BOLYAI SOCIETY
MATHEMATICAL STUDIES, 3

**Extremal Problems for Finite Sets**
**Visegrád (Hungary), 1991,** pp. 229–250.

# The Kruskal-Katona Theorem, some of its Analogues and Applications

P. FRANKL and N. TOKUSHIGE

## 1. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of all positive integers. Set $[i] = \{1, 2, \ldots, i\}$. For a family $\mathcal{F} = \{F_1, \ldots, F_m\}$, consisting of finite subsets of $\mathbb{N}$ let $\sigma(\mathcal{F})$ denote the *shadow* of $\mathcal{F}$, that is

$$\sigma(\mathcal{F}) = \{G \subset \mathbb{N} : \exists F \in \mathcal{F},\ G \subset F\}.$$

Set also

$$\sigma_i(\mathcal{F}) = \{G \in \sigma(\mathcal{F}) : |G| = i\}.$$

We use the notation

$$\binom{\mathbb{N}}{k} = \{H \subset \mathbb{N} : |H| = k\}.$$

If $\mathcal{F} \subset \binom{\mathbb{N}}{k}$, then $\mathcal{F}$ is called *k-uniform.* A family $\mathcal{F}$ is called a *complex* if $\sigma(\mathcal{F}) = \mathcal{F}$ hold. The $f$-vector $f = (f_0, f_1, \ldots, f_k, \ldots)$ of a family $\mathcal{F}$ is defined by

$$f_i = \left|\mathcal{F} \cap \binom{\mathbb{N}}{i}\right|.$$

Set also $\mathcal{F}^{(i)} = \mathcal{F} \cap \binom{\mathbb{N}}{i}$.

Note that this differs slightly from the usual definition in combinatorial topology. An important problem in combinatorial topology is the characterisation of all possible $f$-vectors of complexes. This is done by the — by now classical — Kruskal-Katona Theorem. We shall present it together with (one of) its proof in the next section. For the proof we use the following operation, called *shifting* and first defined by Erdős-Ko-Rado [7]. Given a family $\mathcal{F}$ and integers $1 \le i \le j \le n$ define

$$S_{ij}(F) = \begin{cases} F' = (F - \{j\}) \cup \{i\} & \text{if } j \in F,\ i \notin F \text{ and } F' \notin \mathcal{F} \\ F & \text{otherwise .} \end{cases}$$

Set

$$S_{i,j}(\mathcal{F}) = \{S_{i,j}(F) : F \in \mathcal{F}\}.$$

Note that $f$-vectors of $\mathcal{F}$ and $S_{ij}(\mathcal{F})$ are identical.

**Proposition 1.1.** (Katona [20]) *If $\mathcal{F}$ is a complex then $S_{ij}(\mathcal{F})$ is a complex, too.*

**Proof.** Let $G \subset H$, $|G| = |H| - 1$, $H \in S_{ij}(\mathcal{F})$. Clearly, it is sufficient to prove, that for all such pairs, $G \in S_{ij}(\mathcal{F})$ holds. By definition $H = S_{ij}(F)$ holds for some $F \in \mathcal{F}$. We consider some cases according $F \cap \{i,j\}$.

(i) $\{i,j\} \subset F$

In this case $F = S_{ij}(F)$ and $F - \{i\}$, $F - \{j\}$ are both proper subsets of $F$. Consequently, $S_{ij}(F - \{i\}) = F - \{i\}$.

We infer $G \in S_{ij}(\mathcal{F})$ from this and the fact that $\mathcal{F}$ is a complex.

(ii) $j \notin F$

In this case $S_{ij}(E) = E$ holds for all subsets $E$ of $F$. In particular, $G \in S_{ij}(\mathcal{F})$.

(iii) $j \in F$, $i \notin F$ but $F = H$.

From the definition of $S_{ij}$ it follows that $F' = (F - \{j\}) \cup \{i\}$ satisfies $F' \in \mathcal{F}$ and consequently $S_{ij}(E) = E$ for all subsets $E$ of $F$. In particular, $G \in S_{ij}(\mathcal{F})$.

(iv) $j \in F$, $i \notin F$, $S_{ij}(F) = H = F'$

Define $x$ by $G = F' - \{x\}$. If $x \ne i$ then $j \in F$ and the definition of $S_{ij}$ implies $G \in S_{ij}(\mathcal{F})$ in view of $(F - \{x\}) \in \mathcal{F}$.

Finally, if $G = H - \{i\}$ then $G = F \cap F'$. Thus $G \in \mathcal{F}$ and $S_{ij}(G) = G \in S_{ij}(\mathcal{F})$ follow. ■

For an integer $i$ and for a family $\mathcal{F}$ we define

$\mathcal{F}(i) = \{F - \{i\} : i \in F \in \mathcal{F}\}$ and

$\mathcal{F}(\bar{i}) = \{F \in \mathcal{F} : i \notin \mathcal{F}\}$.

Note that $|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$ holds.



## 2. The colex order and the Kruskal-Katona theorem

The colex order $<_c$ is a total order on all finite subsets of $\mathbb{N}$ defined by:

$A <_c B$ if and only if either $A$ is a proper subset of $B$ or the maximal element of $A - B$ is smaller than that of $B - A$.

The first few sets in the colex order are

$$\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}.$$

The following fact easily follows from the definition but it is very important.

**Fact 2.1.** *The sets in any initial segment of the colex order form a complex.*

**Proof.** Suppose that $G \subset H$ and $H$ is in the initial segment. Then $G <_c H$ by definition and therefore $G$ belongs to the initial segment, too. ■

For positive integers $m$, $k$ let $\mathcal{A}(k,m)$ denote the family of the first $m$ sets of size $k$ in the colex order.

**Fact 2.2.** $\sigma_i(\mathcal{A}(k,m)) = \mathcal{A}(i, |\sigma_i(\mathcal{A}(k,m))|)$, *that is the i'th shadow of $\mathcal{A}(k,m)$ forms an initial segment among i-element sets in the colex order.*

**Proof.** Let $A = \{a_1, \ldots, a_k\}$ be the largest element (in colex order) of $\mathcal{A}(k,m)$, $1 \le a_1 < \ldots < a_k$. Set $B = \{a_{k-i+1}, \ldots, a_k\}$. If $G$ is an i-set with $B <_c G$ then $A <_c G$ and consequently, $A <_c H$ follows for all sets $H$, containing $G$. Consequently, $B$ is the largest element in $\sigma_i(\mathcal{A}(k,m))$. Let now $D$ be an arbitrary i-set satisfying $D <_c B$. Suppose that $D = \{d_{k-i+1}, \ldots, d_k\}$ with $d_{k-i+1} < \ldots < d_k$. By definition of the colex order there is an $a$ $k - i < a \le k$ such that

$d_a < b_a$ and

$d_j = b_j$ for $a < j \le k$.

If $a < b_a$, then take an arbitrary $a$-element set $E \subset \{1, 2, \ldots, b_a - 1\}$ containing $\{d_{k-i+1}, \ldots, d_a\}$. Now $E \cup \{d_{a+1}, \ldots, d_k\} <_c A$ and it contains $D$, proving $D \in \sigma_i(\mathcal{A}(k,m))$.

