AN L-SYSTEM ON THE SMALL WITT DESIGN

NORIHIDE TOKUSHIGE

Abstract. We construct a 12-uniform hypergraph on n vertices with size $(n/12)^6$ which satisfies $|F \cap F'| \in \{0, 1, 2, 3, 4, 6\}$ for all distinct edges F and F'.

1. Introduction

Let k and n be positive integers and let $L \subset \{0, 1, \ldots, k-1\}$. A k-uniform hypergraph $\mathcal{F}$ is called a (k, L)-system (or an L-system for short) if $|F \cap F'| \in L$ holds for all distinct $F, F' \in \mathcal{F}$. Let $m(n, k, L)$ be the maximum size of (k, L)-systems on n vertices. If there exist positive constants α, c, c', and n_0 depending only on k and L such that $cn^\alpha < m(n, k, L) < c'n^\alpha$ holds for $n > n_0$, then we define $\alpha(k, L) = \alpha$ and we say that (k, L)-systems have exponent α. In [4] the following upper bound for the size of (k, L)-systems is obtained.

Theorem 1. *For $n > n_0(k, L)$ it follows that*

$$m(n, k, L) \leq \prod_{l \in L} \frac{n-l}{k-l}.$$

In particular, the above upper bound gives $\alpha(k, L) \leq |L|$ if $\alpha(k, L)$ exists. In this note, we construct some (k, L)-systems satisfying $\alpha(k, L) = |L|$. Among other results, we show that $\alpha(12, \{0, 1, 2, 3, 4, 6\}) = 6$. The corresponding system is related to the small Witt design $S(5, 6, 12)$ and our construction uses the embedding of the design into $PG(5, 3)$.

In the next section we explain our main idea by solving a toy problem. Then we state the general construction scheme in section 3. We deal with the L-systems related to the small Witt designs in section 4, and some other L-systems with geometric structures in section 5. Finally in section 6 we consider "intersection structures" which control L-systems. The results in section 6 together with previously known results give the complete tables of exponents of (k, L)-systems for $k = 11$ and $k = 12$. These tables are presented in the Appendix and settle all the open cases in [7].

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2. A toy problem

As a toy problem, let us consider a $(7, \{0, 1, 3\})$-system. Let A be the following 3×7 matrix over $GF(2)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

This is a parity check matrix of the Hamming $[7, 4, 3]_2$-code. This matrix has the following properties.

(i) Any two columns are linearly independent over $GF(2)$.
(ii) For any two columns c_p, c_q of A, the subspace spanned by c_p, c_q contains precisely three columns c_p, c_q and c_r of A. In fact, $c_r = c_p + c_q$.

Let us choose 3 columns of A. There are $\binom{7}{3} = 35$ choices. Among those choices, $\binom{7}{2}/\binom{3}{2} = 7$ of them span 2-dimensional subspaces and the other 28 span the entire 3-dimensional space. The triples (the indices of 3 columns) corresponding to 2-dimensional subspaces form the Steiner triple system $S(2, 3, 7) = \{124, 135, 167, 236, 257, 347, 456\}$ or the Fano plane.

Now we shall construct a $(7, \{0, 1, 3\})$-system $\mathcal{F}$ using the matrix A. This is going to be a 7-partite hypergraph on the vertex partition $X = V_1 \cup \cdots \cup V_7$ where $V_i \cong \mathbb{F}_2^d$ for $1 \le i \le 7$. For $a, b, c \in \mathbb{F}_2^d$, define the ordered 7-tuple $F(a, b, c)$ by

$$F(a, b, c) = (a, b, c)A = (a, b, c, a + b, a + c, b + c, a + b + c) \in (\mathbb{F}_2^d)^7.$$

Then define

$$\mathcal{F} = \{F(a, b, c) : a, b, c \in \mathbb{F}_2^d\}.$$

Let us check that $\mathcal{F}$ is a $\{0, 1, 3\}$-system. Choose $F = F(a, b, c), F' = F(a', b', c') \in \mathcal{F}$. Suppose that $i := |F \cap F'| \ge 2$ and $F = F'$ on $V_p \cup V_q$. Then by (ii), $F = F'$ on V_r as well, where r is given by $c_r = c_p + c_q$. This means $i \ge 3$ (and $i = 2$ cannot happen). Next suppose that $i \ge 4$. Then by (ii) we can choose p, q, s where $1 \le p < q < s \le 7$ such that $F = F'$ on $V_p \cup V_q \cup V_s$ and the 3×3 minor matrix $B = (c_p, c_q, c_s)$ of A is non-singular. Since $(a - a', b - b', c - c')B = 0$ we have $(a, b, c) = (a', b', c')$ and $F = F'$. This means $i = 7$ (and $i = 4, 5$, or 6 cannot occur). Therefore, $\mathcal{F}$ is a $(7, \{0, 1, 3\})$-system.

The size of this hypergraph is $|\mathcal{F}| = (n/7)^3$ where $n = |V(\mathcal{F})| = 7 \cdot 2^d$. We will generalize the construction in the next section.

