathcal{F}_1$ with $x_1 < \cdots < x_k$ and $\{y_1, \dots, y_k\} \in \mathcal{F}_2$ with $y_1 < \cdots < y_k$. Then we have $x_{k-1} < y_2$.*

Proof. Suppose instead that $y_2 \le x_{k-1}$. This implies $y_1 < x_{k-1}$ and $y_2 < x_k$. Let $A := \{x_1, \dots, x_{k-2}\}$. Recall from (4) that no member in $\mathcal{F}_1$ contains $\{y_1, y_2\}$, and in particular $A \not\supset \{y_1, y_2\}$. If $y_2 \in A$ then by shiftedness $\{y_1, y_2\} \subset A \cup \{y_1, x_k\} \in \mathcal{F}_1$, if $y_1 \in A$ then $\{y_1, y_2\} \subset A \cup \{y_2, x_k\} \in \mathcal{F}_1$, and if neither then $A \cup \{y_1, y_2\} \in \mathcal{F}_1$, a contradiction. □

Let $m_1 := \max\{x_{k-1} : x_1 < \cdots < x_k \text{ and } \{x_1, \dots, x_k\} \in \mathcal{F}_1\}$ and $m_2 := \min\{y_2 : y_1 < \cdots < y_k \text{ and } \{y_1, \dots, y_k\} \in \mathcal{F}_2\}$. By Lemma 2, we have $m_2 > m_1$.

Lemma 3. $m_2 > m_1 + 1$.

Proof. Suppose instead that $m_2 = m_1 + 1$. Then the defining sets $\{x_1, \dots, x_k\} \in \mathcal{F}_1$ and $\{y_1, \dots, y_k\} \in \mathcal{F}_2$ satisfy $y_1 \le m_2 - 1 = m_1$ and $x_k \ge m_1 + 1 = m_2$. Thus by shiftedness $\{x_1, \dots, x_{k-2}, m_1, m_2\} \in \mathcal{F}_1$ and $\{m_1, m_2, y_3, \dots, y_k\} \in \mathcal{F}_2$, contradicting (4). □

We call $p \in [n]$ a *peak for* $\mathcal{F}_1$ if for each $F_1 \in \mathcal{F}_1$, $p \in F_1$ implies that p is the largest element of F_1, and we call p a *peak for* $\mathcal{F}_2$ if for each $F_2 \in \mathcal{F}_2$, $p \in F_2$ implies that p is the smallest element of F_2. By the definition, if $p > m_1$ (resp. $p < m_2$) then p is a peak for $\mathcal{F}_1$ (resp. $\mathcal{F}_2$). We simply call p a *peak* if it is a peak for both $\mathcal{F}_1$ and $\mathcal{F}_2$. By Lemma 3 there is at least one peak, in fact, every p with $m_1 < p < m_2$ is a peak.

Let $\pi = (z_1, z_2, \dots, z_n)$ be a unimodal permutation of $[z]$, and let

$$\mathcal{G}_1(\pi) := \{\{j_1, \dots, j_k\} : j_1 < \cdots < j_k \text{ and } z_{j_1} < \cdots < z_{j_k}\},$$
$$\mathcal{G}_2(\pi) := \{\{j_1, \dots, j_k\} : j_1 < \cdots < j_k \text{ and } z_{j_1} > \cdots > z_{j_k}\}.$$

Then clearly $\mathcal{G}_1(\pi)$ and $\mathcal{G}_2(\pi)$ satisfy (4). Now we are ready to prove a structure result.

Lemma 4. *Let the (oppositely) shifted families $\mathcal{F}_1, \mathcal{F}_2 \subset \binom{[n]}{k}$ satisfy* (4)*. Then there is a unimodal permutation π such that $\mathcal{F}_i \subset \mathcal{G}_i(\pi)$ holds for $i = 1, 2$.*

Proof. We construct π inductively. First we fix a peak, say, $p \in [n]$ and define $z_p = n$. Suppose now that we have fixed $z_r, z_{r+1}, \dots, z_s$ $(r \le p \le s)$ forming an interval and the value set being $\{n, n-1, \dots, n-s+r\}$.

Claim 1. *At least one of $r-1$ and $s+1$ is a peak for the sets disjoint from $[r, s]$.*

Proof. Suppose that $r-1$ is not a peak for $\mathcal{F}_1$. Then by shiftedness $[k-2] \cup \{r-1, s+1\} \in \mathcal{F}_1$. Similarly, if $s+1$ is not a peak for $\mathcal{F}_2$, then $\{r-1, s+1\} \cup [n-k+3, n] \in \mathcal{F}_2$. Since these two sets intersect in 2 elements, at least one of them is missing. Thus we may suppose that $s+1$ is a peak for $\mathcal{F}_2$. If it was not a peak for $\mathcal{F}_1$, then by shiftedness $[k-2] \cup \{s+1, s+2\} \in \mathcal{F}_1$. However, by shiftedness we can replace $s+1$ by p and deduce $[k-2] \cup \{p, s+2\} \in \mathcal{F}_1$, contradicting the choice of p as a peak. This proves the claim. □

By the claim we can take $r-1$ or $s+1$ as a peak and define accordingly $z_{r-1} = n-s+r-1$ or $z_{s+1} = n-s+r-1$. Continuing in this way, eventually we obtain the desired unimodal permutation. This completes the proof of the lemma. $\square$

Now Theorem 2 immediately follows from Lemma 4.

2.2. The Gale–Ryser Theorem

To analyze the auxiliary complete bipartite graph defined in the second paragraph of the previous subsection, we will use the Gale–Ryser theorem on the degree sequence of bipartite graphs. To state their result we need some definitions.