If $a = b_a$, then $A = \{1, 2, \ldots, a\} \cup \{d_{a+1}, \ldots, d_k\}$, implying $D \subset A$. ■

**Kruskal-Katona Theorem.** (Kruskal [23], Katona [20]) *For all $\mathcal{F} \subset \binom{\mathbb{N}}{k}$ one has*

$$|\sigma_i(\mathcal{F})| \ge |\sigma_i(\mathcal{A}(k, |\mathcal{F}|))|, \quad 0 \le i < k.$$

Before giving the proof of the theorem let us explore the colex order.

For positive integers $k, m$ let $A = \{a_1, \dots, a_k\}$, $a_1 < \dots < a_k$ be the last element in $\mathcal{A}(k,m)$. Since $A$ completely determines $\mathcal{A}(k,m)$ we shall use the notation:

$$\mathcal{A}(B) = \{D : |D| = |B|,\ D \leq_c B\}.$$

By definition of the colex order $\binom{[a_k-1]}{k} \subset \mathcal{A}(k,m)$ holds. Furthermore, $a_k \in B$ holds for the remaining members of $\mathcal{A}(A)$. This implies

**Fact 2.3.**

$$\mathcal{A}(A) = \binom{a_k - 1}{k} \bigcup \{\{a_k\} \cup B \ :\ B \in \mathcal{A}(\{a_1, \dots, a_{k-1}\})\}. \qquad \blacksquare$$

Applying this fact $k$ times we obtain the following expression for $m$.

$$m = \binom{a_k - 1}{k} + \binom{a_{k-1} - 1}{k-1} + \dots + \binom{a_1 - 1}{1} + \binom{0}{0}.$$

This is called the *inproper cascade form* of $m$.

Choose the minimal $j \geq 1$ with the property $a_j + 1 < a_{j+1}$. If no such $j$ exist set $j = k$. Then

$$\binom{a_j - 1}{j} + \dots + \binom{a_1 - 1}{1} + \binom{0}{0} =$$
$$= \binom{a_j - 1}{j} + \binom{a_j - 2}{j-1} + \dots + \binom{a_j - i}{1} + \binom{a_j - i - 1}{0} = \binom{a_j}{j}.$$

That is,

$$m = \binom{a_k - 1}{k} + \dots + \binom{a_{j+1} - 1}{j+1} + \binom{a_j}{j}.$$

This is called the (proper) *cascade form* of $m$.

**Fact 2.4.** *The cascade form is unique among all representations of $m$ in the form*

$$m = \binom{b_k}{k} + \dots + \binom{b_j}{j} \quad \textit{satisfying} \quad b_k > \dots > b_j \geq j \geq 1.$$

**Proof.** Suppose that $m = \binom{d_k}{k} + \dots + \binom{d_i}{i}$ is another representation. Suppose by symmetry that $\{b_k, \dots, b_j\} >_c \{d_k \dots, d_i\}$ holds. That is there exist $l \leq k$ with $b_l > d_l$ and $b_u = d_u$ for all $l < u \leq k$. Then using the two representations, we obtain

$$0 = m - m = \binom{b_l}{l} + \dots + \binom{b_j}{j} - \binom{d_l}{l} - \dots - \binom{d_i}{i} \geq$$
$$\geq \binom{b_l}{l} - \binom{b_l - 1}{l} - \binom{b_l - 2}{l-1} - \dots - \binom{b_l - (l-1)}{1} = 1$$

a contradiction. $\blacksquare$



For $m = \binom{a_k-1}{k} + \dots + \binom{a_{j+1}-1}{j+1} + \binom{a_j}{j}$, written in proper cascade form define

$$\partial_l(m) = \binom{a_k - 1}{l} + \dots + \binom{a_{j+1} - 1}{j + 1 - (k - l)} + \binom{a_j}{j - (k - l)}, \quad 0 \leq l < k.$$

Similarly, for $m$ written in inproper cascade form $m = \binom{a_k-1}{k} + \dots + \binom{a_1-1}{1} + \binom{0}{0}$ we define

$$\bar{\partial}_l(m) = \binom{a_k - 1}{l} + \dots + \binom{a_1 - 1}{1 - (k - l)}.$$

Noting that $\binom{a+1}{b} = 0$ for $b < 0$ and using $\binom{a}{b} = \sum_{0 \leq i \leq b} \binom{a-i}{b-i}$ we obtain

**Fact 2.5.**

$$\partial_l(m) = \bar{\partial}_l(m).$$

From this and from the way we defined the inproper cascade form we obtain

**Fact 2.6.**

$$|\sigma_l(\mathcal{A}(k,m))| = \partial_l(m)$$

Thus the Kruskal-Katona theorem is equivalent to

$$|\sigma_l(\mathcal{F})| \geq \partial_l(m) \text{ for all } \mathcal{F} \subset \binom{\mathbb{N}}{k}. \tag{2.1}$$

To avoid misunderstanding sometimes we write $\partial_l^{(k)}(m)$, to indicate that the function arose via $k$-sets.

Combining Facts 2.2 and 2.6 gives

$$\partial_l^{(k)}(m) = \partial_l^{(l+1)}(\dots \partial_{k-2}^{(k-1)}(\partial_{k-1}^{(k)}(m)) \dots). \tag{2.2}$$

In view of (2.2) and monotonicity of $\partial_l(m)$ as a function of $m$, we see that it suffices to prove (2.1) for $l = k-1$.

To do so we first apply the shifting operation $S_{1j}$ to $\mathcal{F}$ for $2 \le j$, $j$ is contained in some member $F$ of $\mathcal{F}$.

After repeated applications of $S_{1j}$ we obtain a *1-shifted* family $\mathcal{G}$, that is, $S_{1j}(\mathcal{G}) = \mathcal{G}$ for all $j \ge 2$. Moreover, $|\mathcal{G}| = |\mathcal{F}|$, $|\sigma_\ell(\mathcal{F})| \ge |\sigma_\ell(\mathcal{G})|$. To see this last inequality consider the complex $\mathcal{F}_*$ generated by $\mathcal{F}$ : $\mathcal{F}_* = \{F_* : \exists F \in \mathcal{F},\ F_* \subset F\}$. Then

$$f_\ell(\mathcal{F}_*) = |\sigma_\ell(\mathcal{F})|.$$

In view of Proposition 1.1 the family $S_{1j}(\mathcal{F}_*)$ is a complex, too. Furthermore $f_\ell(\mathcal{F}_*) = f_\ell(S_{1j}(\mathcal{F}_*))$. Since $S_{1j}(\mathcal{F}_*)$ is a complex $|\sigma_\ell(\mathcal{F})| \ge |\sigma_\ell(S_{ij}(\mathcal{F}))|$ follows . Repeated applications yield the desired inequality $|\sigma_\ell(\mathcal{F})| \ge |\sigma_\ell(\mathcal{G})|$.