In this particular case, we can construct a larger $(7,\{0,1,3\})$-system by taking $\mathcal{F}$ to be the set of projective planes in $PG(d,2)$. Then,

$$|\mathcal{F}| = \frac{(2^{d+1}-1)(2^{d+1}-2)(2^{d+1}-4)}{(2^3-1)(2^3-2)(2^3-4)} = \frac{n(n-1)(n-3)}{7\cdot 6\cdot 4}$$

where $n = |PG(d,2)| = 2^{d+1}-1$. This size attains the upper bound in Theorem 1, so this construction is best possible.

3. Generating matrix for a (k,L)-system

A $(t,b,k)_q$-matrix is a $(t+1)\times k$ matrix over $GF(q)$ satisfying the following properties:

(P1) Any t columns are linearly independent over $GF(q)$.
(P2) For any t columns, the t-dimensional subspace spanned by these columns contains precisely b columns of A.

The matrix in the previous section was a $(2,3,7)_2$-matrix. For a $(t,b,k)_q$-matrix A, there are $\binom{k}{b}$ ways of taking a $(t+1)\times b$ minor matrix of A. Among $\binom{k}{b}$ ways, $\binom{k}{t}/\binom{b}{t}$ of them have rank t, and the others have rank $t+1$. Each of those rank t minor matrices gives a b-set (block) consisting of the indices of corresponding columns. The set of these blocks is a Steiner system $S(t,b,k)$. To represent this situation, we say that a $(t,b,k)_q$-matrix supports $S(t,b,k)$.

Theorem 2. *If there exists a $(t,b,k)_q$-matrix then there exists a (k,L)-system on n-vertices with size $(n/k)^{t+1}$ where $L=\{0,1,\dots,t-1,b\}$.*

Proof. Let $A=(a_{ij})=(c_1,\dots,c_k)$ $(1\le i\le t+1,\ 1\le j\le k)$ be a $(t,b,k)_q$-matrix. We shall construct a (k,L)-system $\mathcal{F}$ which is k-partite on the vertex partition $X=V_1\cup\cdots\cup V_k$ where $V_i\cong\mathbb{F}_q^d$ for $1\le i\le k$. For $(x_1,\dots,x_{t+1})\in(\mathbb{F}_q^d)^{t+1}$, let us define the k-set $F(x_1,\dots,x_{t+1})\in(\mathbb{F}_q^d)^k$ by setting

$$F(x_1,\dots,x_{t+1}) = (x_1,\dots,x_{t+1})A = (\sum_{i=1}^{t+1} a_{ij}x_i)_{j=1}^k \in V_1\times\cdots\times V_k.$$

Then define

$$\mathcal{F} = \{F(x_1,\dots,x_{t+1}) : x_1,\dots,x_{t+1}\in\mathbb{F}_q^d\}. \tag{1}$$

By construction, $\mathcal{F}$ is k-partite and k-uniform. Let us check that $\mathcal{F}$ is an L-system. Choose two edges $F=F(x_1,\dots,x_{t+1})$ and $F'=F(x'_1,\dots,x'_{t+1})$ of $\mathcal{F}$. Let $i:=|F\cap F'|$ and let I be the corresponding i-set such that $F=F'$ on $\bigcup_{i\in I}V_i$. Set $y=(x_1-x'_1,\dots,x_{t+1}-x'_{t+1})$ then $y\cdot c_i=0$ holds for $i\in I$.

Suppose that $b \geq i \geq t$. Then by (P2) there exists $B \supset I$, $|B| = b$ such that $c_j = \sum_{i \in I} \gamma_{ij} c_i$ holds for $j \in B$ where $\gamma_{ij} \in \mathbb{F}_q$. Thus for $j \in B$ it follows that $y \cdot c_j = y \cdot \sum_i \gamma_{ij} c_i = \sum_i \gamma_{ij}(y \cdot c_i) = 0$. So we have $F = F'$ on V_i, $i \in B$. This means $i = b$.

Next suppose that $i \geq b+1$. Then by (P2) we can choose $t+1$ columns from $\{c_i : i \in I\}$ so that the corresponding $(t+1) \times (t+1)$ minor matrix C is non-singular. Then we have $yC = 0$, which implies $y = 0$, i.e., $F = F'$. This means $i = k$, which concludes that $\mathcal{F}$ is an L-system.

The (k, L)-system $\mathcal{F}$ has $n = k \cdot q^d$ vertices and size $|\mathcal{F}| = (q^d)^{t+1} = (n/k)^{t+1}$. □

It is now appropriate to say that a $(t, b, k)_q$-matrix is a generating matrix for a (k, L)-system where $L = \{0, 1, \ldots, t-1, b\}$. The row vectors of a $(t, b, k)_q$-matrix span a $(t+1)$-dimensional subspace in a k-dimensional space. This fact together with (P2) implies that a $(t, b, k)_q$-matrix is a parity check matrix of a $[k, k-t-1, t+1]_q$-code.

Note that Theorem 1 and Theorem 2 imply that $\alpha(k, L) = |L|$ where $L = \{0, 1, \ldots, t-1, b\}$ if a $(t, b, k)_q$-matrix exists.