Let $a = (a_1, \dots, a_s)$ and $c = (c_1, \dots, c_t)$ be non-increasing non-negative integer sequences satisfying $n = a_1 + \cdots + a_s = c_1 + \cdots + c_t$. It is sometimes convenient to consider infinite sequences, and for this purpose we let $a_i = 0$ for $i > s$ and $c_j = 0$ for $j > t$. Also we identify $(a_1, \dots, a_s)$ and $(a_1, \dots, a_s, 0, 0, \dots)$. We say that a is dominated by c, written $a \prec c$, if $\sum_{i=1}^k a_i \le \sum_{i=1}^k c_i$ for all $k \ge 1$. The sequence a determines a Young diagram. For example, the diagram for $a = (5, 4, 2, 1)$ is shown on the left in Figure 1. (In the diagram we read the length of rows from top to bottom.)

Figure 1: Young diagrams for $a = (5, 4, 2, 1)$ and $a^* = (4, 3, 2, 2, 1)$

By transposing the figure, we obtain its conjugate sequence a^*. If $a = (5, 4, 2, 1)$ then $a^* = (4, 3, 2, 2, 1)$. Formally, $a^* = (a_1^*, a_2^*, \dots)$ is defined by $a_j^* := \#\{j : a_i \ge j\}$. We notice that $a \prec c$ iff one can get the diagram for c from the diagram a by moving small squares in the direction of upper right. Then the Gale–Ryser theorem [5, 4] states as follows.

Theorem 3. *Let $a = (a_1, \dots, a_s)$ and $c = (c_1, \dots, c_t)$ be non-increasing non-negative integer sequences with the same sum. Then there is a bipartite graph G with partition $V(G) = A \cup C$ such that a and c are the degree sequences of A and C if and only if $a \prec c^*$.*

2.3. Packed degree sequences

We go back to our coloring problem. Let $s + t = n$ and let $z = (x_s, \dots, x_1, y_t, \dots, y_1)$ be a permutation of $V(K_n) = [n]$. Suppose that z is unimodal, that is,

$$x_s < x_{s-1} < \cdots < x_1 < y_t > y_{t-1} > \cdots > y_1.$$

Let $V_r = \{x_1, \dots, x_s\}$ and $V_b = \{y_1, \dots, y_t\}$ be the corresponding partition. Namely, the pairs in V_r are all red, and the pairs in V_b are all blue. We want to find the optimal s, t and coloring of a complete bipartite graph $K_{s,t}$ which gives $f_k(n)$ (and the goal is to show that the best construction comes from Example 1). Let χ be the desired a coloring. Then the number of red k-cliques in the K_n is

$$T_r := \binom{|V_r|}{k} + \sum_{y\in V_b} \binom{|N_{\text{red}}(y)|}{k-1}, \tag{5}$$

where $N_{\text{red}}(y)$ denotes the red neighborhood of y (in V_r). Similarly the number of blue k-cliques is

$$T_b := \binom{|V_b|}{k} + \sum_{x\in V_r} \binom{|N_{\text{blue}}(x)|}{k-1}. \tag{6}$$

Now we forget about all edges inside V_r and V_b, and we will only consider the edges of $K_{s,t}$. Let $a_i = \deg_{\text{blue}}(x_i)$ and $b_j = \deg_{\text{red}}(y_j)$. Then $a = (a_1, \dots, a_s)$ is the degree sequence of blue edges from V_r with $a_1 \geq a_2 \geq \dots \geq a_s$, and $b = (b_1, \dots, b_t)$ is the degree sequence of red edges from V_b with $b_1 \leq b_2 \leq \dots \leq b_t$. By (5) and (6), we have

$$\frac{1}{k!}\big(s^k + k\sum_{j=1}^{t} b_j^{k-1} - O(n^{k-1})\big) = T_r < \frac{1}{k!}\big(s^k + k\sum_{j=1}^{t} b_j^{k-1}\big), \tag{7}$$

and

$$\frac{1}{k!}\big(t^k + k\sum_{i=1}^{s} a_i^{k-1} - O(n^{k-1})\big) = T_b < \frac{1}{k!}\big(t^k + k\sum_{i=1}^{s} a_i^{k-1}\big). \tag{8}$$

Let $c_j = s - b_j$ for $1 \leq j \leq t$. Then $c = (c_1, \dots, c_t)$ is the degree sequence of blue edges from V_b with $c_1 \geq c_2 \geq \dots \geq c_t$. Notice that $a_1 + \dots + a_s = c_1 + \dots + c_t$, which is the number of blue edges in $K_{s,t}$. So, by Theorem 3, we have $a \prec c^*$.

We observe that if $a \prec \tilde{a}$ then $\sum a_i^{k-1} \leq \sum \tilde{a}_i^{k-1}$. (This is because a_i^{k-1} is a convex function. In fact if $i < j$ then we have $a_i^{k-1} + a_j^{k-1} \geq (a_i - 1)^{k-1} + (a_j + 1)^{k-1}$.) Suppose that b is fixed (and thus c is fixed too). In this situation, T_r is fixed and we want to maximize T_b, or equivalently, we want to maximize $\sum a_i^{k-1}$. Since $a \prec c^*$ we need to choose $a = c^*$. Then we can pack a and $b = (s, \dots, s) - c$ into an s by t rectangle. For example, if $s = 4, t = 5$ and $b = (0, 1, 2, 2, 3)$ then $c = (4, 4, 4, 4, 4) - b = (4, 3, 2, 2, 1)$ and $c^* = a = (5, 4, 2, 1)$ and the packing is shown in Figure 2. In the figure, a_i ($1 \leq i \leq s$) is the length of the ith row from the left side, and b_j ($1 \leq j \leq t$) is the height of the jth column from the bottom.