1-shifted families have the following nice property.

**Fact 2.7.** *Suppose that $\mathcal{G} \subset \binom{\mathbb{N}}{k}$ is 1-shifted. Then*

$$\sigma_{k-1}(\mathcal{G}) = \mathcal{G}(1) \cup \{H \cup \{1\} : H \in \sigma_{k-2}(\mathcal{G}(1))\}.$$

**Proof.** Clearly the RHS is a subset of the LHS. To prove the inclusion in the other direction, take $E \subset G \in \mathcal{G}$, $|E| = k-1$. If $1 \in E$, then $E - \{1\}$ is a $(k-2)$-subset of $(G - \{1\}) \in \mathcal{G}(1)$. If $1 \notin E$ but $F = E \cup \{1\}$, then $E \in \mathcal{G}(1)$. Finally if $1 \notin E$ and $F = E \cup \{j\}$ for some $2 \le j$ then 1-shiftedness implies $S_{1j}(F) = F$ thus $(F - \{j\}) \cup \{1\} = E \cup \{1\}$ has to be in $\mathcal{G}$. That is, $E \in \mathcal{G}(1)$. ■

Now we prove (2.1) by the proof given in [8], for the case $l = k-1$, by double introduction on $m$ and $k$. The cases $m = 1$ or $k = 1$ are trivial. Suppose that $|\mathcal{G}| = \binom{a_k}{k} + \dots + \binom{a_j}{j}$ in proper cascade form.

**Claim.**

$$|\mathcal{G}(1)| \ge \binom{a_k-1}{k-1} + \dots + \binom{a_j-1}{j-1}.$$

**Proof of the claim.** Suppose the contrary, $|\mathcal{G}(\bar{1})| = |\mathcal{G}| - |\mathcal{G}(1)| \ge \binom{a_k-1}{k} + \dots + \binom{a_j-1}{j} + 1$ follows. If $a_j - 1 \ge j$ then the induction hypothesis, Fact 2.7 and monotonicity yields $|\mathcal{G}(1)| \ge |\sigma_{k-1}(\mathcal{G}(\bar{1}))| \ge \binom{a_k-1}{k-1} + \dots + \binom{a_j-1}{j-1}$, contradicting the indirect assumption.



If $a_j = j$ then let $r$ be the greatest integer $j \le r \le k$ with $a_r = r$. We have

$$|\mathcal{G}(\bar{1})| \ge \binom{a_k-1}{k} + \dots + \binom{a_{r+1}-1}{r+1} + 1.$$

Writing 1 as $\binom{r}{r}$ and using Fact 2.7 together with the induction hypothesis gives

$$|\mathcal{G}(1)| \ge |\sigma_{k-1}(\mathcal{G}(\bar{1}))| \ge \binom{a_k-1}{k-1} + \dots + \binom{a_{r+1}-1}{r} + \binom{r}{r-1} \ge$$

$$\ge \binom{a_k-1}{k-1} + \dots + \binom{a_{r+1}-1}{r} + \binom{r-1}{r-1} + \dots + \binom{j-1}{j-1}$$

in contradiction with the indirect assumption. ■

Now the proof of (2.1) is easy.

Set $\tilde{m} = \binom{a_k-1}{k-1} + \dots + \binom{a_j-1}{j-1}$. This is a proper or inproper cascade form. Using the claim, Facts 2.5, 2.6, 2.7 and monotonicity gives

$$|\sigma_{k-1}(\mathcal{G})| \ge \tilde{m} + \partial^{(k-1)}_{k-2}(\tilde{m}) = \binom{a_k}{k-1} + \dots + \binom{a_j}{j-1} = \partial_l(m). \ ■ ■$$

As an immediate consequence we have.

**Corollary 2.8.** *A sequence $(f_0, f_1, \dots)$ of non-negative integers is the $f$-vector of a complex if and only if*

$$\partial^{(k)}_{k-1}(f_k) \le f_{k-1} \qquad (2.3)$$

*holds for all $k \ge 1$.*

**Proof.** Suppose that $\mathcal{F}$ is a complex with $f$-vector $(f_0, \dots)$. Then

$$\sigma_{k-1}(\{F \in \mathcal{F} : |F| = k\}) \subset \{F \in \mathcal{F} : |F| = k-1\}$$

holds. Thus (2.1) and the definition of $(f_0, \dots)$ gives the inequality (2.3).

Suppose next that (2.3) is satisfied and consider $\mathcal{A} = \cup_k \mathcal{A}(k, f_k)$.

In view of Fact 2.2 and 2.6 we have

$$\sigma_{k-1}(\mathcal{A}(k, f_k)) = \mathcal{A}(k-1, \partial^{(k)}_{k-1}(f_k)),$$

that is, $\mathcal{A}$ is a complex. ■

## 3. Canonical antichains

A family $\mathcal{F} \subset 2^{[n]}$ is called an *antichain* if $F \not\subset G$ holds for all distinct $F, G \in \mathcal{F}$.

Let us introduce the notation

$$\mathcal{F}^{(k)} = \{F \in \mathcal{F} : |F| = k\}.$$

Suppose that $f = (f_0, \dots, f_n)$ is the $f$-vector of $\mathcal{F}$. Then $|\mathcal{F}^{(k)}| = f_k$ holds.

**Claim.** *$\mathcal{F}$ is an antichain if and only if for all $0 \le l < k \le n$ one has*

$$\sigma_l(\mathcal{F}^{(k)}) \cap \mathcal{F}^{(l)} = 0.$$

**Proposition 3.1.** (Sperner [30]) *Suppose that $\mathcal{F} \subset 2^{[n]}$ is an antichain with $f$-vector $(f_0, \dots, f_n)$. If $k$ is the maximal integer with $f_k > 0$, then*

$$(\mathcal{F} - \mathcal{F}^{(k)}) \cup \sigma_{k-1}(\mathcal{F})$$

*is an antichain, too.*

**Proof.** A member, $G$ of $\sigma_{k-1}(\mathcal{F})$ is of maximal size in the new family, so it cannot be properly contained in any other member. On the other hand the existence of $F \in \mathcal{F}$ with $G \subset F$ implies that $G$ contains no member of $\mathcal{F}$. ∎

Given the $f$-vector $(f_0, \dots, f_n)$ of an antichain define the sequence $(s_0, \dots, s_n)$ inductively, by setting $s_n = f_n$, and if $s_k$ is defined set $s_{k-1} = \partial_{k-1}^{(k)}(s_k) + f_{k-1}$ for $k \ge 1$.