4. A $(12, 012346)$-system and its derived system

Let us construct a $(5, 6, 12)_3$-matrix A. We use a geometric structure due to Havlicek[10] originated by Coxeter[2]. Let $\varphi : PG(2,3) \to PG(5,3)$ be the Veronese mapping, that is, $\varphi(x, y, z) = (x^2, xy, xz, y^2, yz, z^2)$. Choose a line ℓ in $PG(2,3)$, say, $\ell = \{001, 010, 011, 012\}$. Its Veronese image $\Gamma := \varphi(\ell) = \{000001, 000100, 000111, 000121\}$ is a planar quadrangle with the diagonal triangle $\Delta = \{000101, 000211, 000221\}$. Now we take $(\varphi(PG(2,3)) - \Gamma) \cup \Delta$ as the 12 column vectors of A. More concretely, we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The 12 points in $PG(5,3)$ determined by A have many interesting geometric properties. In particular, A is a $(5, 6, 12)_3$-matrix. See [2, 10] for details.

The matrix is a parity check matrix of a $[12, 6, 6]_3$-code. Noting that this is the unique extended ternary Golay code G_{12}, the following standard parity check matrix B of G_{12} is also a $(5,6,12)_3$-matrix.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

These matrices support the Witt design $S(5,6,12)$.

By deleting the top row and the left-most column from A or B, we obtain a $(4,5,11)_3$-matrix. This corresponds to a $(11,\{0,1,2,3,5\})$-system, $S(4,5,11)$, and the perfect ternary Golay code G_{11}.

Consequently, we have the following bounds for the size of L-systems on the small Witt designs. (Lower bounds follow from the constructions above and Theorem 2, while upper bounds follow from Theorem 1.)

Theorem 3. *For n sufficiently large, we have*

$$\left(\frac{n}{12}\right)^6 \le m(n,12,\{0,1,2,3,4,6\}) \le \frac{n^6}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 6},$$

$$\left(\frac{n}{11}\right)^5 \le m(n,11,\{0,1,2,3,5\}) \le \frac{n^5}{11\cdot 10\cdot 9\cdot 8\cdot 6}.$$

By deleting the first two rows and the first two columns from B, we obtain a $(3,4,10)_3$-matrix C which supports $S(3,4,10)$. This gives $(n/10)^4$ as lower bound for a $(10,\{0,1,2,4\})$-system on n vertices. In [5], $n^4/65610$ is obtained as lower bound by a construction using sections of elliptic quadrics over $GF(3)$. We can use quadrics to construct generating matrices. In fact, each column vector (x,y,z,w) of the matrix C satisfies $xy - xz + xw - yz - yw + zw = 0$, which means the 10 points are on the Möbius plane. We will extend the construction in the next section.

5. More examples of a generating matrix

Let q be a prime power. Here we present some examples of a $(t,b,k)_q$-matrix and its corresponding (k,L)-system on n vertices. Similar constructions are also given in [5], but our constructions are simpler and give better lower bounds for (k,L)-systems.

5.1. **Affine plane.** Let $d \geq m$ and set $n = q^d$, $k = q^m$, $L = \{0, 1, q, \ldots, q^{m-1}\}$. Let $\mathcal{F}$ be the set of m-dimensional affine subspaces in $\mathbb{F}_q^d$. This is a (k, L)-system on n vertices with size

$$|\mathcal{F}| = \frac{q^d(q^d-1)(q^d-q)\cdots(q^d-q^{m-1})}{q^m(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})} = \prod_{\ell \in L} \frac{n-\ell}{k-\ell}.$$

Comparing to Theorem 1, it follows that $|\mathcal{F}| = m(n, k, L)$, in particular, $\alpha(q^m, \{0, 1, q, \ldots, q^{m-1}\}) = m$.

Now let $m = 2$. Then the above $\mathcal{F}$ is a $(q^2, \{0, 1, q\})$-system. By taking q^2 points of an affine plane as column vectors, we obtain a $(2, q, q^2)_q$-matrix. For example, the case $q = 3$ gives the following $(2, 3, 9)_3$-matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Taking $m = 3$, $q = 2$, we get a $(3, 4, 8)_2$-matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

5.2. **Projective plane.** Let $d \geq m$ and set $n = q^{d+1} - 1$, $k = \frac{q^{m+1}-1}{q-1}$, $L = \{0, 1, q+1, \ldots, \frac{q^m-1}{q-1}\}$. Let $\mathcal{F}$ be the set of m-dimensional projective subspaces in $PG(d, q)$. This is a (k, L)-system on n vertices with size

$$|\mathcal{F}| = \frac{(q^{d+1}-1)(q^{d+1}-q)\cdots(q^{d+1}-q^m)}{(q^{m+1}-1)(q^{m+1}-q)\cdots(q^{m+1}-q^m)},$$

which gives $|\mathcal{F}| = \Omega(n^{m+1})$ as d (and hence n) grows. This together with Theorem 1 implies $\alpha(\frac{q^{m+1}-1}{q-1}, \{0, 1, q+1, \ldots, \frac{q^m-1}{q-1}\}) = m + 1$.