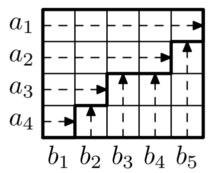

Figure 2: Packing $a = (5, 4, 2, 1)$ and $b = (0, 1, 2, 2, 3)$ into 4×5 rectangle

We restate our problem formally. Let $t \geq a_1 \geq a_2 \geq \dots \geq a_s \geq 0$ and $0 \leq b_1 \leq b_2 \leq \dots \leq b_t \leq s$. We say that $a = (a_1, \dots, a_s)$ and $b = (b_1, \dots, b_t)$ are packed each other if

$a = c^*$ where $c = (s - b_1, s - b_2, \ldots, s - b_t)$. Taking (7) and (8) into account, let us define

$$g_k(s,t) = \frac{1}{k!} \max \Big\{ \min \Big(s^k + k \sum_{j=1}^{t} b_j^{k-1},\ t^k + k \sum_{i=1}^{s} a_i^{k-1} \Big) :\ a \in \mathbb{N}^s \text{ and } b \in \mathbb{N}^t \text{ are packed} \Big\}$$

and $g_k(n) = \max\{g_k(s,t) : 1 \le s \le t \le n,\ s + t = n\}$. Evidently we have the following.

Theorem 4. $g_k(n) - O(n^{k-1}) \le f_k(n) < g_k(n)$.

For each s, t and $b = (b_1, \ldots, b_t)$ we can uniquely determine the packing partner $a = (a_1, \ldots, a_s)$. So our problem is to find optimal s, t and b which give $g_k(n)$.

2.4. A continuous version of $g_k(n)$

The rectangle in Figure 2 has a border which separates the sequences a and b. This border is a zigzag line connecting $(0,0)$ and $(5,4) = (t,s)$, and the line is monotone non-decreasing, and each segment is either a horizontal or vertical line of an integral length. Here we will consider similar borders, but as a length of horizontal or vertical line segments we allow any real numbers. To be more precise, we will define a border as follows. Let p and q be positive reals with $p + q = 1$. Let $x : [0,1] \to [0,q]$ and $y : [0,1] \to [0,p]$ be piecewise C^1 functions. We say that a zigzag line $\ell = \{(x(t), y(t)) : 0 \le t \le 1\}$ is a border if

- $x(0) = y(0) = 0$, $x(1) = q$ and $y(1) = p$, namely, ℓ connects $(0,0)$ and (q,p),
- $x'(t) \ge 0$ and $y'(t) \ge 0$ for all $t \in [0,1]$, namely, ℓ is non-decreasing, and
- $x'(t)y'(t) = 0$ for all $t \in [0,1]$, namely, each segment of ℓ is a horizontal or vertical line.

Let $\mathcal{L} = \mathcal{L}(q)$ be the set of borders, and let $\mathcal{L} = \bigcup_{i \ge 1} \mathcal{L}_i$ be a partition, where $\mathcal{L}_i$ denotes the set of borders having exactly i corners. For example, the border ℓ in Figure 2 has 4 right-up corners and 3 up-right corners, and it belongs to $\mathcal{L}_7$ (for $q = 5/9$, after shrinking so that the right-upper corner is $(5/9, 4/9)$). As a continuous version of $g(s,t)$ we define

$$h_k(q) = \frac{1}{k!} \max_{\ell \in \mathcal{L}} \min \big\{ p^k + kI_X(\ell),\ q^k + kI_Y(\ell) \big\}, \tag{9}$$

where $\ell = \{(x(t), y(t)) : 0 \le t \le 1\} \in \mathcal{L}$, and

$$I_X(\ell) = I_X := \int_0^1 (y(t))^{k-1} x'(t) dt, \quad I_Y(\ell) = I_Y := \int_0^1 (x(t))^{k-1} y'(t) dt.$$

Informally, if we write the border ℓ as $y = u(x)$, then $I_X(\ell) = \int_0^q (u(x))^{k-1} dx$ and $I_Y(\ell) = \int_0^p (u^{-1}(y))^{k-1} dy$. In particular, we have

$$I_X(\ell) \le qp^{k-1} \text{ for all } \ell \in \mathcal{L} \text{ with equality holding only if } u(x) \equiv p. \tag{10}$$

Also we have $I_Y(\ell) \le pq^{k-1}$ for all $\ell \in \mathcal{L}$. It follows from the definition (9) that

$$h_k(q) = h_k(1-q) = h_k(p). \tag{11}$$

We note that p, q and $h_k(q)$ are corresponding to s/n, t/n and $g_k(s,t)/n^k$, respectively. Indeed we have

$$f_k(n) < g_k(n) \le \max_{q\in(0,1)} h_k(q)n^k. \tag{12}$$

We will show the following continuous version of Theorem 1.

Theorem 5. *For all $k \ge 3$ we have*

$$\max_{q\in(0,1)} h_k(q) = \frac{\mu_k^k}{k!}.$$

The optimal borders are (I) $\ell \in \mathcal{L}(\mu_k)$ which has only one (up-right) corner at $(0, 1-\mu_k)$, and (II) $\ell \in \mathcal{L}(1-\mu_k)$ which has only one (right-up) corner at $(1-\mu_k, 0)$.

Then (12) and Theorem 5 immediately imply our main result Theorem 1. We notice that the border (I) in Theorem 5 consists of two line segments connecting $(0,0)$, $(0, 1-\mu_k)$ and $(\mu_k, 1-\mu_k)$, and this border corresponds to the construction in Example 1. On the other hand, the border (II) corresponds to the one obtained from the same construction by exchanging the role of red and blue.