Repeated application of Proposition 3.1 together with the Kruskal-Katona Theorem gives that

$$s_i \le \binom{n}{i}. \tag{3.1}$$

**Theorem 3.2.** *([2], [6]) Let $\mathcal{F} \subset 2^{[n]}$ be an antichain. The family $\mathcal{A} = \mathcal{A}(\mathcal{F})$ defined by*

$$\mathcal{A} = \bigcup_{0 \le i \le n} (\mathcal{A}(i, s_i) - \mathcal{A}(i, s_i - f_i))$$



*is an antichain of [n] having the same $f$-vector as $\mathcal{F}$.*

**Proof.** From (3.1) it follows that $\mathcal{A} \subset 2^{[n]}$. Define

$$\mathcal{S}^{(i)} = \{S \in \binom{[n]}{i},\ \exists A \in \mathcal{A},\ S \subset A\}.$$

We prove by descending induction on $n$ that $\mathcal{S}^{(k)} = \mathcal{A}(k, s_k)$ holds. For $k = n$ this holds trivially.

On the other hand, by definition,

$$\mathcal{S}^{(k-1)} = \sigma_{k-1}(\mathcal{S}^{(k)}) \cup \mathcal{A}^{(k-1)}.$$

Facts 2.2 and 2.6 together with the definition of $s_{k-1}$ imply

$$\sigma_{k-1}(\mathcal{S}^{(k)}) = \mathcal{A}(k-1, s_{k-1} - f_{k-1}).$$

Consequently, the sets in $\mathcal{A}(k-1, s_{k-1}) - \mathcal{A}(k-1, s_{k-1} - f_{k-1})$ are not contained in the other members of $\mathcal{A}$, proving that $\mathcal{A}$ is an antichain. The identity of the $f$-vectors of $\mathcal{A}$ and $\mathcal{F}$ is immediate from the definition. ∎

Even that this theorem is a simple consequence of the Kruskal-Katona Theorem, it has many applications in extremal set theory. The family $\mathcal{A}(\mathcal{F})$ is called a *canonical* antichain.

Let us mention the basic extremal theorem for antichains.

**Theorem 3.3.** (Sperner [30]) *Suppose that $\mathcal{H} \subset 2^{[n]}$ is an antichain. Then $|\mathcal{H}| \le \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$ holds with equality if and only if $\mathcal{H} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{H} = \binom{[n]}{\lceil n/2 \rceil}$ holds.*

Sperner's proof is based upon Proposition 3.1 and the following Kruskal-Katona type lemma.

**Proposition 3.4.** (Sperner [30]) *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $1 \le k \le n$. Then $|\sigma_{k-1}(\mathcal{F})| / |\mathcal{F}| \ge \binom{n}{k-1} \big/ \binom{n}{k}$ holds with equality holding if and only if $\mathcal{F} = \binom{[n]}{k}$.*

**Proof.** Consider the bipartite graph $B$ with parts $\binom{[n]}{k}$, $\binom{[n]}{k-1}$ defined in a way that a $k$-set $F$ and a $(k-1)$-set $G$ are joined by an edge if and only if $G \subset F$. Every $k$-set $F$ has degree $k$ and every $(k-1)$-set degree $n - k + 1$. This implies

$$|\sigma_{k-1}(\mathcal{F})| \ge |\mathcal{F}|k / (n - k + 1), \tag{3.2}$$

with equality holding if and only if $\mathcal{F}$ and $\sigma_{k-1}(\mathcal{F})$ define a connected component of $B$. Now the proposition is equivalent to (3.2) and uniqueness follows from the fact that $B$ is connected. ∎

## 4. The Erdős-Ko-Rado theorem

A family $\mathcal{F} \subset 2^{[n]}$ is called *intersecting* if $F \cap F' \neq \emptyset$ holds for all $F, F' \in \mathcal{F}$. Probably the simplest result in extremal set theory is the following.

**Proposition 4.1.** *If $\mathcal{F}$ is intersecting then*

$$|\mathcal{F}| \leq 2^{n-1} \tag{4.1}$$

*holds.*

**Proof.** One can partition $2^{[n]}$ into $2^{n-1}$ complementary pairs $(A, B)$, $B = [n] - A$. Since $A \cap B = \emptyset$, at most one set out of each pair can belong to the intersecting family $\mathcal{F}$. ■

Suppose next that $\mathcal{F}$ is *k-uniform*, $\mathcal{F} \subset \binom{[n]}{k}$. If $2k > n$, then $\mathcal{F}$ is automatically intersecting.

For $2k \leq n$ probably the most natural way of defining an intersecting family is the following:

$$\mathcal{E}(k,n) = \left\{ E \in \binom{[n]}{k} : 1 \in E \right\}$$

Clearly $|\mathcal{E}(k,n)| = \binom{n-1}{k-1}$ holds.

**Erdős-Ko-Rado Theorem.** ([7]) *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, $2k \leq n$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{4.2}$$

**Proof.** (Daykin [4]) Consider $\mathcal{G} = \{[n] - F : F \in \mathcal{F}\} \subset \binom{[n]}{n-k}$. Since $\mathcal{F}$ is intersecting, $\sigma_k(\mathcal{G}) \cap \mathcal{F} = \emptyset$ holds. Suppose now that $|\mathcal{F}| \geq \binom{n-1}{k-1} = \binom{n-1}{n-k}$ holds. Since $|\mathcal{G}| = |\mathcal{F}|$, the Kruskal-Katona Theorem implies $|\sigma_k(\mathcal{G})| \geq \binom{n-1}{k}$. Consequently, $|\mathcal{F}| \leq \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$ holds. ■



## 5. Estimating functionals on complexes

Let $g : \{0, 1, \ldots, n, \ldots\} \to \mathbb{R}$ be an arbitrary function.

**Problem 5.1.***Given $1 \leq m < 2^n$ and $g$, find a complex $\mathcal{F} \subset 2^{[n]}$, $|F| = m$ minimizing $\sum_{F \in \mathcal{F}} g(|f|)$.*

This problem was solved for some special case by Lindström (see [25]) without using the Kruskal-Katona Theorem. Ahlswede-Katona [1] (see also [21]) solved it for decreasing functions using the Kruskal-Katona Theorem. Let $\mathcal{A}(m)$ be the family consisting of the first $m$ sets in the colex order.

**Theorem 5.2.** (Ahlswede-Katona [1]) *Suppose that $g$ is decreasing, then the family $\mathcal{A}(m)$ gives a solution to Problem 5.1.*

**Proof.** Let $(f_0, f_1, \ldots, f_n)$ be the $f$-vector of some complex $\mathcal{F} \subset 2^{[n]}$, $|\mathcal{F}| = m$ and let $(a_0, \ldots, a_n)$ be the $f$-vector of $\mathcal{A}(m)$.

**Claim.** *For all $0 \leq k \leq n$ one has*

$$\sum_{k \leq i \leq n} a_i \geq \sum_{k \leq i \leq n} f_i \tag{5.1}$$

**Proof of the claim.** Suppose the contrary and let $k$ be the smallest integer for which (5.1) fails. Then $a_k < f_k$ holds.

Now we prove $a_i \leq f_i$ for all $i < k$ by descending induction on $i$. If it is done we obtain the contradiction

$$m = a_0 + \ldots + a_{k-1} + \sum_{k \leq i \leq n} a_i < f_0 + \ldots + f_{k-1} + \sum_{k \leq i \leq n} f_i = m.$$

Suppose that $a_j \leq f_j$ for some $j$.