Now let $m = 2$. Then the above construction gives a $(q^2 + q + 1, \{0, 1, q + 1\})$-system. By taking $p^2 + p + 1$ points of the projective plane as column vectors, we get a $(2, q + 1, q^2 + q + 1)_q$-matrix. The $(2, 3, 7)_2$-matrix in section 2 is one of the examples.

5.3. **Möbius plane.** Set $k = q^2 + 1$, $L = \{0, 1, 2, q + 1\}$, and let

$$\mathcal{M} = \{(x, y, z, w) \in PG(3, q) : f(x, y) + zw = 0\},$$

where $f(x, y)$ is an $\mathbb{F}_q$-irreducible quadratic form. Then $|\mathcal{M}| = q^2 + 1$ and we obtain a $(3, q + 1, q^2 + 1)_q$-matrix where the column vectors come from $\mathcal{M}$ (cf. Example 26.5 in [11]). Therefore we have $\alpha(q^2+1,\{0,1,2,q+1\})=4$.

For example, setting $q=3$, we consider $x^2+y^2+zw=0$ in $PG(3,3)$, which has 10 solutions giving a $(3,4,10)_3$-matrix:

$$\begin{pmatrix} 0&0&0&0&1&1&1&1&1&1\\ 0&0&1&1&0&0&1&1&2&2\\ 0&1&1&2&1&2&1&2&1&2\\ 1&0&2&1&2&1&1&2&1&2 \end{pmatrix}.$$

The following examples are a $(3,5,17)_4$-matrix ($q=4$ and $\mathbb{F}_4=\mathbb{F}_2(\beta)$ where $\beta^2+\beta+1=0$, and $f(x,y)=x^2+\beta xy+y^2$):

$$\begin{pmatrix} 0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&0&0&0&1&1&1&\beta&\beta&\beta&\beta^2&\beta^2&\beta^2\\ 0&1&1&\beta&\beta^2&1&\beta&\beta^2&1&\beta&\beta^2&1&\beta&\beta^2&1&\beta&\beta^2\\ 1&0&1&\beta^2&\beta&1&\beta^2&\beta&\beta&1&\beta^2&1&\beta^2&\beta&\beta&1&\beta^2 \end{pmatrix},$$

and a $(3,6,26)_5$-matrix ($q=5$ and $f(x,y)=x^2+xy+y^2$):

$$\begin{pmatrix} 0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4\\ 0&1&1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4\\ 1&0&4&2&3&1&4&2&3&1&2&1&4&3&3&4&1&2&2&1&4&3&4&2&3&1 \end{pmatrix}.$$

Let $\mathcal{F}$ be a (k,L)-system on n vertices constructed by a $(3,q+1,q^2+1)_q$-matrix. Then we have $|\mathcal{F}|=(\frac{n}{k})^4$ where $n=kq^d$, $k=q^2+1$. For fixed q we have $n\to\infty$ as d grows, and

$$\lim_{n\to\infty}\frac{|\mathcal{F}|}{\prod_{\ell\in L}\frac{n-\ell}{k-\ell}}=\frac{k(k-1)(k-2)(k-q-1)}{k^4}=\frac{q^2(q^2-1)(q^2-q)}{(q^2+1)^3}=:C_q.$$

Moreover we have $\lim_{q\to\infty}C_q=1$. Therefore our construction is asymptotically best possible in this sense. In [5] an (n,k,L)-system ($k=q^2+1$, $L=\{0,1,2,q+1\}$) $\mathcal{F}'$ with $\lim_{n\to\infty}|\mathcal{F}'|/\prod_{\ell\in L}\frac{n-\ell}{k-\ell}=C'_q$ is constructed. In [5] and [3] they claim that $\lim_{q\to\infty}C'_q=1$, but it seems that their construction only gives $\lim_{q\to\infty}C'_q=\frac{1}{2}$.

5.4. **Unital.** Set $k=q^3+1$, $L=\{0,1,q+1\}$, and let

$$\mathcal{U}=\{(x,y,z)\in PG(2,q^2):x^{q+1}+y^{q+1}+z^{q+1}=0\}.$$

Then $|\mathcal{U}|=q^3+1$ and we obtain a $(2,q+1,q^3+1)_{q^2}$-matrix (cf. Example 26.8 in [11], [1]). Therefore we have $\alpha(q^3+1,\{0,1,q+1\})=3$.

For example, setting $q=2$ and $\mathbb{F}_4=\mathbb{F}_2(\beta)$ where $\beta^2+\beta+1=0$, we get the following $(2,3,9)_4$-matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & \beta & \beta^2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & \beta & \beta^2 \\ 1 & \beta & \beta^2 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The next example is a $(2,4,28)_9$-matrix ($q=3$ and $\mathbb{F}_9=\mathbb{F}_3(\beta)$ where $\beta^2+\beta+2=0$, $\beta^4=2$, $\beta^8=1$):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \beta & \beta^3 & \beta^5 & \beta^7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \beta & \beta^3 & \beta^5 & \beta^7 & \beta^2 & \beta^2 & \beta^2 & \beta^2 & 2 & 2 & 2 & 2 & \beta^6 & \beta^6 & \beta^6 & \beta^6 & 1 & 1 & 1 & 1 \\ \beta & \beta^3 & \beta^5 & \beta^7 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \beta^2 & 2 & \beta^6 & 1 & \beta^2 & 2 & \beta^6 & 1 & \beta^2 & 2 & \beta^6 & 1 & \beta^2 & 2 & \beta^6 & 1 \end{pmatrix}.$$