Recall that μ_k is a root of (1), that is, $\theta(z) := z^k - (1-z)^k - kz(1-z)^{k-1} = 0$. Since $\theta'(z) = k(z^{k-1} + (k-1)z(1-z)^{k-2}) > 0$ for $z \in (0,1)$, and $\theta(1/2) < 0 < \theta(1)$ (in fact $\theta(1/2) = -k(1/2)^k$ and $\theta(1) = 1$), we have $\mu_k > 1/2$. Thus $1 - \mu_k < \mu_k$.

To prove Theorem 5 we distinguish two cases $q \in (1-\mu_k, \mu_k)$ and $q \in (0, 1-\mu_k] \cup [\mu_k, 1)$. First we deal with the latter case, which is easier. The extremal configurations come from only this latter case.

Lemma 5. *If $q \in (0, 1-\mu_k] \cup [\mu_k, 1)$ then $\max_q h_k(q) = \mu_k^k/k!$. The optimal borders are (I) and (II) in Theorem 5.*

Proof. First suppose that $q \ge \mu_k$. Since $\theta(z)$ is monotone increasing and $\theta(\mu_k) = 0$, it follows that $\theta(q) \ge 0$, that is, $q^k \ge p^k + kqp^{k-1}$. On the other hand, using (10), we have $p^k + kI_X(\ell) \le p^k + kqp^{k-1} (\le q^k \le q^k + kI_Y(\ell))$ for any $\ell \in \mathcal{L}$. This gives

$$\max_{\ell} \min\{p^k + kI_X(\ell),\, q^k + kI_Y(\ell)\} = \max_{\ell}\{p^k + kI_X(\ell)\} \le p^k + kqp^{k-1} =: H(q).$$

We have $H(\mu_k) = (1-\mu_k)^k + k\mu_k(1-\mu_k)^{k-1} = \mu_k^k$ by (1). Since $H'(q) = -k(k-1)qp^{k-2} < 0$ we have $H(q) \le H(\mu_k) = \mu_k^k$, and

$$h_k(q) = \frac{1}{k!}\max_{\ell} \min\{p^k + kI_X(\ell),\, q^k + kI_Y(\ell)\} \le \frac{H(q)}{k!} \le \frac{H(\mu_k)}{k!} = \frac{\mu_k^k}{k!}.$$

Moreover, $h_k(q) = \mu_k^k/k!$ iff $H(q) = H(\mu_k) = p^k + kI_X(\ell)$. Then we have $q = \mu_k$ and $H(\mu_k) = \mu_k^k = (1-\mu_k)^k + k\mu_k(1-\mu_k)^{k-1} = p^k + kqp^{k-1}$. Thus we have $I_X(\ell) = qp^{k-1}$ and we can conclude from (10) that the border ℓ is type (I) in this case.

Next suppose that $q \le 1 - \mu_k$. Using the symmetry (11) we obtain the desired inequality and the corresponding border is type (II) in this case. □

From now on we assume that

$$q \in (1-\mu_k, \mu_k) \tag{13}$$

unless otherwise explicitly stated. Then we will show that $h_k(q) < \mu_k^k/k!$. Thus the extremal configurations will not appear in this range. In this sense the case (13) is less interesting, but the proof is somewhat more involved, though we will use elementary calculus only.

First we will show that an optimal border giving $h_k(q)$ has at most two corners (one right-up corner and one up-right corner). In other words, the border divides the $q \times p$ rectangle into two rectangles (possibly one of them is empty). (This fact is true for all $q \in (0,1)$ not only for (13).)

Lemma 6. *Let $n \geq 3$. For every $\ell \in \mathcal{L}_n$ there is an $\ell' \in \mathcal{L}_{n'}$ with $n' < n$ such that*

$$I_X(\ell') > I_X(\ell) \text{ and } I_Y(\ell') > I_Y(\ell). \tag{14}$$

Proof. Let $\ell \in \mathcal{L}_n$ be given. Let $C_0 = (x_0, y_0)$ and let C_1, C_2, and C_3 be three consecutive corners on the border ℓ. Consider the case when $C_1 = (x_1, y_0)$ and $C_3 = (x_2, y_1)$ are right-up corners. Then $C_2 = (x_1, y_1)$ is an up-right corner, and $C_4 = (x_2, y_2)$ could be also an up-right corner or $C_4 = (q, p)$, see Figure 3, left. (One can deal with the case that C_1 and C_3 are up-right corners in a similar way and we omit this case.) Notice that $0 \leq x_0 < x_1 < x_2 \leq q$ and $0 \leq y_0 < y_1 < y_2 \leq p$.

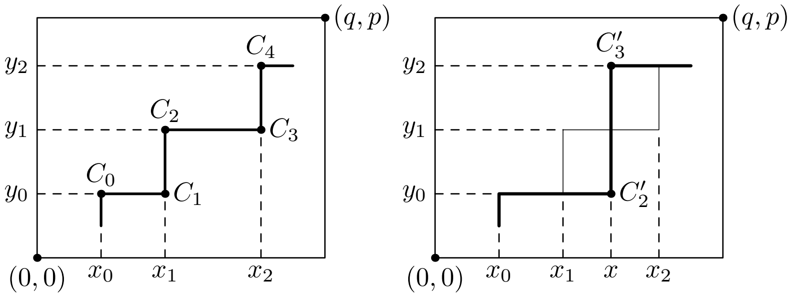

Figure 3: A border having two right-up corners and a new border

Choose $x_1 < x < x_2$ and let $C_2' = (x, y_0)$ and $C_3' = (x, y_2)$. Now we construct a new border $\ell' = \ell'(x)$ from ℓ by replacing a path $C_1C_2C_3C_4$ with a path $C_1C_2'C_3'C_4$, see Figure 3, right. Let $m = k - 1$. Then we have

$$I_X(\ell') = I_X(\ell) - (x - x_1)(y_1^m - y_0^m) + (x_2 - x)(y_2^m - y_1^m).$$