Then $\mathcal{A}(m)^{(j)} = \mathcal{A}(j, a_j)$ and $|\mathcal{F}^{(j)}| = f_j$. By the Kruskal-Katona Theorem

$$a_{j-1} = |\sigma_{j-1}(\mathcal{A}(j, a_j))| \leq |\sigma_{j-1}(\mathcal{F}^{(j)})| \leq f_{j-1},$$

proving $a_{j-1} \leq f_{j-1}$. This concludes the proof of the claim. ■

Since $\sum a_i = \sum f_i = m$, (5.1) can be rewritten as

$$\sum_{0 \leq i \leq k} a_i \leq \sum_{0 \leq i < k} f_i \quad \text{for all} \quad 0 \leq k \leq n \tag{5.2}$$

Using Abel-summation we can write setting $g(n+1) = g(n)$

$$\sum_{A\in\mathcal{A}(m)} g(|A|) = \sum_{0\le i\le n} a_i g(i) = \sum_{k=0}^{n}\sum_{0\le i\le k} a_i(g(k) - g(k+1)) + mg(n) \le$$

$$\sum_{k=0}^{n}\sum_{0\le i\le k} f_i(g(k) - g(k+1)) + mg(n) =$$

$$\sum_{F\in\mathcal{F}} g(|F|). \quad \blacksquare$$

For a family $\mathcal{F}$ and a set $Y$, define the trace $\mathcal{F}_Y$ by $\mathcal{F}_Y = \{F\cap Y : F\in \mathcal{F}\}$.

If $\mathcal{F}\subset 2^{[n]}$ is a complex, $1\le i\le n$, $Y = [n]-\{i\}$, then $\mathcal{F}_Y = \mathcal{F}(\bar{i})$, that is $|\mathcal{F}_Y| = |\mathcal{F}| - |\mathcal{F}(i)|$ holds.

Suppose that $n = dt$. Let $[n] = Y_1\cup\ldots\cup Y_d$ be a partition with $|Y_i| = t$ $1\le i\le d$. Define $D = 2^{Y_1}\cup\ldots\cup 2^{Y_d}$. Then $|D| = 1 + d(2^t-1)$ and $|D(i)| = 2^{t-1}$ for all $i\in[n]$.

Theorem 5.2. was used in [12] to prove the following.

**Theorem Theorem 5.3.** *Suppose that $\mathcal{F}\subset 2^n$, $|\mathcal{F}|\le 1+n(2^t-1)/t$. Then there exists some $i\in[n]$ such that*

$$|\mathcal{F}_{[n]-\{i\}}| \ge |\mathcal{F}| - 2^{t-1}$$

*holds.*

For the proof we refer to [12].

## 6. Lovász's numerical version of the Kruskal-Katona theorem

This complexity of the form, makes the Kruskal-Katona Theorem often awkward for concrete applications. This following version is more handy for computations.

**Theorem 6.1.** (Lovász [24]) *Suppose that $\mathcal{F}\subset\binom{\mathbb{N}}{k}$, $|\mathcal{F}|\ge\binom{x}{k}$ with $x\ge k$, real. Then*

$$|\sigma_{k-1}(\mathcal{F})| \ge \binom{x}{k-1} \quad (6.1)$$



*with equality holding if and only if $x$ is an integer and $\mathcal{F} = \binom{x}{k}$ for some $x$-element set $X$.*

**Proof.** The cases $k = 1,2$ are almost trivial, we may suppose that $k\ge 3$. As in the proof of the Kruskal-Katona Theorem we first use shifting to obtain a family $\mathcal{F}$, satisfying $S_{1j}(\mathcal{F}) = \mathcal{F}$ for all $j\ge 2$. To fully justify this we have to prove the following:

**Proposition 6.2.** *Suppose that $\mathcal{F}$ is not of the form $\binom{Y}{k}$ for $|Y| = x$ but $S_{1j}(\mathcal{F}) = \binom{x}{k}$. Then*

$$|\sigma_{k-1}(\mathcal{F})| > \binom{x}{k-1}.$$

**Proof of the proposition.** By the assumption $j\notin X$, $\binom{X-\{1\}}{k}\subset\mathcal{F}$. Define $\mathcal{G} = \mathcal{F}(1)\cap\binom{X-\{1\}}{k-1}$ and $\mathcal{H} = \mathcal{F}(j)\cap\binom{X-\{1\}}{k-1}$. Again by $S_{1j}(\mathcal{F}) = \binom{X}{k}$ the two families $\mathcal{G}$ and $\mathcal{H}$ form a partition of $\binom{X-\{1\}}{k-1}$. Since $|\sigma_{k-1}(\mathcal{F})|\ge\binom{x-1}{k-1} + |\sigma_{k-2}(\mathcal{G})| + |\sigma_{k-2}(\mathcal{H})|$, all we have to show is

$$|\sigma_{k-2}(\mathcal{G})| + |\sigma_{k-2}(\mathcal{H})| > \binom{x-1}{k-2}. \quad (6.2)$$

Look at the bipartite graph with parts $\binom{X-\{1\}}{k-1}$ and $\binom{X-\{1\}}{k-2}$ the edges defined by containment. For $k\ge 2$ this graph is connected, consequently $\sigma_{k-2}(\mathcal{G})\cap\sigma_{k-2}(\mathcal{H})\ne\emptyset$. This implies (6.2). $\blacksquare$

Now we return to the proof of (6.1). We apply double induction on $|\mathcal{F}|$ and $k$. We claim that

$$|\mathcal{F}(1)| \ge \binom{x-1}{k-1}. \quad (6.3)$$

Otherwise $|\mathcal{F}(\bar{1}) = |\mathcal{F}| - |\mathcal{F}(1)| > \binom{x-1}{k}$ follows. Choose $y$ such that $|\mathcal{F}(\bar{1})| = \binom{y}{k}$. Then $y > x-1$ and by the induction hypothesis and 1-shiftedness $|\mathcal{F}(1)|\ge\sigma_{k-1}(\mathcal{F}(\bar{1}))|\ge\binom{y}{k-1} > \binom{x}{k-1}$ follows proving (6.3). By the induction hypothesis and Fact 2.7 we infer

$$|\sigma_{k-1}(\mathcal{F})| = |\mathcal{F}(1)| + |\sigma_{k-2}(\mathcal{F}(1))| \ge \binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1},$$

as desired. If equality holds, then by the induction hypothesis $\mathcal{F}(1) = \binom{X-\{1\}}{k-1}$ follows. Now 1-shiftedness implies $\mathcal{F}\subset\binom{X}{k}$. $\blacksquare\ \blacksquare$

Theorem 6.1 shows the uniqueness of optimal families in the Kruskal-Katona Theorem for the case $|\mathcal{F}| = \binom{a}{k}$, $a \geq k$, integer. Applying the same result $k-l$ times proves $\sigma_l(\mathcal{F}) \geq \binom{x}{l}$ and uniqueness for all $1 \leq l < k$.