Let $\mathcal{F}$ be a (k,L)-system on n vertices constructed by a $(2,q+1,q^3+1)_{q^2}$-matrix. Then we have $|\mathcal{F}|=(\frac{n}{k})^3$ where $n=kq^d$, $k=q^3+1$. For fixed q we have $n\to\infty$ as d grows, and

$$\lim_{n\to\infty}\frac{|\mathcal{F}|}{\prod_{\ell\in L}\frac{n-\ell}{k-\ell}}=\frac{k(k-1)(k-q-1)}{k^3}=\frac{q^3(q^3-q)}{(q^3+1)^2}=:C_q.$$

Moreover we have $\lim_{q\to\infty}C_q=1$. Therefore our construction is asymptotically best possible in this sense.

5.5. **No $(2,3,13)_q$-matrix exists.** There are two non-isomorphic Steiner triple systems $S(2,3,13)$. But it is known [6] that there is no $(2,3,13)_q$-matrix for any q. It would be interesting to know which Steiner systems are supported by a generating matrix.

6. Intersection structure

In [7] the authors tried to determine all the exponents of (k,L)-systems for $k\leq 12$, but they could not find the exact values for 36 cases (16 cases for $k=11$, and 20 cases for $k=12$) out of $\sum_{k=2}^{12}2^{k-1}=4094$ cases. In this section, we settle all the remaining cases completely.

Let $k\in\mathbb{N}$ and $L\subset\{0,1,\dots,k-1\}$ be given. A family $\mathcal{I}\subset 2^{[k]}$ is called a closed L-system if $|I|\in L$ for all $I\in\mathcal{I}$ and $I\cap I'\in\mathcal{I}$ for all $I,I'\in\mathcal{I}$. Let us define the rank of $\mathcal{I}$ by

$$\operatorname{rank}(\mathcal{I}):=\min\{t\in\mathbb{N}:\Delta_t(\mathcal{I})\neq\binom{[k]}{t}\},$$

where Δ_t denotes the t-th shadow, i.e., $\Delta_t(\mathcal{I}):=\{J\in\binom{[k]}{t}:J\subset I \text{ for some } I\in\mathcal{I}\}$. Then the rank of (k,L)-system is defined by

$$\operatorname{rank}(k,L):=\max\{\operatorname{rank}(\mathcal{I}):\mathcal{I}\subset 2^{[k]}\text{ is a closed } L\text{-system}\}.$$

We say that $\mathcal{I} \subset 2^{[k]}$ is an intersection structure of a (k, L)-system if $\mathcal{I}$ is a closed L-system whose rank is $\mathrm{rank}(k, L)$. A generator set $\mathcal{I}^*$ of $\mathcal{I}$ is the collection of all maximal elements of $\mathcal{I}$, that is

$$\mathcal{I}^* := \{I \in \mathcal{I} : \nexists I' \in \mathcal{I} \text{ such that } I \subset I',\, I \neq I'\}.$$

We can retrieve $\mathcal{I}$ from $\mathcal{I}^*$ by taking all possible intersections.

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and an edge $F \in \mathcal{F}$ define

$$\mathcal{I}(F, \mathcal{F}) := \{F \cap F' : F' \in \mathcal{F} - \{F\}\} \subset 2^F.$$

Moreover, if $\mathcal{F}$ is k-partite with k-partition $[n] = X_1 \cup \cdots \cup X_k$ then we define the projection $\pi(I)$ of $I \in \mathcal{I}(F, \mathcal{F})$ by $\pi(I) := \{i : I \cap X_i \neq \emptyset\} \subset 2^{[k]}$ and set $\pi(\mathcal{I}(F, \mathcal{F})) := \{\pi(I) : I \in \mathcal{I}(F, \mathcal{F})\}$. Füredi[8] proved the following fundamental result.

Theorem 4. *Given $k \geq 2$ and $L \subset \{0, 1, \ldots, k-1\}$ there exists a positive constant $c = c(k, L)$ such that every (k, L)-system $\mathcal{F} \subset \binom{[n]}{k}$ contains a k-partite subfamily $\mathcal{F}^* \subset \mathcal{F}$ with k-partition $[n] = X_1 \cup \cdots \cup X_k$ satisfying (1)–(3).*

(1) $|\mathcal{F}^*| > c|\mathcal{F}|$.
(2) *For any two edges $F_1, F_2 \in \mathcal{F}^*$, $\pi(\mathcal{I}(F_1, \mathcal{F}^*)) = \pi(\mathcal{I}(F_2, \mathcal{F}^*))$.*
(3) *For all $F \in \mathcal{F}^*$, $\mathcal{I}(F, \mathcal{F}^*)$ is a closed L-system.*