Thus $I_X(\ell') = I_X(\ell)$ is equivalent to

$$x = \frac{x_1(y_1^m - y_0^m) + x_2(y_2^m - y_1^m)}{y_2^m - y_0^m} =: \xi. \tag{15}$$

Since $I_X(\ell')$ is a decreasing function of x we have that

$$x < \xi \text{ implies } I_X(\ell') > I_X(\ell). \tag{16}$$

Similarly we have

$$I_Y(\ell') = I_Y(\ell) + (y_1 - y_0)(x^m - x_1^m) - (y_2 - y_1)(x_2^m - x^m),$$

and $I_Y(\ell') = I_Y(\ell)$ holds iff

$$x = \left(\frac{x_1^m(y_1 - y_0) + x_2^m(y_2 - y_1)}{y_2 - y_0} \right)^{1/m} =: \xi'. \tag{17}$$

Since $I_Y(\ell')$ is an increasing function of x we have that

$$x > \xi' \text{ implies } I_Y(\ell') > I_Y(\ell). \tag{18}$$

Therefore, by (16) and (18), if $\xi' < \xi$, then for all $x \in (\xi', \xi)$ we get (14). But $\xi' \geq \xi$ can, in fact, happen.

Next we choose $y_0 < y < y_1$ and let $C_1'' = (x_0, y)$ and $C_2'' = (x_2, y)$. We construct yet another border $\ell'' = \ell''(y)$ from ℓ by replacing a path $C_0C_1C_2C_3$ with a path $C_0C_1''C_2''C_3$. Then we have

$$I_Y(\ell'') = I_Y(\ell) - (y - y_0)(x_1^m - x_0^m) + (y_1 - y)(x_2^m - x_1^m),$$

and $I_Y(\ell'') = I_Y(\ell)$ iff

$$y = \frac{y_0(x_1^m - x_0^m) + y_1(x_2^m - x_1^m)}{x_2^m - x_0^m} =: \eta \in (y_0, y_1). \tag{19}$$

Since $I_Y(\ell'')$ is a decreasing function of y we have that

$$y < \eta \text{ implies } I_Y(\ell'') > I_Y(\ell).$$

Similarly we have

$$I_X(\ell'') = I_X(\ell) + (x_1 - x_0)(y^m - y_0^m) - (x_2 - x_1)(y_1^m - y^m),$$

and $I_X(\ell'') = I_X(\ell)$ iff

$$y = \left(\frac{y_0^m(x_1 - x_0) + y_1^m(x_2 - x_1)}{x_2 - x_0} \right)^{1/m} =: \eta'. \tag{20}$$

Since $I_X(\ell'')$ is an increasing function of y we have that

$$y > \eta' \text{ implies } I_X(\ell'') > I_X(\ell).$$

Therefore if $\eta' < \eta$, then by choosing $y \in (\eta', \eta)$ we get (14). But $\eta' \geq \eta$ can happen.

We may assume that $x_0 = y_0 = 0$. Then we are able to show that one of $\xi' < \xi$ and $\eta' < \eta$ necessarily holds in a slightly more general setting as in the next Lemma 7, which will complete the proof of this Lemma 6. □

Let $x_0 = y_0 = 0$, $s := x_1/x_2$ and $t := y_1/y_2$. Then we have

$$0 < s < 1, \quad 0 < t < 1. \tag{21}$$

We can rewrite (15) and (19) as

$$\xi/x_2 = st^m + 1 - t^m, \quad \eta/y_1 = 1 - s^m,$$

and similarly (17) and (20) as

$$\xi'/x_2 = (s^m t + 1 - t)^{1/m}, \quad \eta'/y_1 = (1-s)^{1/m}.$$

Thus $\xi > \xi'$ and $\eta > \eta'$ are equivalent to

$$(st^m + 1 - t^m)^m > s^m t + 1 - t, \text{ and} \tag{22}$$
$$(1 - s^m)^m > 1 - s. \tag{23}$$

Now to complete the proof of Lemma 6 it suffices to show the following. (In the proof of Lemma 6, $m = k - 1$ was an integer, but in the next lemma m is not necessarily an integer. See also Theorem 6.)

Lemma 7. *Let m, s, t be reals with $m > 1$ and (21). Then one of the two inequalities (22) and (23) holds.*

We defer a rather technical proof of the above lemma in the next subsection. Lemma 6 immediately gives the following.

Lemma 8. *Let $q \in (0, 1)$. Then $h_k(q)$ is attained by a border $\ell \in \mathcal{L}_1 \cup \mathcal{L}_2$.*

Thus we may assume that the optimal border giving $h_k(q)$ is either (i) a curve $\ell_H(\alpha)$ with only one horizontal line $y = \alpha$ $(0 \le \alpha \le p)$, or (ii) a curve $\ell_V(\beta)$ with only one vertical line $x = \beta$ $(0 \le \beta \le q)$. (We see $\ell_H(p) = \ell_V(0)$.)

First consider the case (i). In this case we have $I_X(\ell_H(\alpha)) = q\alpha^{k-1}$ and $I_Y(\ell_H(\alpha)) = (p - \alpha)q^{k-1}$. Let

$$\tilde{F}(q, \alpha) := \min\{p^k + kI_X(\ell_H(\alpha)), q^k + kI_Y(\ell_H(\alpha))\},$$

and we will find $\alpha = \alpha(q)$ which maximize $\tilde{F}(q, \alpha)$ for given $q \in [1 - \mu_k, \mu_k]$. Recall that $\theta(z)$ is monotone increasing and $\theta(\mu_k) = 0$. Thus $1 - q \le \mu_k$ gives $\theta(1 - q) = \theta(p) \le 0$, that is, $p^k \le q^k + kpq^{k-1}$. Similarly it follows from $q \le \mu_k$ that $\theta(q) \le 0$, that is, $p^k + kqp^{k-1} \ge q^k$. We also notice that

- $I_X(\ell_H(\alpha))$ is an increasing function of α,
- $I_Y(\ell_H(\alpha))$ is a decreasing function of α,
- $p^k + kI_X(\ell_H(0)) = p^k \le q^k + kpq^{k-1} = q^k + kI_Y(\ell_H(0))$, and
- $p^k + kI_X(\ell_H(p)) = p^k + kqp^{k-1} \ge q^k = q^k + kI_Y(\ell_H(p))$.