The values of $|\mathcal{F}| = m$ for given $k$ and $l$ such that $\mathcal{A}(k,m)$ is the only optimal family in the Kruskal-Katona Theorem were determined independently by Füredi-Griggs [15] and Mörs [26].

Combining the full version of the Kruskal-Katona Theorem with Lovász's version of it gives.

**Theorem 6.3.** *Suppose that* $\mathcal{F} \subset \binom{\mathbb{N}}{k}$, $|\mathcal{F}| = \binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\ldots+\binom{a_{l+1}}{l+1}+\binom{x}{l}$ *with* $a_k, \ldots, a_{l+1}$ *integers and* $x$ *real, satisfying* $a_k > \ldots > a_{l+1} \geq x+1 \geq l+1$, *then*

$$|\sigma_{k-1}(\mathcal{F})| \geq \binom{a_k}{k-1} + \ldots + \binom{a_{l+1}}{l} + \binom{x}{l-1}$$

*holds.*

## 7. A product version of the Erdős-Ko-Rado theorem

Let $1 \leq k,\ l \leq n$ be integers, $X$ an $n$-element set. Two families $\mathcal{F} \subset \binom{X}{k}$ and $\mathcal{G} \subset \binom{X}{l}$ are called *cross-intersecting* if $F \cap G \neq \emptyset$ holds for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. The following observation is due to Katona [19].

**Fact 7.1.** *For* $n \geq k+l$ $\mathcal{F}$, $\mathcal{G}$ *are cross-intersecting if and only if*

$$\sigma_l(\{X-F : F \in \mathcal{F}\}) \cap \mathcal{G} = \emptyset.$$

This shows that for $|\mathcal{F}| = m$ fixed the maximum size of $|\mathcal{G}|$ is $\binom{n}{l} - \partial_l^{(n-k)}(m)$. Thus

$$\max |\mathcal{F}||\mathcal{G}| = \max_{o \leq m \leq \binom{n}{k}} m\left(\binom{n}{l} - \partial_l^{(n-k)}(m)\right) \tag{7.1}$$

holds.

However, due to the complexity of computing and estimating $\partial_l$, (7.1) is difficult to use.



**Theorem 7.2.** (Matsumoto-Tokushige [27]) *Suppose that* $\mathcal{F} \subset \binom{X}{k}$, $\mathcal{G} \subset \binom{X}{l}$, $2k \leq n$, $2l \leq n$ *and* $\mathcal{F}$, $\mathcal{G}$ *are cross-intersecting. Then*

$$|\mathcal{F}||\mathcal{G}| \leq \binom{n-1}{k-1}\binom{n-1}{l-1} \tag{7.2}$$

*holds; moreover, equality holds if and only if* $|\mathcal{F}| = \binom{n-1}{k-1}$ *and* $|\mathcal{G}| = \binom{n-1}{l-1}$. *Further,* $\mathcal{F} = \left\{G \in \binom{X}{k} : x \in F\right\}$ *and* $\mathcal{G} = \left\{G \in \binom{X}{l} : \in G\right\}$ *must hold for some* $x \in X$, *unless* $n = 2k = 2l$.

The proof of (7.2) uses Fact 7.1, Theorem 6.3 and the following purely analytical inequality

**Lemma 7.3.** ([27]) *For* $1 \leq t \leq n-k$ *and* $n-k-t \leq x \leq n-t-1$,

$$\left(\binom{n-1}{n-k} + \binom{n-2}{n-k-1} + \ldots + \binom{n-t}{n-k-t+1} + \binom{x}{n-k-t}\right) \times$$
$$\left(\binom{n}{l} - \binom{n-1}{l} - \binom{n-2}{l-1} - \ldots - \binom{n-t}{l-t+1} - \binom{x}{l}\right) < \binom{n-1}{k-1}\binom{n-1}{l-1}.$$

Let us mention that Theorem 7.2 was proved by Pyber [29] in the cases $k = l$ and $2k+l \leq n$, $k \geq l$. The proof of the second case is by a nice application of Katona's cyclic permutation method ([22]).

Setting $k = l$ and $\mathcal{F} = \mathcal{G}$ in Theorem 7.2 we obtain the Erdős-Ko-Rado-Theorem (4.2).

In [17] the following inequalites for cross-intersecting families were obtained using the Kruskal-Katona Theorem.

For each inequality $0 < a \leq b$, $a+b \leq n$, $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are nonempty cross-intersecting families

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \binom{n-a}{b} + 1 \tag{7.3}$$

Suppose that $|\mathcal{A}| \geq \binom{x}{n-a}$ where $n-a \leq x \leq n-1$. If

$$\binom{x}{n-a} \leq |\mathcal{A}| \leq \binom{n-1}{n-a}$$

then

$$|\mathcal{A}| + |\mathcal{B}| \leq \begin{cases} \binom{n}{b} - \binom{x}{b} + \binom{x}{n-a} & \text{if } a < b \text{ or } x \leq n-2 \\ 2\binom{n-1}{a-1} & \text{if } a = b \text{ and } n-2 \leq x. \end{cases} \tag{7.4}$$

Suppose that $|\mathcal{A}| \geq \binom{n-1}{a-1}$ then

$$|\mathcal{A}| + |\mathcal{B}| \leq \begin{cases} \binom{n}{a} - \binom{n-a}{a} + 1 & \text{if } a = b \geq 2 \\ \binom{n-1}{a-1} - \binom{n-1}{b-1} & \text{otherwise.} \end{cases} \tag{7.5}$$

Let us note that all these inequalities are best possible. In the case $a = b$ inequality (7.3) was already proved by Hilton-Milner [18] using different methods.

## 8. Katona's shadow theorem for intersecting families

A family $\mathcal{F}$ is called *t-intersecting* if $|F \cap F'| \geq t$ holds for all $F, F' \in \mathcal{F}$.

A simple example of $t$-intersecting families is $\binom{[2k-t]}{k}$. This shows that the following result is best possible.

**Theorem 8.1.** (Katona [19]) *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and $k - t \leq g < k$. Then*

$$|\sigma_g(\mathcal{F})|/|\mathcal{F}| \geq \binom{2k-t}{g} \Big/ \binom{2k-t}{k} \tag{8.1}$$

*holds.*

**Proof.** The proof is based upon shifting. By Proposition 1.1. it does not increase the size of the shadow, also it maintains the $t$-intersecting property (cf. [19], [7] for the easy proof). We apply induction on $n$. The case $n = 2k - t$ follows from Proposition 3.4. Suppose now that $n > 2k - t$ and that $S_{in}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < n$.

Clearly, $\mathcal{F}(\bar{n}) \subset \binom{[n-1]}{k}$ is $t$-intersecting.