In the above situation, we say that $\mathcal{I}(F, \mathcal{F}^*)$ is the intersection structure of $\mathcal{F}^*$. Let us see how the rank is related to the exponent of a (k, L)-system. Set $\mathcal{I} = \pi(\mathcal{I}(F, \mathcal{F}^*))$ and $t = \mathrm{rank}(\mathcal{I})$, and consider $\mathcal{F}^*$ in the above theorem. We can find some $A \in \binom{[k]}{t}$ such that $A \notin \Delta_t(\mathcal{I})$. Then for every $B \in \prod_{a \in A} V_a$ with $\pi(B) = A$ there is at most one member F of the family $\mathcal{F}^*$ such that $B \subset F$. Thus the size $|\mathcal{F}^*|$ is at most the number of choices for B, that is $\prod_{a \in A} |V_a| = O(n^t)$. In other words, if $\alpha(k, L)$ exists then we have

$$\alpha(k, L) \leq \mathrm{rank}(k, L).$$

On the other hand, Füredi[9] conjectures that

$$\alpha(k, L) > \mathrm{rank}(k, L) - 1.$$

This is true if $\mathrm{rank}(k, L) = 2$ (cf. [9]). As we will see in the next subsections, the conjecture is also true if $k \leq 12$ for all L. If there exists a Steiner system $S(t, b, k)$ then we have $\mathrm{rank}(k, L) = t + 1$ for $L = [0, t-1] \cup \{b\}$. Rödl and Tengan[12] found a construction which shows $\alpha(k, L) > t$ in this situation. However there is no general lower bound known for $\alpha(k, L)$ in terms of $\mathrm{rank}(k, L)$.

6.1. **The case $k = 11$.** Let $L_0 = \{0,1,2,3,5\}$. We consider (k,L)-systems with $k = 11$ and L listed below (16 cases).

(I) L_0, $L_0\cup\{6\}$, $L_0\cup\{8\}$, $L_0\cup\{9\}$, $L_0\cup\{6,8\}$, $L_0\cup\{6,9\}$, $L_0\cup\{8,9\}$, $L_0\cup\{6,8,9\}$, $L_0\cup\{11\}$, $L_0\cup\{6,11\}$, $L_0\cup\{8,11\}$, $L_0\cup\{9,11\}$, $L_0\cup\{6,8,11\}$, $L_0\cup\{6,9,11\}$, $L_0\cup\{8,9,11\}$, $L_0\cup\{6,8,9,11\}$.

By computer search, we found that $\operatorname{rank}(11,L) = 5$ for all L in (I), and the Steiner system $S(4,5,11)$ is the unique generator set of the corresponding intersection structure. As we saw in section 4 that $\alpha(11,L_0) = 5$, we now have $\alpha(11,L) = 5$ for all L in (I).

Next we consider the case $L = \{0,1,2,3,5,7\}$. By computer search, we found that $\operatorname{rank}(11,L) = 5$ and there are precisely two intersection structures — one is generated by $S(4,5,11)$ and the other is $\mathcal{I}_{11}$ described below.

Let $J_i = \{2i, 2i+1\}$ for $i = 1,\dots,5$ and set

$$\mathcal{P} = \{\{1\}\cup J_a\cup J_b\cup J_c : \{a,b,c\}\in\tbinom{[5]}{3}\}\subset\tbinom{[11]}{7},$$

$$\mathcal{Q} = \{\{j_1,j_2,j_3,j_4,j_5\} : j_i\in J_i \text{ and } \textstyle\sum j_i = \text{even}\}\subset\tbinom{[11]}{5}.$$

Then we define $\mathcal{I}_{11}^* = \mathcal{P}\cup\mathcal{Q}$ which is the generator set of $\mathcal{I}_{11}$.

Now we construct a 11-partite $(11,L)$-system $\mathcal{F}$ whose intersection structure is $\mathcal{I}_{11}$. Let A be the following generating matrix over $GF(2)$:

$$A = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&0&0&1&1\\ 0&0&0&1&1&1&1&1&1&1&1\\ 0&1&1&0&0&1&1&1&1&1&1\\ 1&0&1&0&1&0&1&0&1&0&1 \end{pmatrix}.$$

This is not a $(t,b,k)_q$-matrix, but we can define the family $\mathcal{F}$ on $V_1\cup\cdots\cup V_{11}$ where $V_i\cong\mathbb{F}_2^d$, by (1) in section 3.

Let c_i be the i-th column vector of A. We note the following two properties of A. One is that

$$c_{2i} + c_{2i+1} = c_1 \text{ for all } 1\le i\le 5. \tag{2}$$

The other is that for $j_i\in J_i = \{2i,2i+1\}$ $(1\le i\le 5)$ we have

$$c_{j_1} + c_{j_2} + \cdots + c_{j_5} = 0 \text{ iff } \textstyle\sum j_i = \text{even}. \tag{3}$$

Let us check that $\mathcal{F}$ is a $(11,\{0,1,2,3,5,7\})$-system. Choose $F,F'\in\mathcal{F}$ $(F\ne F')$ and set $I = \pi(F\cap F')$ where π is the projection. We shall show that $|I|\in\{0,1,2,3,5,7\}$. By construction, if c_j is a linear combination of c_i, $i\in I$ then we have $c_j\in I$.