Thus there exists a unique point $\alpha = \alpha_k \in [0, p]$ such that

$$p^k + kI_X(\ell_H(\alpha_k)) = q^k + kI_Y(\ell_H(\alpha_k)),$$

and we can define

$$F(q) := p^k + kq\alpha_k^{k-1} = q^k + k(p - \alpha_k)q^{k-1}. \tag{24}$$

Then $F(q) = \max_{0 \le \alpha \le p} \tilde{F}(q, \alpha)$. By the definition of $\alpha_k = \alpha_k(q)$, this is a function of $q \in [1 - \mu_k, \mu_k]$ with $\alpha_k(1 - \mu_k) = 0$ and $\alpha_k(\mu_k) = 1 - \mu_k$.

Next we consider the case (ii). Similarly as above there exists a unique point $\beta = \beta_k \in [0, p]$ such that $p^k + kI_X(\ell_V(\beta_k)) = q^k + kI_Y(\ell_V(\beta_k))$, and we can define

$$G(q) := p^k + k(q - \beta_k)p^{k-1} = q^k + kp\beta_k^{k-1}, \tag{25}$$

which gives $\max_{0\le\beta\le q} \min\{p^k + kI_X(\ell_V(\beta)), q^k + kI_Y(\ell_H(\beta))\}$. Note that $\beta_k = \beta_k(q)$ satisfies $\beta_k(1 - \mu_k) = 1 - \mu_k$ and $\beta_k(\mu_k) = 0$.

Here we list some properties about functions defined above. We defer the purely analytic proof of these properties in the next subsection.

Lemma 9.

$$\alpha_k(1 - q) = \beta_k(q) \text{ and } F(1 - q) = G(q). \tag{26}$$

$$G(q) < F(q) \quad \text{for } q \in (1/2, \mu_k). \tag{27}$$

$$\tfrac{d}{dq}F(q) > 0 \quad \text{for } q \in [1/2, \mu_k). \tag{28}$$

The following result is a complement of Lemma 5, which easily follows from Lemma 9.

Lemma 10. *If $q \in (1 - \mu_k, \mu_k)$ then $h_k(q) < \mu_k^k/k!$.*

Proof. By the symmetry (11) it suffices to show $k!h_k(q) < \mu_k^k$ for $q \in [1/2, \mu_k)$. So let $1/2 \le q < \mu_k$. Using $\alpha_k(\mu_k) = 1 - \mu_k$ and (1) we have $F(\mu_k) = (1 - \mu_k)^k + k\mu_k(1 - \mu_k)^{k-1} = \mu_k^k$. By (28) we have $F(q) < F(\mu_k) = \mu_k^k$. We also have $F(1/2) = G(1/2)$ by (26), and so $G(q) \le F(q)$ by (27). Therefore we have

$$k!h_k(q) = \max\{F(q), G(q)\} = F(q) < F(\mu_k) = \mu_k^k$$

as desired. □

Now Theorem 5 follows from Lemma 5 and Lemma 10. Finally our main result Theorem 1 follows from (12) with Theorem 5.

2.5. Proof of Lemma 7 and Lemma 9

Let $m > 1$ be a real. We define the following two functions the unit interval $[0, 1]$:

$$\phi_m(x) := (1 - x^m)^m - (1 - x) \quad \text{and} \quad \psi_m(x) := x^m + x - 1.$$

These functions will play an important role for our proofs below. Let $\nu_m \in (0, 1)$ be the unique real root of the equation $\psi_m(x) = 0$.

Claim 2.

$$\phi_m(x) > 0 \quad \text{iff } 0 < x < \nu_m. \tag{29}$$

$$2\alpha_k(1/2) = \nu_{k-1}. \tag{30}$$

$$\nu_{k-1} < \alpha_k(q)/p < 1 \quad \text{for } q \in (1/2, \mu_k). \tag{31}$$

$$\nu_m > m^{-\frac{1}{m-1}}. \tag{32}$$

Proof. We have

$$\phi_m(0) = \phi_m(1) = \phi_m(\nu_m) = 0. \tag{33}$$

We used $\phi_m(\nu_m) = (1-\nu_m^m)^m - (1-\nu_m) = \nu_m^m - (1-\nu_m) = \psi_m(\nu_m)$ for the last equality. Since

$$\phi_m'(x) = m(1-x^m)^{m-1}(-mx^{m-1}) + 1 = -m^2\{x(1-x^m)\}^{m-1} + 1$$

we have $\phi_m'(x) \geq 0$ if $x(1-x^m) \leq (1/m^2)^{\frac{1}{m-1}}$ (we need $m > 1$ here), that is, if

$$\Phi(x) := x^{m+1} - x + (1/m^2)^{\frac{1}{m-1}} \geq 0.$$

Then $\Phi'(x) = (m+1)x^m - 1$ gives $\Phi'(x) = 0$ if $x = \tilde{\nu} := (\frac{1}{m+1})^{\frac{1}{m}} \in (0,1)$. Moreover, $\Phi'(x) < 0$ for $0 < x < \tilde{\nu}$, and $\Phi'(x) > 0$ for $\tilde{\nu} < x < 1$. If $\Phi(\tilde{\nu}) \geq 0$ then $\Phi(x) > 0$ and thus $\phi_m'(x) > 0$ for all $x \in (0,1)$. So $\phi_m(x)$ is monotone increasing, contradicting (33). Consequently we must have $\Phi(\tilde{\nu}) < 0$. Then $\phi_m(x)$ is increasing-decreasing-increasing in this order in $(0,1)$-interval, and $\phi_m(x)$ satisfies (33) as well. This shows (29).