**Claim.** $\mathcal{F}(n) \subset \binom{[n-1]}{k-1}$ *is t-intersecting, too.*

**Proof of the claim.** Choose $E, E' \in \mathcal{F}(n)$. Then $E \cup \{n\}$, $E' \cup \{n\}$ are sets in the $t$-intersecting family $\mathcal{F}$, consequently $|E \cap E'| \geq t-1$. Suppose that we have equality. Than $|E \cup E'| = 2(k-1)-(t-1) = 2k-t-1 < n-1$. Therefore we can find $i \in ([n-1] - (E \cup E'))$. Since $S_{in}(\mathcal{F}) = \mathcal{F}$, $F' = E' \cup \{i\}$ is a member of $\mathcal{F}$. However, $|(E \cup \{n\}) \cap F'| = t - 1$, a contradiction. ■

To continue with the proof of (8.1) observe that

$$|\sigma_g(\mathcal{F})| \geq |\sigma_g(\mathcal{F}(\bar{n}))| + |\sigma_{g-1}(\mathcal{F}(n))|.$$



Applying the induction hypothesis to both $\mathcal{F}(\bar{n})$ and $\mathcal{F}(n)$ we obtain

$$|\sigma_g(\mathcal{F})| \geq |\mathcal{F}(\bar{n})| \binom{2k-t}{g} \Big/ \binom{2k-t}{k} + |\mathcal{F}(n)| \binom{2k-t-2}{g-1} \Big/ \binom{2k-t-2}{k-1}.$$

To conclude the proof of (8.1) we only need to notice $|\mathcal{F}| = |\mathcal{F}(\bar{n})| + |\mathcal{F}(n)|$ and that the coefficient of $|\mathcal{F}(n)|$ is not less than that of $|\mathcal{F}(\bar{n})|$. This latter fact is equivalent to

$$k(k-t) \leq g(2k-t-g). \qquad ■$$

For a different proof of (8.1) and some sharpening of it see [9].

The inequality (8.1) was used by Katona [19] to give a short proof of the Erdős-Ko-Rado Theorem in its dual form: let $\mathcal{F} \subset \binom{[n]}{k}$, $2k \leq n$ and suppose that $F \cap F' \neq \emptyset$ holds for all $F, F' \in \mathcal{F}$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. The proof is as follows: the condition implies that $\mathcal{F}^c = \{[n] - F : F \in \mathcal{F}\}$ $t$-intersecting with $t = n - 2k + 1$. Apply now (8.1) with $g = (n-k) - (t-1) = k$.

Since $\sigma_k(\mathcal{F}^c) \cap \mathcal{F} = \emptyset$ we obtain

$$\binom{n}{k} \geq |\mathcal{F}| \left(1 + \binom{n-1}{k} \Big/ \binom{n-1}{n-k}\right) = |\mathcal{F}| n/k,$$

that is $|\mathcal{F}| \leq \binom{n-1}{k-1}$, as desired.

Actually Katona invented (8.1) in order to determine the maximum size of a $t$-intersecting family $\mathcal{F} \subset 2^{[n]}$. For $n + t$ even define

$$\mathcal{K}(n,t) = \{K \subset [n] : |K| \geq (n+t)/2\}.$$

For $n + t$ odd define

$$\mathcal{K}(n,t) = \{F \subset [n] : |F \cap [n-1]| \geq ((n-1)+t)/2\}.$$

**Katona Theorem.** ([19]) *Suppose that $\mathcal{F} \subset 2^{[n]}$ is t-intersecting. Then $|\mathcal{F}| \leq |\mathcal{K}(n,t)|$ holds. Moreover, for $t \geq 2$ equality is possible only for $\mathcal{K}(n,t)$.*

We only give the proof in the case $n + t$ even. The other case is very similar but slightly more complicated. Let $(f_0, \ldots, f_n)$ be the $f$-vector of $\mathcal{F}$. Note that $f_i = 0$ for $i < t$. For $t \leq k < (n+t)/2$ consider the families

$$\sigma_{k-(t-1)}(\mathcal{F}^{(k)}) \text{ and } \mathcal{G}^{(k-t+1)} \stackrel{\text{def}}{=} \left\{[n] - F : F \in \mathcal{F}^{(n-k+(t-1))}\right\}.$$

Since $\mathcal{F}$ is $t$-intersecting, they are disjoint. Now (8.1) implies

$$f_k \cdot k/(k-t+1) + f_{n-k+t-1} \le \binom{n}{n-k+t-1} \qquad (8.2)$$

Adding these inequalities and $f_n \le 1$, and using $f_i = 0$ for $i < t$ gives

$$|\mathcal{F}| = \sum_{0 \le i \le n} f_i \le \sum_{j \ge (n+t)/2} \binom{n}{j} = |\mathcal{K}(n,t)|,$$

proving the theorem. Uniqueness for $t \ge 2$ follows from the fact that in this case the coefficient of $f_k$ in (8.2) is greater than 1. ■

Let us mention that for general $n, k, t$ the maximum size of $t$-intersecting families, $\mathcal{F} \subset \binom{[n]}{k}$ is not always known.

**Conjecture.** ([10]) *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is $t$-intersecting $n \ge 2k-t$. Then $|\mathcal{F}| \le \max_{0 \le i \le k-t} \mathcal{B}_i$ where*

$$\mathcal{B}_i = \left\{ B \in \binom{[n]}{k} : |B \cap [t+2i]| \ge t+i \right\}.$$

*It is know (cf. [7], [10] and [32]) that $|\mathcal{F}| \le |\mathcal{B}_0|$ holds for $n \ge (k-t+1)(t+1)$.*

Let us note that (8.1) implies for all $n, k, t$ the bound $|\mathcal{F}| \le \binom{n}{k-t}$.

Let us also mention that in [13] an extension of (8.1) is proved using linear algebra, while in [11] it is extended in other directions.

Let us also mention that a product version of the Katona Theorem is proved in [28].

## 9. The original application: matching $k$ sets into $(k-1)$-sets

Katona's discovery of the Kruskal-Katona Theorem, as so many other results in combinatorics, was motivated by a problem of Erdős. What is the maximum number $m = m(k)$ such that to any collection of at most $m$ sets of size $k$ one can find a matching into the $(k-l)$-sets in a way that each $k$-subset is matched onto a proper subset.

**Theorem 9.1.** (Katona [20]) *Suppose that $\mathcal{F} = \{F_1, \dots, F_m\}$ is a collection of distinct $k$-sets and*

$$m \le \binom{2k-1}{k} + \binom{2k-3}{k-1} + \dots + \binom{3}{2} + \binom{1}{1} \stackrel{\text{def}}{=} m(k). \qquad (9.1)$$

*Then there exist $m$ distinct sets $G_1, \dots, G_m$, each of size $k-1$ such that $G_i \subset F_i$ holds for $1 \le i \le m$.*



**Proof.** In view of the Köning-Hall theorem it is sufficient to show that for all $\mathcal{F} \subset \binom{\mathbb{N}}{k}$, $|\mathcal{F}| \le m(k)$ one has

$$|\sigma_{k-1}(\mathcal{F})| \ge |\mathcal{F}|.$$

Now in view of the Kruskal-Katona Theorem, it suffices to show

$$\partial^{(k)}_{k-1}(m) \ge m \quad \text{for all } m \le m(k). \qquad (9.2)$$

We prove (9.2) by induction on $k$. The case $k=1$ is trivial. In general, if $|\mathcal{F}| \le \binom{2k-1}{k}$ then $|\sigma_{k-1}(\mathcal{F})| \ge |\mathcal{F}|$ follows from Proposition 3.4. Suppose next that $m > \binom{2k-1}{k}$.