First suppose that $1\in I$. Then by (2) we must have $|I\cap J_i| = 0$ or 2 for all $1\le i\le 5$, in particular, $|I| = \text{odd}$. We can accept

$|I| \in \{1,3,5,7\}$. Suppose that $|I| = 9$. Since $|I \cap J_i| \neq 1$, I contains precisely 4 of $J_1, \ldots, J_5$. But then by (3) and (2), I must contain all of $J_1, \ldots, J_5$, which is a contradiction.

Next we suppose that $1 \notin I$. Then by (2) we must have $|I \cap J_i| \leq 1$ for all $1 \leq i \leq 5$, in particular, $|I| \leq 5$. By (3) we cannot have $|I| = 4$. This concludes that $\mathcal{F}$ is a $(11, \{0,1,2,3,5,7\})$-system.

By construction, we have $|\mathcal{F}| = (n/11)^5 = n^5/161051$ where $n = |V(\mathcal{F})| = 11 \cdot 2^d$. On the other hand, a $(11, \{0,1,2,3,5,7\})$-system with size $|\mathcal{F}| = n^5/(15 \cdot 2^{17}) = n^5/1966080$ was already constructed in [5]. (Unfortunately some of the values in the tables contained in [5] seem to be inaccurate.) Both constructions use

$$\mathcal{V} = \{(x,y,z,w,\lambda) \in \mathbb{F}_2^5 - \{0\} : x^2 + xy + y^2 + zw = 0\}.$$

In fact $\mathcal{V}$ coincides with the set of column vectors of the matrix A.

6.2. **The case** $k = 12$**.** Let $L_0 = \{0,1,2,3,4,6\}$. We consider (k,L)-systems with $k = 12$ and L listed below $(16 + 4 = 20$ cases).

(I) L_0, $L_0 \cup \{7\}$, $L_0 \cup \{9\}$, $L_0 \cup \{10\}$, $L_0 \cup \{7,9\}$, $L_0 \cup \{7,10\}$, $L_0 \cup \{9,10\}$, $L_0 \cup \{7,9,10\}$, $L_0 \cup \{12\}$, $L_0 \cup \{7,12\}$, $L_0 \cup \{9,12\}$, $L_0 \cup \{10,12\}$, $L_0 \cup \{7,9,12\}$, $L_0 \cup \{7,10,12\}$, $L_0 \cup \{9,10,12\}$, $L_0 \cup \{7,9,10,12\}$.

(II) $L_0 \cup \{8\}$, $L_0 \cup \{8,9\}$, $L_0 \cup \{8,12\}$, $L_0 \cup \{8,9,12\}$.

By computer search, we found that $\mathrm{rank}(12, L) = 6$ for all L in (I) and (II). As we saw in section 4 that $\alpha(12, L_0) = 6$, we now have $\alpha(12, L) = 6$ for all L in (I) and (II). The Steiner system $S(5,6,12)$ is the unique generator set of the corresponding intersection structure for all L in (I). There are precisely two intersection structures for all L in (II) — one is generated by $S(5,6,12)$ and the other is $\mathcal{I}_{12}$ described below.

Let $J_i = \{2i-1, 2i\}$ for $i = 1, \ldots, 6$ and set

$$\mathcal{P} = \{J_a \cup J_b \cup J_c \cup J_d : \{a,b,c,d\} \in \tbinom{[6]}{4}\} \subset \tbinom{[12]}{8},$$

$$\mathcal{Q} = \{\{j_1, j_2, j_3, j_4, j_5, j_6\} : j_i \in J_i \text{ and } \textstyle\sum j_i = \text{even}\} \subset \tbinom{[12]}{6}.$$

Then we define $\mathcal{I}_{12}^* = \mathcal{P} \cup \mathcal{Q}$ which is the generator set of $\mathcal{I}_{12}$. The corresponding generating matrix over $GF(2)$ is the following:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Let $\mathcal{F}$ be the 12-partite family generated by A with vertex partition $[n] = V_1 \cup \cdots \cup V_{12}$ where $V_i \cong \mathbb{F}_2^d$. Let c_i be the i-th column vector of A. We note the following two properties of A. One is that

$$(c_{2s-1} + c_{2s}) + (c_{2t-1} + c_{2t}) = 0 \text{ for all } 1 \leq s < t \leq 6. \tag{4}$$

The other is that for $j_i \in J_i = \{2i-1, 2i\}$ $(1 \leq i \leq 6)$ we have

$$c_{j_1} + c_{j_2} + \cdots + c_{j_6} = 0 \text{ iff } \sum j_i = \text{even}. \tag{5}$$

Let us check that $\mathcal{F}$ is a $(12, \{0,1,2,3,4,6,8\})$-system. Choose $F, F' \in \mathcal{F}$ $(F \neq F')$ and set $I = \pi(F \cap F')$ where π is the projection. We shall show that $|I| \in \{0,1,2,3,4,6,8\}$. By construction, if c_j is a linear combination of c_i, $i \in I$ then we have $c_j \in I$.

If I contains one of $J_1, \ldots, J_6$, then by (4) we must have $|I \cap J_i| = 0$ or 2 for all i, in particular, $|I| =$ even. We can accept $|I| \in \{0,2,4,6,8\}$. Suppose that $|I| = 10$. Then I contains precisely 5 of $J_1, \ldots, J_6$. But then by (5) and (4), I must contain all of $J_1, \ldots, J_6$, which is a contradiction.