By (24) we have $\frac{1}{k}(F(1/2) - (1/2)^k) = \frac{1}{2}\alpha_k(1/2)^{k-1} = (\frac{1}{2} - \alpha_k(1/2))(\frac{1}{2})^{k-1}$, that is,

$$\psi_{k-1}(2\alpha_k(1/2)) = (2\alpha_k(1/2))^{k-1} + 2\alpha_k(1/2) - 1 = 0.$$

This gives (30).

Let $m = k-1$. By differentiating both sides of (24) with respect to q and rearranging, we have

$$A\alpha_k' = B, \tag{34}$$

where $A = mq\alpha_k^{m-1} + q^m$, $B = (p^m - \alpha_k^m) + m(p - \alpha_k)q^{m-1}$, and $\alpha_k' = \frac{d\alpha_k}{dq}$. Since $A > 0$ and $B > 0$, $\alpha_k(q)$ is an increasing function of q. Thus, using (30), we have $\nu_{k-1} = \alpha_k(1/2)/(1/2) < \alpha_k(q)/p < \frac{\alpha_k(\mu_k)}{\mu_k} = \frac{1-\mu_k}{\mu_k} < 1$, which proves (31).

Finally we show (32). By a direct computation one can verify that $(1+\frac{1}{m})^{m-1}/m$ is a decreasing function for $m > 1$ with supremum 1. This gives $(1+\frac{1}{m})^{m-1} < m$, or, $1 + \frac{1}{m} - m^{\frac{1}{m-1}} < 0$. Then we have

$$\psi_m(m^{-\frac{1}{m-1}}) = m^{-\frac{m}{m-1}} + m^{-\frac{1}{m-1}} - 1 = m^{-\frac{1}{m-1}}\left(\frac{1}{m} + 1 - m^{\frac{1}{m-1}}\right) < 0.$$

This implies (32) because $\psi_m(q)$ is an increasing function of q with $\psi_m(\nu_m) = 0$. □

Proof of Lemma 7. It follows from (29) that (23) holds for $0 < s < \nu_m$. So from now on we assume that

$$\nu_m \leq s < 1, \quad 0 < t < 1 \tag{35}$$

and we will show (22), or equivalently, we will show $f(s,t) > 0$, where

$$f(s,t) := (st^m + 1 - t^m)^m - (s^m t + 1 - t).$$

If $f(s,t)$ takes a minimal value in the following open set

$$\nu_m < s < 1, \quad 0 < t < 1,$$

then $\frac{\partial f}{\partial s} = 0$ must hold. By a direct computation we have

$$\frac{\partial f}{\partial s} = mt\{(t(st^m + 1 - t^m))^{m-1} - s^{m-1}\} = mts^{m-1}\left(\left(\frac{t(st^m + 1 - t^m)}{s}\right)^{m-1} - 1\right).$$

Now $\frac{t(st^m+1-t^m)}{s} = t(t^m + \frac{1-t^m}{s})$ is a decreasing function of s, and $\frac{\partial f}{\partial s}|_{s=1} < 0$. This means that (by increasing s for fixed t) f is either monotone decreasing, or it is increasing-decreasing in this order. In neither case does $f(s,t)$ take a minimal value (as a function of s).

Therefore, the minimum value of $f(s,t)$ in the following compact set

$$\nu_m \le s \le 1, \quad 0 \le t \le 1$$

can be attained only on the boundaries. Now we look at the boundaries. It is easy to see

$$f(1,t) = f(s,0) = f(s,1) = 0.$$

As for $f(\nu_m, t)$, using $\nu_m^m = 1 - \nu_m$, we have

$$\begin{aligned} \nu_m t^m + 1 - t^m &= 1 - (1-\nu_m)t^m = 1 - (\nu_m t)^m, \\ \nu_m^m t + 1 - t &= (1-\nu_m)t + 1 - t = 1 - (\nu_m t), \end{aligned}$$

and finally we get

$$\begin{aligned} f(\nu_m, t) &= (\nu_m t^m + 1 - t^m)^m - (\nu_m^m t + 1 - t) \\ &= (1 - (\nu_m t)^m)^m - (1 - (\nu_m t)) = \phi_m(\nu_m t). \end{aligned}$$

Then, by (29), we have $\phi_m(\nu_m t) > 0$ for $0 < t < 1$. This shows $f(s,t) > 0$ for (35), which completes the proof of Lemma 7. □

Proof of Lemma 9. Fix k and let $m = k - 1$.

From Eqs. (24) and (25), we have (26).

To prove (27), suppose, to the contrary, that $F(q) \le G(q)$ for some $q \in (1/2, \mu_k)$. Then, from (24) and (25), we have

$$q\alpha_k^m \le (q - \beta_k)p^m \text{ and } (p - \alpha_k)q^m \le p\beta_k^m.$$

Let $x = \beta_k(q)/q$ and $y = \alpha_k(q)/p$. Then the above inequalities reduce to

$$y^m \le 1 - x \ \text{ and } 1 - y \le x^m,$$

which give $1 - y \le x^m \le (1 - y^m)^m$. This means $y \le \nu_m$ by (29), which contradicts (31).