Let $m = \binom{a_k}{k} + \dots + \binom{a_j}{j}$ be the proper cascade form of $m$. Then $m \le m(k)$ implies that $a_k = 2k-1$ and that $m - \binom{2k-1}{k} = \binom{a_{k-1}}{k-1} + \dots + \binom{a_j}{j} \le m(k-1)$. Using the induction hypothesis it gives

$$\partial^{(k)}_{k-1}(m) = \binom{2k-1}{k-1} + \partial^{(k-1)}_{k-2}\left(m - \binom{2k-1}{k}\right)$$

$$\ge \binom{2k-1}{k-1} + \left(m - \binom{2k-1}{k}\right) = m \qquad ■$$

Let us remark that (9.1) is best possible. Namely for $\mathcal{A} = \mathcal{A}(m(k)+1, k)$ one has $|\sigma_{k-1}(\mathcal{A})| = m(k) < |\mathcal{A}|$.

Almost identical proof gives the following generalisation.

**Theorem 9.2.** *Suppose that $\mathcal{F} = \{F_1, \dots, F_m\}$ is a collection of $k$-sets, $0 \le l \le k$ and*

$$|\mathcal{F}| = m \le \binom{k+l}{k} + \binom{(k-1)+(l-1)}{k-1} + \dots + \binom{k-l}{k-l}.$$

*Then there exist distinct $l$-sets $G_1, \dots G_m$ satisfying $G_i \subset F_i$, $1 \le i \le m$.*

In Daykin-Frankl [5] Theorem 9.1 is extended in the following way. Define for integer $d \ge 2$

$$m(k,d) = \binom{dk-1}{k} + \binom{d(k-1)-1}{k-1} + \dots + \binom{2d-1}{2} + \binom{d-1}{1}.$$

Then $m(k,d)$ is the largest integer with the property $|\sigma_{k-1}(\mathcal{F})|/|\mathcal{F}| \ge 1/(d-1)$ holding for every collection $\mathcal{F}$ of $k$-sets such that $|\mathcal{F}| \le m(k,d)$.

## 10. The shadow function and the Takagi function

The results in this section explain to some extent, why is it difficult to use the Kruskal-Katona Theorem for computations. Let us define the *excess function*

$$e(m,k,l) = \partial^{(k)}_l(m) - m.$$

By the Kruskal-Katona Theorem if $\mathcal{F}$ is a collection of $k$-element sets $|\mathcal{F}| = m$ then

$$|\sigma_l(\mathcal{F})| \geq m + e(m, k, l)$$

holds.

The *shadow function* $s_k(x)$ is defined by normalizing the excess functions.

$$s_k(x) = k\binom{2k-1}{k}^{-1} e\left(\left\lfloor \binom{2k-1}{k} x \right\rfloor, k, k-1\right) \quad \text{for } 0 \leq x \leq 1.$$

In 1903, Takagi [31] constructed a nowhere differentiable continuous functions $t(x)$, which is called the Takagi function. To define it, first set

$$\varphi_1(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{if } 1/2 \leq x \leq 1 \end{cases},$$

$$\varphi_n(x) = \varphi_{n-1}(\varphi_1(x)).$$

Now we can define $t(x)$.

$$t(x) = \sum_{n=1}^{\infty} \varphi_n(x) 2^{-n} \quad \text{for} \quad 0 \leq x \leq 1.$$

This function has several interesting properties including self-similarity.

**Theorem 10.1** ([16]) *The shadow functions converge uniformly to the Takagi function, i.e.,*

$$\lim_{n\to\infty} \sup_{0 \leq x \leq 1} |s_n(x) - t(x)| = 0.$$

The proof of this result uses rather involved computations and we refer the interested reader to the paper [16].

## 11. Some related and open problems

Clearly the general Kruskal-Katona problem is of the following type. Given a bipartite graph $\mathcal{B}$ with part $X$ and $Y$, an integer $m$, $1 \leq m \leq |X|$, determine or estimate the function $\partial(m) = \partial(m, \mathcal{B})$ such that for every $X_0 \subset \binom{X}{m}$ the neighbourhood of $X_0$ consists of at least $\partial(m)$ vertices.

This problem is too difficult to have a general solution, e.g., it includes the problem of expanders. By considering non-bipartite graphs, we come to the isoperimetric problem which is the subject of another chapter of this volume.

Recently, the Kruskal-Katona problem was solved for some subclasses of the $k$-element sets of $\mathbb{N}$.

Let $r \geq k$ be an integer and define $\mathbb{N} = X_1 \cup \ldots \cup X_r$ where

$$X_i = \{j \in \mathbb{N} : j \equiv i \ (\text{mod r})\}.$$



Define $\mathcal{K}(k, r)$ the complete $r$-chromatic $k$-graph by

$$\mathcal{K}(k, r) = \{F \in \binom{\mathbb{N}}{k} : |F \cap X_i| \leq 1\}.$$

Since $\mathcal{K}(k, r) \subset \binom{\mathbb{N}}{k}$, the $k$-sets in $\mathcal{K}(k, r)$ are naturally ordered by the colex order. Let $\mathcal{A}_r(k, m)$ denote the first $m$ sets in this induced order.

**Theorem 11.1.** ([14]) *Suppose that* $\mathcal{F} \subset \mathcal{K}(k, r)$, $|\mathcal{F}| = m$. *Then* $|\sigma_l(\mathcal{F})| \geq |\sigma_l(\mathcal{A}_r(k, m))|$ *holds.*

This result says that the Kruskal-Katona Theorem is "true" for $\mathcal{K}(k, r)$.

**Problem.** *Find other subclasses of* $\binom{\mathbb{N}}{k}$ *for which the Kruskal-Katona Theorem is true.*

Let us conclude this problem with another open problem. For a family $\mathcal{F} \subset \binom{\mathbb{N}}{k}$ and integer $1 \leq l < k$ define the higher incidence matrix $M_l(\mathcal{F})$ as a $|\sigma_l(\mathcal{F})|$ by $|\mathcal{F}|$ matrix whose rows are indexed by $G \in \sigma_l(\mathcal{F})$, columns by $F \in \mathcal{F}$ and the entry $m(G, F)$ by

$$m(G, F) = \begin{cases} 1 & \text{if } G \subset F \\ 0 & \text{if } G \not\subset F. \end{cases}$$

**Problem.** *Determine or estimate the minimum of the rank of* $M_l(\mathcal{F})$ *over all* $\mathcal{F}$ *with* $|\mathcal{F}| = m$.

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Péter Frankl
*University of Paris VII*
*2 Place Juissieu*
*Paris 75 005, France*

N. Tokushige
*Dept. of Computer Science*
*Meiji University*
*1-1-1 Higashimita, Takamaku*
*Kawasaki 214, Japan*