Suppose now that $|I| =$ odd. Then by (4) we need $|I \cup J_i| \leq 1$ for all $1 \leq i \leq 6$, which implies $|I| \leq 5$. But $|I| = 5$ is impossible because of (5).

Therefore, $\mathcal{F}$ is a $(12, \{0,1,2,3,4,6,8\})$-system with size $|\mathcal{F}| = (n/12)^6$ where $n = |V(\mathcal{F})| = 12 \cdot 2^d$.

Appendix: Tables of exponents

Here we present the complete tables of exponents for $k = 11$ and $k = 12$. We found it convenient to present the exponents $\alpha(k, L)$ in rectangular arrays with the rows indexed by subsets of $[0, \lfloor k/2 \rfloor]$ and the columns by subsets of $[\lfloor k/2 \rfloor + 1, k-1]$ and the (A, B) entry being $\alpha(k, A \cup B)$.

$k = 11$

		6	7	7 6	8	8 6	8 7	8 7 6	9	9 6	9 7	9 7 6	9 8	9 8 6	9 8 7	9 8 7 6	10	10 6	10 7	10 7 6	10 8	10 8 6	10 8 7	10 8 7 6	10 9	10 9 6	10 9 7	10 9 7 6	10 9 8	10 9 8 6	10 9 8 7	10 9 8 7 6
0	1	1	1	2	1	1	2	3	1	1	2	2	2	2	3	4	1	1	1	2	1	1	2	3	2	2	2	2	3	3	4	5
0 1	2	2	2	2	2	2	2	3	2	2	2	2	2	2	3	4	2	2	2	2	2	2	2	3	2	2	2	2	3	3	4	5
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0 1 2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	5
0 3	1	1	2	2	2	2	2	3	1	1	2	2	2	2	3	4	1	1	2	2	2	2	2	3	2	2	2	2	3	3	4	5
0 1 3	2	2	2	2	2	2	2	3	2	2	2	2	3	3	3	4	2	2	2	2	2	2	2	3	2	2	2	2	3	3	4	5
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0 1 2 3 4 5	6	7	6	8	6	7	6	9	6	7	6	8	6	7	6	10	6	7	6	8	6	7	6	9	6	7	6	8	6	7	6	11

$k = 12$

		6	7	7 6	8	8 6	8 7	8 7 6	9	9 6	9 7	9 7 6	9 8	9 8 6	9 8 7	9 8 7 6	10	10 6	10 7	10 7 6	10 8	10 8 6	10 8 7	10 8 7 6	10 9	10 9 6	10 9 7	10 9 7 6	10 9 8	10 9 8 6	10 9 8 7	10 9 8 7 6
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0 1 3 4 5	3	4	3	5	3	4	3	6	3	4	3	5	4	4	4	7	3	4	3	5	3	4	3	6	3	4	3	5	4	4	4	8
0 2 3 4 5	4	5	4	6	4	5	4	7	4	5	4	6	4	5	5	8	4	5	4	6	4	6	4	7	4	5	4	6	4	6	5	9
0 1 2 3 4 5	6	7	6	8	6	7	6	9	6	7	6	8	6	7	6	10	6	7	6	8	6	7	6	9	6	7	6	8	6	7	6	11

$k = 12$ (continued)

	11	11 6	11 7	11 7 6	11 8	11 8 6	11 8 7	11 8 7 6	11 9	11 9 6	11 9 7	11 9 7 6	11 9 8	11 9 8 6	11 9 8 7	11 9 8 7 6	11 10	11 10 6	11 10 7	11 10 7 6	11 10 8	11 10 8 6	11 10 8 7	11 10 8 7 6	11 10 9	11 10 9 6	11 10 9 7	11 10 9 7 6	11 10 9 8	11 10 9 8 6	11 10 9 8 7	11 10 9 8 7 6
0	1	2	1	2	1	2	2	3	1	2	1	2	2	2	3	4	2	2	2	2	2	3	2	3	3	3	3	3	4	4	5	6
0 1	2	2	2	3	2	2	2	3	2	2	2	3	2	2	3	4	2	2	2	3	2	3	2	3	3	3	3	3	4	4	5	6
0 2	2	2	2	2	2	3	2	3	2	2	2	2	2	3	3	4	2	2	2	2	2	3	2	3	3	3	3	3	4	4	5	6
0 1 2	3	3	3	4	3	3	3	4	3	3	3	4	3	3	3	4	3	3	3	4	3	3	3	4	3	3	3	4	4	4	5	6
0 3	2	3	2	3	2	3	2	3	2	4	2	4	2	4	3	4	2	3	2	3	2	3	2	3	3	4	3	4	4	4	5	6
0 1 3	2	3	2	3	2	3	2	3	2	4	2	4	2	4	3	4	2	3	2	3	2	3	2	3	3	4	3	4	4	4	5	6
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College of Education, Ryukyu University, Nishihara, Okinawa, 903-0213 Japan

E-mail address: hide@edu.u-ryukyu.ac.jp