Finally we prove (28). From (24) we have

$$F'(q)/k = -(p^m - \alpha_k^m) + mq\alpha_k^{m-1}\alpha_k'. \tag{36}$$

It follows from (34) and (36) that

$$A(F'(q)/k) = -A(p^m - \alpha_k^m) + mq\alpha_k^{k-2}B = q^m C,$$

where $C = m^2(p - \alpha_k)\alpha_k^{m-1} - (p^m - \alpha_k^m)$. We show that $C > 0$ which will give $F'(q) > 0$. Let $y = \alpha_k(q)/p$. By (31) and (32) we have $1 - y > 0$ and $my^{m-1} - 1 \ge m\nu_m^{m-1} - 1 > 0$, and

$$\begin{aligned} \frac{C}{p^m} &= m^2(1-y)y^{m-1} - (1 - y^m) = (1-y)\left(m^2 y^{m-1} - \frac{1-y^m}{1-y}\right) \\ &= m(1-y)\left\{my^{m-1} - \tfrac{1}{m}(1 + y + \cdots + y^{m-1})\right\} > m(1-y)(my^{m-1} - 1) > 0, \end{aligned}$$

as needed. □

We mention that Lemma 7 can be extended as follows.

Theorem 6. *Let* $\ell = \{(x(t), y(t)) : 0 \le t \le 1\}$ *be a monotone increasing curve connecting* $(0,0)$ *and* $(1,1)$*, where* x *and* y *are monotone increasing piecewise* C^1 *functions from* $[0,1]$ *to* $[0,1]$*. Then, for all reals* $m > 1$*, one of the following inequalities holds:*

$$\int_0^1 y\, d(x^m) \ge \left(\int_0^1 y^m\, dx\right)^{1/m} \quad \text{and} \quad \int_0^1 x\, d(y^m) \ge \left(\int_0^1 x^m\, dy\right)^{1/m}.$$

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Note added in proof

After this paper was written we learned that H. Huang, N. Linial, H. Naves, Y. Peled and B. Sudakov obtained closely related results independently of our work.

References

[1] F. Franěk. On Erdős's conjecture on multiplicities of complete subgraphs: lower upper bound for cliques of size 6. Combinatorica 22 (2002) 451–454.

[2] F. Franěk, V. Rödl. 2-colorings of complete graphs with a small number of monochromatic K_4 subgraphs. Combinatorics and algorithms (Jerusalem, 1988). Discrete Math. 114 (1993) 199–203.

[3] A. W. Goodman. On sets of acquaintances and strangers at any party. Amer. Math. Monthly 66 1959 778–783.

[4] M. Krause. A simple proof of the Gale–Ryser theorem. Amer. Math. Monthly 103 (1996) 335–337.

[5] H. J. Ryser. "The Class $\mathcal{A}(R, S)$." Combinatorial Mathematics. Buffalo, NY: Math. Assoc. Amer. pp. 61–65, 1963.

[6] A. Thomason. A disproof of a conjecture of Erdős in Ramsey theory. J. London Math. Soc. (2) 39 (1989) 246–255.

Appendix

Proof of Theorem 6. We only include a sketch of a proof. Let $I_X(\ell) := \int_0^1 y^m dx$ and $J_X(\ell) := \int_0^1 yd(x^m)$, and let $I_Y(\ell), J_Y(\ell)$ be defined similarly. We will show that either

$$J_X(\ell) \ge I_X(\ell)^{1/m} \text{ or } J_Y(\ell) \ge I_Y(\ell)^{1/m}$$

holds. It follows from integration by parts that

$$J_X(\ell) = 1 - I_Y(\ell) \text{ and } J_Y(\ell) = 1 - I_X(\ell). \tag{37}$$

First we consider the case that ℓ is a step-function. Let $\mathcal{L}_n$ be the set of increasing step-functions connecting $(0,0)$ and $(1,1)$, and having exactly n corners. If $\ell \in \mathcal{L}_1 \cup \mathcal{L}_2$, then a direct computation shows that both $J_X(\ell) = I_X(\ell)^{1/m}$ and $J_Y(\ell) = I_Y(\ell)^{1/m}$ hold.

Let $n \ge 3$ and we will show a slightly stronger inequality (without equality). Namely, we prove the following statement by induction on n:

if $\ell \in \mathcal{L}_n$ $(n \geq 3)$, then either $J_X(\ell) > I_X(\ell)^{1/m}$ or $J_Y(\ell) > I_Y(\ell)^{1/m}$ holds.

The initial step $n = 3$ is Lemma 7 itself. Now let $n > 3$ and we proceed the induction step for n. Suppose the contrary. Then there is an $\ell \in \mathcal{L}_n$ such that both

$$J_X(\ell) \leq I_X(\ell)^{1/m} \text{ and } J_Y(\ell) \leq I_Y(\ell)^{1/m} \tag{38}$$

hold. By Lemma 6 there is an $\ell' \in \mathcal{L}_{n'}$ with $n' < n$ such that

$$I_X(\ell') > I_X(\ell) \text{ and } I_Y(\ell') > I_Y(\ell). \tag{39}$$

By (37) and (39) we have

$$J_X(\ell') = 1 - I_Y(\ell') < 1 - I_Y(\ell) = J_X(\ell).$$

Then (38) and (39) give

$$J_X(\ell) \leq I_X(\ell)^{1/m} < I_X(\ell')^{1/m}.$$

Thus we have $J_X(\ell') < I_X(\ell')^{1/m}$. Similarly we also get $J_Y(\ell') < I_Y(\ell')^{1/m}$. But these two inequalities contradict the induction hypothesis.

A general curve ℓ can be approximated by step-functions, and one can show the inequalities for general case based on a result for step-functions in a standard way. □